Note: Students are expected to work independently on homework. You should not discuss the homework with other students. However, you can ask other students for help with R.

A challenging problem for statistical inference is finding confidence intervals for a parameter known to satisfy certain bounds. A lower bound of zero is a common occurrence. For example, physicists are interested in determining whether neutrinos have either zero mass or a positive mass, but their mass clearly is not negative. However, because of the complex and indirect methods of measuring the mass of particles, measured values of mass can be negative.

We know that for certain simple problems, e.g., a normal mean without constraints, Bayesian credible intervals using a non-informative prior coincide with the standard confidence interval. In this homework assignment, we will investigate the coverage probabilities of Bayesian credible intervals of a normal mean with known constraints. That is, we will see if credible intervals can also be interpreted as confidence intervals, or, at least, approximate confidence intervals in this situation.

As a simple example, we will assume that $X \sim N(\mu, \sigma^2)$ and $0 \leq \mu \leq 1$ and $\sigma$ known. Although we assume a single $X$, this is without loss of generally because $X$ could actually be a sample mean. This problem can be handled without MCMC, but we will use it as an introduction to MCMC calculations.

I have written an R function `BoundedMeanMCMC.R` which is in a file by the same name. The function is listed below. You can use this function by downloading the file from the course website into some directory on your computer and then “sourcing” the file. This function produces a sample from the posterior distribution using a random-walk Metropolis algorithm with a normal proposal density. A beta prior is assumed for $\mu$. The function returns the MCMC sample from the posterior and the estimated probability that the proposal is accepted. The latter is useful for tuning the choice of the proposal standard deviation.

```r
BoundedMeanMCMC <- function(x,a,b,sig_prop,sig_x,niter) {
  mu_curr = min( c(max(c(x,0)),1) )
  mu_samp = rep(0,niter)
  accept = rep(0,niter)
  for (i in 1:niter) {
    mu_prop = rnorm(1,mean=mu_curr,sd=sig_prop)
    num = dbeta(mu_prop,a,b)*dnorm(x,mean=mu_prop,sd=sig_x)
    den = dbeta(mu_curr,a,b)*dnorm(x,mean=mu_curr,sd=sig_x)
    ratio = num/den
    if (ratio > 1) { mu_samp[i] = mu_prop; accept[i] = 1 }
    else { mu_samp[i] = mu_curr; accept[i] = 0 }
  }
  return(list(x=x,sig_x=sig_x,a=a,b=b,mu=mu_samp,accept=accept))
}
```

accept_prob = min(c(ratio,1))
ind = (runif(1) < accept_prob)
mu_curr = (1-ind)*mu_curr + ind*mu_prop
mu_samp[i] = mu_curr
accept[i] = ind
}
list(mu = mu_samp,accept_prob = mean(accept))
}

Run the following program which is on the course web site and called hw4_prog1.R. The program will produce an MCMC sample from the posterior when \( \mu = 0 \) and \( \sigma = 0.3 \) and the prior is uniform(0,1) = beta(1,1). The program will also produce four plots based upon the sample from the posterior: a trace plot of every 20th element, a kernel density estimate, a histogram and loess smooth of the histogram, and the acf (autocorrelation function). The kernel density estimator doesn’t handle the known bounds on \( \mu \), and the loess smooth of the histogram is preferable because it can handle the constraints.

```r
a = 1
b = 1
sig_prop = .5
sig_x = .3
niter = 20000
mu = 0
x = rnorm(1,mean=mu,sd=sig_x)
out = BoundedMeanMCMC(x,a,b,sig_prop,sig_x,niter)
mu_samp = out$mu
print(out$accept_prob)
par(mfrow=c(2,2))
plot(mu_samp[20*(1:1000)],type='l')
plot(density(mu_samp))
hs = hist(mu_samp,probability=TRUE,breaks=seq(from=0,to=1,by=.01))
lo = loess(hs$density ~ hs$mids)
lines(hs$mids,lo$fitted)
acf(mu_samp)
```

1. Investigate how the acceptance probability and the 1st order autocorrelation coefficient depend on \( \text{sig\_prop} \), with \( \mu, a, b \), and \( \text{sig\_x} \) fixed at the values in this program. Summarize your results; graphs could be helpful here, but also include a written summary. Assume that you want to minimize the 1st order autocorrelation coefficient. What value of \( \text{sig\_prop} \) do you recommend? What is the acceptance probability if your recommended value of \( \text{sig\_prop} \) is used?

2. Consider two types of 95% credible intervals: 1) the interval from the 2.5 percentile to the 97.5 percentile of the sample of \( \mu \) from the posterior, and 2) the 95% highest posterior density interval. Investigate the coverage probability of both intervals with \( \mu, a, b, \text{sig\_x}, \) and \( \text{sig\_prop} \) fixed at the values in this program. To get a good estimate of a coverage probability, you need to generate at least 1,000 values of \( X \), preferably more. For each \( X \), sample the posterior, find the credible interval, and check whether the credible interval contains 0 (the true value of \( \mu \)).

3. Repeat part 2. with \( \mu \) equal to 0.2 and \( a, b, \text{sig\_x}, \) and \( \text{sig\_prop} \) as before.