Time (in)consistency of multistage distributionally robust inventory models with moment constraints

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Recently, there has been a growing interest in developing inventory control policies which are robust to model misspecification. One approach is to posit that nature selects a worst-case distribution for any relevant stochastic primitives from some pre-specified family. Several communities have observed that a subtle phenomena known as time inconsistency can arise in this framework. In particular, it becomes possible that a policy which is optimal at time zero (i.e. solution to the multistage-static formulation) may not be optimal for the associated optimization problem in which the decision-maker recomputes her policy at each point in time (i.e. solution to the distributionally robust dynamic programming formulation), which has implications for implementability. If there exists a policy which is optimal for both formulations (w.p.1 under every joint distribution for demand belonging to the uncertainty set), we say that the policy is *time consistent*, and the problem is *weakly time consistent*. If every optimal policy for the multistage-static formulation is time consistent, we say that the problem is *strongly time consistent*.

We study these phenomena in the context of managing an inventory over time, when only the mean, variance, and support are known for the demand at each stage. We provide several illustrative examples showing that here the question of time consistency can be quite subtle. We complement these observations by providing simple sufficient conditions for weak and strong time consistency. We also relate our results to the well-studied notion of *rectangularity* of a family of measures. Interestingly, our results show that time consistency may hold even when rectangularity does not. Although a similar phenomena was previously identified by Shapiro for the setting in which only the mean and support of the demand are known, there the problem was always weakly time consistent, with both formulations having the same optimal value. Here our model is rich enough to exhibit a variety of interesting behaviors, including lack of weak time consistency, strong time consistency even when both formulations have different optimal values, and non-existence of even a single optimal base-stock policy under the static formulation.

Key words: inventory, news vendor, multistage distributionally robust optimization, rectangularity, moment constraints, time consistency, dynamic programming, base-stock policy

1. Introduction

The news vendor problem, used to analyze the trade-offs associated with stocking an inventory, has its origin in a seminal paper by Edgeworth (1888). In its classical formulation, the problem

is stated as a minimization of the expected value of the relevant ordering, backorder, and holding costs. Such a formulation requires a complete specification of the probability distribution of the underlying demand process. However, in applications knowledge of the exact distribution of the demand process is rarely available. This motivates the study of minimax type (i.e. distributionally robust) formulations, where minimization is performed with respect to a worst-case distribution from some family of potential distributions. In a pioneering paper Scarf (1958) gave an elegant solution for the minimax news vendor problem when only the first and second order moments of the demand distribution are known. His work has led to considerable follow-up work (cf. Gallego and Moon (1993, 1994), Gallego (1998, 2001), Popescu (2005), Yue, Chen and Wang (2006), Gallego, Katircioglu and Ramachandran (2007), Perakis and Roels (2008), Chen and Sim (2009), See and Sim (2010), Hanasusanto et al. (2015), Zhu, Zhang and Ye (2013)). For a more general overview of risk analysis for news vendor and inventory models we can refer, e.g., to Ahmed, Cakmak and Shapiro (2007) and Choi, Ruszczyński and Zhao (2011). We also note that a distributionally robust minimax approach is not the only way to model such uncertainty, and that there is a considerable literature on alternative approaches such as the robust optimization paradigm (cf. Kasugai and Kasegai (1961), Ben-Tal et al. (2005), Bertsimas and Thiele (2006), Ben-Tal, Boaz and Shimrit (2009), Bertsimas, Iancu and Parrilo (2010), Carrizosaa, Olivares-Nadal and Ramirez-Cobob (2016), Gabrel, Murat and Thiele (2014)) and Bayesian approach (cf. Scarf (1959, 1960), Lovejoy (1992), Levi, Perakis and Uichanco (2015), Klabjan, Simchi-Levi and Song (2013)).

In practice an inventory must often be managed over some time horizon, and the classical news vendor problem was naturally extended to the multistage setting, for which there is also a considerable literature (see, e.g., Zipkin (2000) and the references therein). Recently, distributionally robust variants of such multistage problems have begun to receive attention in the literature (cf. Gallego (2001), Ahmed, Cakmak and Shapiro (2007), Choi and Ruszczynski (2008), See and Sim (2010), Shapiro (2012), Klabjan, Simchi-Levi and Song (2013)). It has been observed that such multistage distributionally robust optimization problems can exhibit a subtle phenomenon known as time inconsistency. Over the years various concepts of time consistency have been discussed in the economics literature, in the context of rational decision making. This can be traced back at least to the work of Strotz (1955) - for a more recent overview we refer the reader to the recent survey by Etner, Jeleva and Tallon (2012), and the references therein. Questions of time consistency have also attracted attention in the mathematical finance literature, in the context of assessing the risk and value of investments over time, and have played an important role in the associated theory of coherent risk measures (cf. Wang (1999), Artzner et al. (2007), Roorda and Schumacher (2007), Cheridito and Kupper (2009), Ruszczyński (2010)). These concepts have also been studied from the perspective of robust control across various academic communities (cf. Hansen and Sargent

(2001), Iyengar (2005), Nilim and El Ghaoui (2005), Grunwald and Halpern (2011), Carpentier et al. (2012), Wiesemann, Kuhn and Rustem (2013)). Recently, these concepts have also begun to receive attention in the setting of inventory control (cf. Chen et al. (2007), Chen and Sun (2012), Yang (2013), Homem-de-Mello and Pagnoncelli (2016), Shapiro and Xin (2017)).

In this work, we will consider questions of time (in)consistency in the context of managing an inventory over time. We will give a formal definition of time consistency, which is naturally suited to our framework, in Section 4. At this point let us provide the following high-level intuition. A multistage distributionally robust optimization problem can be viewed in two ways. In one formulation, the policy maker is allowed to recompute her policy choice after each stage (we will refer to this as the distributionally robust dynamic programming (DP) formulation), thus taking prior realizations of demand into consideration when performing the relevant minimax calculations at later stages. In that case it follows from known results that there exists a base-stock policy which is optimal. In the second formulation, the policy maker is not allowed to recompute her policy after each stage (we will refer to this as the *multistage-static formulation*), in which case far less is known. If these two formulations have a common optimal policy, i.e. the policy maker would be content with the given policy whether or not she has the power to recompute after each stage (w.p.1 under every joint distribution for demand belonging to the uncertainty set), we say that the policy is *time consistent*, and the problem is *weakly time consistent*. If every optimal policy for the multistage-static formulation is time consistent, i.e. it is impossible to devise a policy which is optimal at time zero yet suboptimal at a later time, we say that the problem is *strongly time consistent*. Such a property is desirable from a policy perspective, as it ensures that previously agreed upon policy decisions remain rational when the policy is actually implemented, possibly at a later time.

Within the optimization and inventory control communities, much of the work on time consistency restricts its discussion of optimal policies to the setting in which the family of distributions from which nature can select satisfies a certain factorization property called *rectangularity*, which endows the associated minimax problem with a DP structure. Outside of this setting, most of the literature focuses on discussing and demonstrating hardness of the underlying optimization problems (cf. Iyengar (2005), Nilim and El Ghaoui (2005), Wiesemann, Kuhn and Rustem (2013)). We note that this is in spite of the fact that previous literature has discussed the importance and relevance of such non-decomposable formulations from a modeling perspective (cf. Iyengar (2005)).

1.1. Our contributions

In this paper, we depart from much of the past literature by seeking both negative *and positive* results regarding time consistency when no such decomposition holds, i.e. the underlying family of

distributions from which nature can select is non-rectangular. Our work is in the spirit of Grunwald and Halpern (2011), in which a definition of (weak) time consistency similar to ours was analyzed in the context of rectangularity and dynamic consistency (a concept defined in Epstein and Schneider (2003)), albeit in a substantially different context motivated by questions in decision theory and artificial intelligence. Our work can also be viewed as providing a more in-depth and inventory-focused study of several notions of time-consistency studied in Homem-de-Mello and Pagnoncelli (2016). In contrast to Homem-de-Mello and Pagnoncelli (2016) and several other works in which all concepts are explained through the language of risk measures, here we explain all relevant concepts purely in the language of (robust) newsvendor models with moment constrainst, a model popular in the operations management community, and hope that in doing so our work brings the concept of time-consistency to a broader audience.

We extend the work of Scarf (1958) (and followup work of Gallego (2001)) by considering the question of time consistency in multistage news vendor problems when the support and first two moments are known for the demand at each stage, and demand is stage-wise independent. In addition to refining multiple definitions related to time-consistency, we provide several illustrative examples showing that here the question of time consistency can be quite subtle. In particular: (i) the problem can fail to be weakly time consistent, (ii) the problem can be weakly but not strongly time consistent, and (iii) the problem can be strongly time consistent even if every associated optimal policy takes different values under the multistage-static and distributionally robust DP formulations. We also prove that, although the distributionally robust DP formulation always has an optimal policy of base-stock form, there may be no such optimal policy for the multistage-static formulation. We complement these observations by providing simple sufficient conditions for weak and strong time consistency.

Interestingly, in contrast to much of the related literature, our results show that time consistency may hold even when rectangularity does not. This stands in contrast to the analysis of Shapiro (2012) for the setting in which only the mean and support of the demand distribution are known, where the problem is always (weakly) time consistent, amenable to a simple DP solution, with both formulations having the same optimal value. Likewise, in the setting in which only the support is known, both formulations reduce to the so-called adjustable robust formulation described in Ben-Tal et al. (2004), where again (weakly) time consistency always holds.

1.2.   Outline of paper

The structure of the rest of the paper is organized as follows. In Section 2, we review the single-stage classical and distributionally robust formulations and their properties, as well as Scarf's solution to the single-stage distributionally robust formulation and various generalizations. In Section 3, we

discuss the extension to the multi-stage setting, formally defining the multistage-static formulation, the relevant notions of time-consistency, and the distributionally robust DP formulation, and review the notion of rectangularity and its relation to our own formulations. In Section 4, we prove our sufficient conditions for weak and strong time consistency, and present several illustrative examples showing that here the question of time consistency can be quite subtle. In Section 5, we provide closing remarks and directions for future research. We include a technical appendix in Section 6.

2. Single-stage formulation

In this section we review both the classical and distributionally robust single-stage formulation, including some relevant results of Scarf (1958) and Natarajan and Zhou (2007).

2.1. Classical formulation

Consider the following classical formulation of the news vendor problem:

$$\inf_{x \geq 0} \mathbb{E}[\Psi(x, D)], \tag{1}$$

where

$$\Psi(x, d) := cx + b[d - x]_+ + h[x - d]_+, \tag{2}$$

and c, b, h are the ordering, backorder penalty, and holding costs, per unit, respectively. Unless stated otherwise we assume that $b > c > 0$ and $h \geq 0$. The expectation is taken with respect to the probability distribution of the demand D, which is modeled as a random variable having nonnegative support. It is well known that this problem has the closed form solution $\bar{x} = F^{-1}\left(\frac{b-c}{b+h}\right)$, where $F(\cdot)$ is the cumulative distribution function (cdf) of the demand D, and F^{-1} is its inverse. Of course, it is assumed here that the probability distribution, i.e. the cdf F, is completely specified.

2.2. Distributionally robust formulation

Suppose now that the probability distribution of the demand D is not fully specified, but instead assumed to be a member of a family of distributions $\mathfrak{M}$. Then we consider the following distributionally robust formulation:

$$\inf_{x \geq 0} \psi(x), \tag{3}$$

where

$$\psi(x) := \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[\Psi(x, D)], \tag{4}$$

and the notation $\mathbb{E}_Q$ emphasizes that the expectation is taken with respect to the distribution Q of the demand D.

We now introduce some additional notations to describe certain families of distributions. For a probability measure (distribution) Q, we let supp(Q) denote the support of the measure, i.e.

the smallest closed set $A \subseteq \mathbb{R}$ such that $Q(A) = 1$. With a slight abuse of notation, for a random variable Z, we also let supp(Z) denote the support of the associated probability measure. For a given closed (and possibly unbounded) subset $\mathcal{I} \subseteq \mathbb{R}$, we let $\mathfrak{P}(\mathcal{I})$ denote the set of probability distributions Q such that supp$(Q) \subseteq \mathcal{I}$. Although we will be primarily interested in the setting that $\mathcal{I} \subseteq \mathbb{R}_+$ (i.e. demand is nonnegative), it will sometimes be convenient for us to consider more general families of demand distributions. By δ_a we denote the probability measure of mass one at $a \in \mathbb{R}$.

In this paper, we will study families of distributions satisfying moment constraints of the form

$$\mathfrak{M} := \left\{ Q \in \mathfrak{P}(\mathcal{I}) : \mathbb{E}_Q[D] = \mu, \mathbb{E}_Q[D^2] = \mu^2 + \sigma^2 \right\}. \tag{5}$$

Unless stated otherwise, it will be assumed that $\mathfrak{M}$ is indeed of the form (5), and is nonempty. We let α denote the left-endpoint of $\mathcal{I}$ (or $-\infty$ if $\mathcal{I}$ is unbounded from below), and let β denote the right-endpoint of $\mathcal{I}$ (or $+\infty$ if $\mathcal{I}$ is unbounded from above); i.e., $\mathcal{I} = [\alpha, \beta]$. Here we note that if α or β equals $\pm\infty$, the interval should be interpreted as being unbounded in the associated direction(s). It may be easily verified that the set $\mathfrak{M}$ is nonempty iff the following conditions hold:

$$\mu \in [\alpha, \beta] \text{ and } \sigma^2 \leq (\beta - \mu)(\mu - \alpha), \tag{6}$$

which will be assumed throughout. (We assume here that $0 \times \infty = 0$, so that if, e.g., $\mu = \alpha$ and $\beta = +\infty$, then the right hand side of (6) is 0.)

Furthermore, one can also identify conditions under which $\mathfrak{M}$ is a singleton.

Observation 1 *If $-\infty < \alpha < \beta < +\infty$, $\mu \in [\alpha, \beta]$, and $\sigma^2 = (\beta - \mu)(\mu - \alpha)$, then $\mathfrak{M}$ consists of the single probability measure which assigns to the point α probability $p = \frac{\beta - \mu}{\beta - \alpha}$, and to the point β probability $1 - p = \frac{\mu - \alpha}{\beta - \alpha}$.*

We now rephrase $\psi(x)$ as the optimal value of a certain optimization problem. For use in later proofs, we define the following more general maximization problem, in terms of a general integrable objective function ζ:

$$\begin{array}{c} \sup\limits_{Q \in \mathfrak{P}(\mathcal{I})} \int \zeta(\tau) dQ(\tau) \\ \text{s.t.} \quad \int \tau dQ(\tau) = \mu, \int \tau^2 dQ(\tau) = \mu^2 + \sigma^2. \end{array} \tag{7}$$

Our definitions imply that for all $x \in \mathbb{R}$, $\psi(x)$ equals the optimal value of Problem (7) for the special case that $\zeta(\tau) = \Psi(x, \tau)$. Problem (7) is a classical problem of moments (see, e.g., Landau 1987). From the Richter-Rogosinski Theorem (e.g., Shapiro, Dentcheva and Ruszczyński 2009, Proposition 6.40) or results in Bertsimas and Popescu (2005), we have the following.

Observation 2 *If Problem (7) possesses an optimal solution, then it has an optimal solution with support of at most three points.*

2.2.1. Review of relevant duality theory. As several of our later proofs will be based on duality theory, we now briefly review duality for Problem (7). The dual of Problem (7) can be constructed as follows (cf. Isii 1962). Consider the Lagrangian

$$L(Q, \lambda) := \int \Big[\zeta(\tau) - \sum_{i=0}^{2} \lambda_i \tau^i \Big] dQ(\tau) + \lambda_0 + \lambda_1 \mu + \lambda_2 (\mu^2 + \sigma^2).$$

By maximizing $L(Q, \lambda)$ with respect to $Q \in \mathfrak{P}(\mathcal{I})$, and then minimizing with respect to λ, we obtain the following Lagrangian dual for Problem (7):

$$\begin{aligned} &\inf_{\lambda \in \mathbb{R}^3} \ \lambda_0 + \lambda_1 \mu + \lambda_2 (\mu^2 + \sigma^2) \\ &\text{s.t.} \ \ \lambda_0 + \lambda_1 \tau + \lambda_2 \tau^2 \geq \zeta(\tau), \ \ \tau \in \mathcal{I}. \end{aligned} \tag{8}$$

We denote by $\mathrm{val}(P)$ and $\mathrm{val}(D)$ the respective optimal values of the primal Problem (7) and its dual Problem (8). By convention, if Problem (7) is infeasible, we set $\mathrm{val}(P) = -\infty$, and if Problem (8) is infeasible, we set $\mathrm{val}(D) = +\infty$. We denote by $\mathrm{Sol}_P(x)$ the set of optimal solutions of the primal problem, and by $\mathrm{Sol}_D(x)$ the set of optimal solutions of the dual problem, and note that these sets may be empty, even when both programs are feasible, e.g. if the respective optimal values are approached but not attained.

Note that $\mathrm{val}(D) \geq \mathrm{val}(P)$. We now give sufficient conditions for there to be no duality gap, i.e. $\mathrm{val}(P) = \mathrm{val}(D)$, as well as conditions for Problems (7) and (8) to have optimal solutions. By specifying known general results for duality of such programs, e.g., (Bonnans and Shapiro 2000, Theorem 5.97), to the considered setting, we have the following.

Proposition 2.1 *If $\bar{Q}$ is a probability measure which is feasible for the primal Problem (7), $\bar{\lambda} = (\bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2)$ is a vector which is feasible for the dual Problem (8), and*

$$\mathrm{supp}(\bar{Q}) \subseteq \big\{ \tau \in \mathcal{I} : \zeta(\tau) = \bar{\lambda}_0 + \bar{\lambda}_1 \tau + \bar{\lambda}_2 \tau^2 \big\}, \tag{9}$$

then $\bar{Q}$ is an optimal primal solution, $\bar{\lambda}$ is an optimal dual solution, and $\mathrm{val}(P) = \mathrm{val}(D)$. Conversely, if $\mathrm{val}(P) = \mathrm{val}(D)$, and $\bar{Q}$ and $\bar{\lambda}$ are optimal solutions of the respective primal and dual problems, then condition (9) holds.

2.2.2. Scarf's solution. We note that the distributionally robust single-stage news vendor problem considered by Scarf (1958) is exactly Problem (3), when $\mathcal{I} = \mathbb{R}_+$. As it will be useful for later proofs, we briefly review Scarf's explicit solution. We actually state a slight generalization of the results of Scarf, and for completeness we include a proof in the technical appendix (Section 6). Define $\mathfrak{f}(z) := \big((z - \mu)^2 + \sigma^2 \big)^{\frac{1}{2}}$ for all $z \in \mathbb{R}$.

Theorem 1 *Suppose that $b > c$, $c + h > 0$, $\mu > 0$, $\sigma > 0$, and $\mathcal{I} = \mathbb{R}_+$. Let $\kappa := \frac{b-h-2c}{b+h}$. Then for each $x \in \mathbb{R}$,*

$$\psi(x) = \begin{cases} c\mu + \frac{b+h}{2}\left((x-\mu)^2 + \sigma^2\right)^{\frac{1}{2}} - \frac{b-h-2c}{2}(x-\mu), & \text{if } x \geq \frac{\mu^2+\sigma^2}{2\mu}, \\ \frac{(h+c)\sigma^2 - (b-c)\mu^2}{\mu^2+\sigma^2}x + b\mu, & \text{if } x \in [0, \frac{\mu^2+\sigma^2}{2\mu}), \\ b\mu - (b-c)x, & \text{otherwise.} \end{cases} \tag{10}$$

As a consequence, a complete solution to the problem $\inf_{x \in \mathbb{R}} \psi(x)$ is as follows.

(i) If $\frac{\sigma^2}{\mu^2} > \frac{b-c}{h+c}$, then the unique optimal solution is $x = 0$, and the optimal value is μb.

(ii) If $\frac{\sigma^2}{\mu^2} < \frac{b-c}{h+c}$, then the unique optimal solution is $x = \mu + \kappa\sigma(1-\kappa^2)^{-\frac{1}{2}}$, and the optimal value is $c\mu + \left((h+c)(b-c)\right)^{\frac{1}{2}}\sigma$.

(iii) If $\frac{\sigma^2}{\mu^2} = \frac{b-c}{h+c}$, then all $x \in [0, \mu + \kappa\sigma(1-\kappa^2)^{-\frac{1}{2}}]$ are optimal, and the optimal value is μb.

Furthermore, in all cases $\arg\max_{Q \in \mathfrak{M}} \mathbb{E}_Q[\Psi(x, D)]$ is nonempty for every $x \in \mathbb{R}$. Also, the optimal solution set and value of the problem $\inf_{x \in \mathbb{R}} \psi(x)$ is identical to that of Problem (3), i.e. optimizing over $x \in \mathbb{R}$, as opposed to $x \in \mathbb{R}_+$, makes no difference.

For use in later proofs (e.g., sufficient conditions for strong time consistency in Theorem 4), it will also be useful to demonstrate a particular variant of Theorem 1. Suppose that in Problem (3), we are not forced to select a deterministic constant x, but can instead select any distribution D_1 for x. Specifically, let us consider the following minimax problem:

$$\inf_{Q_1 \in \mathfrak{P}(\mathcal{I})} \phi(Q_1), \tag{11}$$

where

$$\phi(Q_1) := \sup_{Q_2 \in \mathfrak{M}} \mathbb{E}_{Q_1 \times Q_2}\left[\Psi(D_1, D_2)\right],$$

and the notation $\mathbb{E}_{Q_1 \times Q_2}$ indicates that for any choices for the marginal distributions Q_1, Q_2 of D_1 and D_2, the expectation is taken with respect to the associated product measure, under which D_1 and D_2 are independent. In this case, we have the following result, whose proof we defer to the technical appendix (Section 6).

Proposition 2.2 *Suppose that $b > c$, $c + h > 0$, $\mu > 0$, $\sigma > 0$, $\frac{\sigma^2}{\mu^2} > \frac{b-c}{h+c}$, and $\mathcal{I} = \mathbb{R}$. Then Problem (11) has the unique optimal solution $\bar{Q}_1 = \delta_0$.*

We also note that ψ inherits the property of convexity from Ψ.

Observation 3 *$\Psi(\cdot, d)$ is a convex function for every fixed $d \in \mathcal{I}$, ψ is a convex function on $\mathbb{R}$, and Problem (3) is a convex program.*

2.2.3. Generalization of Natarajan and Zhou (2007) to a class of convex, continuous, piecewise affine functions.

Scarf (1958) gave an explicit solution for Problems (7) and (8) when $\mathcal{I} = \mathbb{R}_+$, and ζ is a convex, continuous piecewise affine function with exactly two pieces, by explicitly constructing a feasible primal - dual solution pair satisfying the conditions of Proposition 2.1 (details of this construction can be found in Section 6). Natarajan and Zhou (2007) generalized Scarf's results to a class of convex, continuous, piecewise affine (CCPA) functions with three pieces. We now state the solution to a special case of the problems studied in Natarajan and Zhou (2007), as we will need the solution to such problems for our later studies of time consistency. For completeness, we provide a proof in the technical appendix (Section 6).

Theorem 2 *[Natarajan and Zhou (2007)] Suppose that there exist $c_1, c_2 > 0$ such that $c_1 < c_2$, and $\zeta(d) = \max\{-d + c_1, 0, d - c_2\}$ for all $d \in \mathbb{R}$. Let $\eta := \frac{1}{2}(c_1 + c_2)$, and recall that $\mathfrak{f}(z) := \left((z - \mu)^2 + \sigma^2\right)^{\frac{1}{2}}$. Further suppose that $\sigma > 0$, $\mathcal{I} = \mathbb{R}_+$,*

$$\frac{1}{4}(2\mu - 3c_1 + c_2)(3c_2 - c_1 - 2\mu) \leq \sigma^2,$$

and $\eta - \mathfrak{f}(\eta) \geq 0$. Then the unique optimal solution to the primal Problem (7) is the probability measure Q having support at two points $h_1 = \eta - \mathfrak{f}(\eta)$ and $h_2 = \eta + \mathfrak{f}(\eta)$, with

$$Q(h_1) = \sigma^2 \left(\sigma^2 + \left(\eta - \mathfrak{f}(\eta) - \mu\right)^2\right)^{-1}, \quad Q(h_2) = 1 - Q(h_1). \tag{12}$$

Also, the unique optimal solution to the dual Problem (8) is

$$\lambda_0 = \tfrac{1}{2}\left(\eta^2 + (\eta - \mu)^2 + \sigma^2\right)\mathfrak{f}^{-1}(\eta) + \frac{c_1 - c_2}{2}, \quad \lambda_1 = -\eta\mathfrak{f}^{-1}(\eta), \quad \lambda_2 = \tfrac{1}{2}\mathfrak{f}^{-1}(\eta), \tag{13}$$

where $\mathfrak{f}^{-1}(\eta)$ represents the reciprocal of $\mathfrak{f}(\eta)$.

3. Multistage formulation

In this section, we study a multistage extension of the distributionally robust news vendor problem discussed in Section 2.2.

3.1. Classical formulation

We begin by giving a quick review of the classical (i.e. non-robust) multistage news vendor problem (also called inventory problem), and start by introducing some additional notations. For a vector $z = (z_1, ..., z_n) \in \mathbb{R}^n$ and $1 \leq i \leq j \leq n$, denote $z_{[i,j]} := (z_i, ..., z_j)$. In particular for $i = 1$ we simply write $z_{[j]}$ for the vector consisting of the first j components of z, and set $z_{[0]} := \emptyset$.

We suppose that there is a finite time horizon T, and a (random) vector of demands $D = (D_1, ..., D_T)$. By $d = (d_1, ..., d_T)$ we usually denote a particular realization of the random vector D. We assume that the components of random vector D are *mutually independent*, and refer to

this as the *stagewise independence* condition. We now define the family of admissible policies Π by introducing two families of functions, $\{y_t,\ t=1,\ldots,T\}$ and $\{x_t,\ t=1,\ldots,T\}$. Conceptually, y_t will correspond to the inventory level at the start of stage t, and x_t will correspond to the inventory level after having ordered in stage t, but before the demand in that stage is realized.

We will consider policies which are nonanticipative, i.e. decisions do not depend on realizations of future demand. We assume that y_1, the initial inventory level, is a given constant. We also require that one can only order a nonnegative amount of inventory at each stage. Thus the set of admissible policies Π should consist of those vectors of (measurable) functions $\pi = \{x_t(d_{[t-1]}), t = 1,\ldots,T\}$, such that $x_t : \mathbb{R}_+^{t-1} \to \mathbb{R}$ satisfies $x_t(d_{[t-1]}) \geq y_t$, for all $d_{[t-1]} \in \mathbb{R}_+^{t-1}$ and $t = 1,...,T$, where

$$y_{t+1} = x_t(d_{[t-1]}) - d_t,\ t = 1,...,T-1. \tag{14}$$

It follows that any given choice of $\pi \in \Pi$, along with the given y_1, completely determines the associated functions $y_1, \ldots, y_T$. Sometimes we will explicitly express x_t and y_t as a function of the associated policy π and demands $d_{[t]}$ with the notations $x_t^\pi(d_{[t-1]})$ and $y_t^\pi(d_{[t-1]})$; other times we will suppress this notation. Combining the above, we can write the classical multistage news vendor problem (inventory problem) as follows:

$$\inf_{\pi \in \Pi} \mathbb{E}\left\{ \sum_{t=1}^T \rho^{t-1}\left[c_t\big(x_t^\pi(D_{[t-1]}) - y_t^\pi(D_{[t-1]})\big) + \Psi_t\big(x_t^\pi(D_{[t-1]}), D_t\big) \right] \right\}. \tag{15}$$

Here $\rho \in (0,1]$ is a discount factor, c_t, b_t, h_t are the ordering, backorder penalty and holding costs per unit in stage t, respectively, and

$$\Psi_t(x_t, d_t) := b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+. \tag{16}$$

Unless stated otherwise, we assume that $b_t > c_t > 0$ and $h_t \geq 0$ for all $t = 1,...,T$.

Problem (15) can be viewed as an optimal control problem in discrete time with state variables y_t, control variables x_t and random parameters D_t. It is well known that Problem (15) can be solved using DP equations, which can be written as

$$V_t(y_t) = \inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \mathbb{E}\big[\Psi_t(x_t, D_t) + \rho V_{t+1}(x_t - D_t) \big] \right\}, \tag{17}$$

$t = 1,...,T$, with $V_{T+1}(\cdot) \equiv 0$ (e.g., Zipkin (2000)). Note that the value functions $V_t(\cdot)$ are convex, and do not depend on the demand data because of the stagewise independence assumption. These equations naturally define a set of policies through the relation $x_t(y_t) \in \mathfrak{X}_t(y_t)$, where $\mathfrak{X}_t(y_t)$, $t = 1,...,T$, is the set of optimal solutions of the problem

$$\inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \mathbb{E}[\Psi_t(x_t, D_t) + \rho V_{t+1}(x_t - D_t)] \right\}, \tag{18}$$

and the optimal value of Problem (15) is given by $V_1(y_1)$. Note that $x_t(y_t)$, $t = 1, \ldots, T$, are functions of y_t, i.e., it suffices to consider policies (measurable functions) of the form $x_t = \pi_t(y_t)$; this fact is well known from optimal control theory (see, e.g., Bertsekas and Shreve (1978) for technical details). Of course, the assumption of stagewise independence is essential for this conclusion.

Under the specified conditions, the objective function of Problem (18) tends to $+\infty$ as $x_t \to \pm\infty$. It thus follows from convexity that this objective function possesses a (possibly non-unique) unconstrained minimizer x_t^* over $x \in \mathbb{R}$, and $\bar{x}_t := \max\{y_t, x_t^*\}$ is an optimal solution of Problem (18). In particular, the so-called *base-stock policy* is optimal for the inventory Problem (15), where we note that such a result is classical in the inventory literature.

Definition 3.1 *A policy $\pi \in \Pi$ is said to be a* base-stock *policy if there exist constants x_t^*, $t = 1, \ldots, T$, such that*

$$x_t^\pi = \max\left\{y_t^\pi, x_t^*\right\}, \, t = 1, \ldots, T, \tag{19}$$

That is, Problem (15) can be solved using the DP formulation (17) and associated policy (18) in the following sense.

Lemma 3.1 *The optimal value of Problem (15) equals $V_1(y_1)$. Any policy π such that $x_t^\pi(d_{[t-1]}) \in \mathfrak{X}_t\big(y_t^\pi(d_{[t-1]})\big)$ are for all $t = 1, ..., T$ and $d_{[t-1]} \in \mathbb{R}_+^{t-1}$, is an optimal solution to Problem (15). Conversely, for any optimal policy π for Problem (15), and any $t \in \{1, ..., T\}$, there exists a set $A \subseteq \mathbb{R}$ such that $\Pr\big(y_t^\pi(D_{[t-1]}) \in A\big) = 1$, and $x_t^\pi(D_{[t-1]}) \in \mathfrak{X}_t\big(y_t^\pi(D_{[t-1]})\big)$ conditional on the event $\{y_t^\pi(D_{[t-1]}) \in A\}$. Furthermore, it follows from the convexity of the relevant cost-to-go functions $V_t(y_t)$ that any set of base-stock constants $\{x_t^*, t = 1, \ldots, T\}$ such that $x_t^* \in \mathfrak{X}_t(0)$ for all $t \in [1, T]$ will yield an optimal policy for Problem (15).*

As we shall see, such an equivalence does not necessarily hold for distributionally robust multistage inventory problems with moment constraints.

3.2. Distributionally robust formulations

Suppose now that the distribution of the demand process is not known, and we only have at our disposal information about the support and first and second order moments. In this case, it is natural to use the framework previously developed for the single-stage problem (see Section 2) to handle the distributional uncertainty at each stage. However, in the multistage setting, there is a nontrivial question of how to model the associated uncertainty in the joint distribution of demand.

We will consider two formulations, one intuitively corresponding to the modeling choices of a policy maker who does not recompute her policy choices after each stage and one corresponding to a policy-maker who does. These two formulations are analogous to the two optimization models

discussed in Iyengar (2005) and Nilim and El Ghaoui (2005) in the framework of robust MDP, and can also be interpreted through the lens of (non)rectangularity of the associated families of priors (cf. Epstein and Schneider (2003), Iyengar (2005), Nilim and El Ghaoui (2005)), as we will explore later in this section. We refer to these formulations as *multistage-static* and distributionally robust DP, respectively. Questions regarding the interplay between the sets of optimal policies of these two formulations are important from an implementability perspective, as a policy deemed optimal at time 0, but which does not remain optimal if the relevant decisions are re-examined at a later time, may not be implemented by the relevant stake-holders. We note that such considerations were one of the original motivations for the study of time consistency in economics (cf. Strotz (1955)). We further note that the particular definitions and formulations we introduce here are by no means the only way to define the relevant notions of time consistency, and again refer the reader to the survey by Etner, Jeleva and Tallon (2012), and other recent papers in the optimization community (cf. Iyengar (2005), Boda and Filar (2006), Carpentier et al. (2012), Iancu, Petrik and Subramanian (2015), Homem-de-Mello and Pagnoncelli (2016), Shapiro and Xin (2017)) for alternative perspectives.

We suppose that we have been given a sequence of closed (possibly unbounded) intervals $\mathcal{I}_t = [\alpha_t, \beta_t] \subset \mathbb{R}$, $t = 1, \ldots, T$, and sequences of the corresponding means $\{\mu_t, t = 1, \ldots, T\}$, and variances $\{\sigma_t^2, t = 1, \ldots, T\}$.

3.2.1.   Multistage-static formulation. We first consider the following formulation, referred to as *multistage-static*, in which the policy maker does not recompute her policy choices after each stage. Let us define

$$\mathfrak{M}_t := \left\{ Q_t \in \mathfrak{P}(\mathcal{I}_t) : \mathbb{E}_{Q_t}[D_t] = \mu_t, \ \mathbb{E}_{Q_t}[D_t^2] = \mu_t^2 + \sigma_t^2 \right\}, \ t = 1, \ldots, T; \tag{20}$$

$$\mathfrak{M} := \left\{ Q = Q_1 \times \cdots \times Q_T : Q_t \in \mathfrak{M}_t, \ t = 1, \ldots, T \right\}. \tag{21}$$

That is, the set $\mathfrak{M}$ consists of probability measures given by direct products of probability measures $Q_t \in \mathfrak{M}_t$. This can be viewed as an extension of the stage-wise independence condition, employed in Section 3.1, to the considered distributionally robust case. In order for the sets $\mathfrak{M}_t$ to be nonempty we assume that (compare with (6))

$$\mu_t \in [\alpha_t, \beta_t] \text{ and } \sigma_t^2 \leq (\beta_t - \mu_t)(\mu_t - \alpha_t), \ t = 1, \ldots, T. \tag{22}$$

According to (21), the associated minimax problem supposes that although the set of associated marginal distributions may be "worst-case", the joint distribution will always be a product measure (i.e. the demand will be independent across stages). The multistage-static formulation for the distributionally robust inventory problem can then be formulated as follows.

$$\inf_{\pi \in \Pi} \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q\left[Z^\pi\right], \tag{23}$$

where $Z^\pi = Z^\pi(D_{[T]})$ is a function of $D_{[T]} = (D_1, ..., D_T)$ given by

$$Z^\pi(D_{[T]}) := \sum_{t=1}^{T} \rho^{t-1} \big[c_t \big(x_t^\pi(D_{[t-1]}) - y_t^\pi(D_{[t-1]}) \big) + \Psi_t \big(x_t^\pi(D_{[t-1]}), D_t \big) \big], \tag{24}$$

and Π is the set of admissible policies defined previously in Section 3.1. Of course, if the set $\mathfrak{M} = \{Q\}$ is a singleton, then formulation (23) coincides with formulation (15) taken with respect to the distribution $Q = Q_1 \times \cdots \times Q_T$ of the demand vector $D_{[T]}$.

We note that the multistage-static formulation (23) is closely related to optimization with risk measures. Indeed, the functional $\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q [Z]$ is a coherent risk measure (cf. Shapiro, Dentcheva and Ruszczyński (2009)).

Very little is known about the set of optimal policies for Problem (23), as this problem does not enjoy a DP formulation along the lines of (17).

3.2.2.   Time consistency and distributionally robust DP equations. As informally referenced earlier, time inconsistency refers to the possibility that policy choices which seemed optimal from the perspective of time 0 no longer seem optimal if one re-performs one's minimax calculations at a later time. Although first addressed within the economics community, the issue of time (in)consistency has recently started to receive attention in the stochastic and robust optimization communities (cf. Riedel (2004), Boda and Filar (2006), Artzner et al. (2007), Shapiro (2009), Ruszczyński (2010), Grunwald and Halpern (2011), Carpentier et al. (2012), Shapiro (2012), Chen, Li and Guo (2013), Iancu, Petrik and Subramanian (2015), Homem-de-Mello and Pagnoncelli (2016)), in which closely related concepts such as Pareto robust optimality (Iancu and Trichakis (2014)) have also been studied. We note that related issues were addressed even in the seminal work of Bellman (1957) on DP, where it is asserted that: *"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."* The same principle has been subsequently reformulated by several authors in a somewhat more precise form, e.g., in the recent work of Carpentier et al. (2012), where it is asserted that *"The decision maker formulates an optimization problem at time t_0 that yields a sequence of optimal decision rules for t_0 and for the following time steps $t_1, ..., t_N = T$. Then, at the next time step t_1, he formulates a new problem starting at t_1 that yields a new sequence of optimal decision rules from time steps t_1 to T. Suppose the process continues until time T is reached. The sequence of optimization problems is said to be dynamically consistent if the optimal strategies obtained when solving the original problem at time t_0 remain optimal for all subsequent problems."* A nearly identical concept, which the authors refer to as the inherited-optimality-property (IOP), is also formalized in Homem-de-Mello and Pagnoncelli (2016).

To motivate the particular definition of time-consistency we will use here, which is very similar to that used in e.g. Homem-de-Mello and Pagnoncelli (2016), let us reason as follows. Suppose we wished to know whether a policy π which was optimal for the multi-stage static formulation had the property that, should one re-perform one's minimax calculation in the final period, one would make the same ordering decision. As she cannot change past decisions, the only policy decision she still has to make is the determination of the function x_T. However, she now has knowledge of $D_{[T-1]}$ and y_T, which she can incorporate into her minimax computations. We note that here we are faced with the modeling question of how to reconcile the use of $D_{[T-1]}$ and y_T's realized values in performing one's minimax computations with the previously assumed stagewise independence of demand. A natural approach, consistent with the economics literature on time consistency, is to reason as follows. As $D_{[T-1]}$ has already been realized, it is unreasonable to enforce independence of D_T on this realization, as it is no longer undetermined. Instead, the relevant minimax computation is carried out with this knowledge of the realization of $D_{[T-1]}$. At that time, such a policy-maker is thus led to the optimization (at time T)

$$\inf_{x_T \geq y_T} \left\{ c_T(x_T - y_T) + \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T}[\Psi_T(x_T, D_T)] \right\},$$

with

$$\mathfrak{Y}_T(y_T) := \arg\min_{x_T \geq y_T} \left\{ c_T(x_T - y_T) + \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T}[\Psi_T(x_T, D_T)] \right\}$$

the corresponding set of optimal policy choices. Here we note that (for example) the inner maximization $\sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T}[\Psi_T(x_T, D_T)]$ is implicitly a function of $D_{[T-1]}$, through the dependence on x_T. Thus time-consistency of an optimal policy π for the multi-stage static formulation should (at least as regards policy decisions in this final period) be equivalent to requiring that $x_T^\pi(D_{[T-1]}) \in \mathfrak{Y}_T\big(y_T^\pi(D_{[T-1]})\big)$. We note that there is a second subtlety here. Indeed, as the exact distribution of $D_{[T]}$ is no longer known with certainty, the question of under which measures (for $D_{[T]}$) the requirement $x_t^\pi(D_{[t-1]}) \in \mathfrak{Y}_T\big(y_T^\pi(D_{[T-1]})\big)$ should hold w.p.1 must be resolved. We note that when all distributions in $\mathcal{M}$ have the same support, such issues do not arise, while for the moment-based uncertainty sets we consider this distinction is important. We also note that many past works do not take this subtlety into account in their definitions. Here, we propose the natural and intuitive interpretation that one should require the inclusion hold w.p.1 for every measure in $\mathcal{M}$, as these are exactly those measures one believes possible.

Distributionally robust DP formulation. Before proceeding with our formal definition of time consistency, let us expand on the distributionally robust DP formulation, which we have defined only in the final period. Carrying out the same logic inductively, we conclude that if a policy is to be deemed time-consistent when the policy-maker is (possibly) given the choice to recompute

her minimax calculations in an arbitrary set of time periods, her choices should be consistent with the following distributionally robust DP equations.

$$V_t(y_t) = \inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \sup_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\Psi_t(x_t, D_t) + \rho V_{t+1}(x_t - D_t)] \right\}, \tag{25}$$

$t = 1, ..., T$, with $V_{T+1}(\cdot) \equiv 0$. The optimal value of DP formulation (25) is given by $V_1(y_1)$. Dynamic equations (25) naturally define a set of policies of the form $x_t = \pi_t(y_t)$, $t = 1, \ldots, T$, with $x_t = \pi_t(y_t)$ being measurable selections $x_t \in \mathfrak{Y}_t(y_t)$ from sets

$$\mathfrak{Y}_t(y_t) := \operatorname*{arg\,min}_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \sup_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\Psi_t(x_t, D_t) + \rho V_{t+1}(x_t - D_t)] \right\}, \ t = 1, ..., T. \tag{26}$$

We refer to (25) as the *distributionally robust DP formulation* and $V_1(y_1)$ as its optimal value.

We now observe that due to certain convexity properties, DP formulation (25) always possesses an optimal base-stock policy. We note that such results are generally well-known to hold in this setting (cf. Ahmed, Cakmak and Shapiro (2007)). Recall Definition 3.1 of a base-stock policy.

Observation 4 *It follows from the convexity of the relevant cost-to-go functions $V_t(y_t)$ that Problem (25) possesses an optimal base-stock policy. Furthermore, any set of base-stock constants $\{x_t^*, t = 1, \ldots, T\}$ such that $x_t^* \in \mathfrak{Y}_t(0)$ for all $t \in [1, T]$ yields an optimal policy. Namely, for any such $\{x_t^*, t = 1, \ldots, T\}$, $\max\{y, x_t^*\} \in \mathfrak{Y}_t(y)$ for all $y \in \mathbb{R}$ and $t = 1, \ldots, T$.*

We note that the same conclusion could also have been drawn by rephrasing our formulation in the language of coherent risk measures, and applying known results for so-called nested risk measures (cf. Ruszczyński and Shapiro (2006), (Shapiro, Dentcheva and Ruszczyński 2009, section 6.7.3)), although we do not pursue such an analysis here.

We note that the question of whether or not there exists such an optimal base-stock policy for the multistage-static formulation is considerably more challenging, and will be central to our discussion of time consistency.

We close our discussion of the distributionally robust DP formulation with a final definition, formalizing our earlier discussion of for which measures one should require optimality (of decisions) under the distributionally robust DP formulation.

Definition 3.2 (Robust-w.p.1-optimal) *Let us say that a policy $\pi \in \Pi$ is robust-w.p.1-optimal for the distributionally robust DP formulation if for all $Q \in \mathcal{M}$, w.p.1 $x_t^\pi(D_{[t-1]}) \in \mathfrak{Y}_t\big(y_t^\pi(D_{[t-1]})\big)$ for all $t \in [1, T]$.*

Formal definition of time consistency. We now formally define time consistency, in light of our earlier discussion. We note that given the motivation behind time consistency, i.e. implementation of policies, a further subtlety must be considered. Clearly, it is desirable for there to exist

at least one policy which is optimal both initially, and if reconsidered at later times. However, it is similarly undesirable for there to exist even one policy which could potentially be selected (i.e. optimal) initially, but deemed sub-optimal (i.e. non-implementable) at a later time. Although such a notion is of course a stringent requirement (as noted informally in passing in Homem-de-Mello and Pagnoncelli (2016)), we believe that its conceptual importance none-the-less makes it worthy of further study. This motivates the following definition(s) of time consistency, where we note that similar definitions were presented in Grunwald and Halpern (2011) in a different context motivated by considerations in decision theory and artificial intelligence. Then our definition of time consistency is as follows.

Definition 3.3 (Time consistency) *If a policy $\pi \in \Pi$ is optimal for the multistage-static formulation* (23), *and robust-w.p.1-optimal for the distributionally robust DP formulation, we say that π is* time consistent. *If there exists at least one optimal policy $\pi \in \Pi$ which is time consistent, we say that Problem* (23) *is* weakly time consistent. *If every optimal policy of Problem* (23) *is time consistent, we say that Problem* (23) *is* strongly time consistent.

Of course the notion of strong time consistency makes sense only if Problem (23) possesses at least one optimal solution. Otherwise it is strongly time consistent simply because the set of optimal policies is empty.

Our definition of time consistency can, in a certain sense, be viewed as an extension of the definition typically used in the theory of risk measures to an optimization context. In Section 4.3.3, we show that it is possible for the multistage-static problem to have an optimal solution and to be strongly time consistent, but with a different optimal value than the distributionally robust DP formulation. That is, it is possible for the multistage-static problem to possess an optimal solution and to be strongly time consistent even when the rectangularity property does not hold. This stands in contrast to the definition of consistency typically used in the theory of risk measures, i.e. the notion of dynamic consistency coming from Epstein and Schneider (2003) and based on a certain stability of preferences over time, which may result in a problem being deemed inconsistent based on the values that a given optimal policy takes under the different formulations, and even the values taken by suboptimal policies (cf. Ruszczyński (2010), Grunwald and Halpern (2011)). In an optimization setting one may be primarily concerned only with the implementability of optimal policies, irregardless of their values and the values of suboptimal policies, and this is the approach we take here. We note that such optimization-oriented formulations have been considered in several recent works, e.g. Carpentier et al. (2012) and Homem-de-Mello and Pagnoncelli (2016), and our definitions are largely consistent with those works.

Before exploring some of the subtle and interesting features of time (in)consistency for our model, we briefly review some previously known results for related models. Note that if the set $\mathfrak{M}$ is a singleton, then both the multistage-static and distributionally robust DP formulations collapse to the classical formulation. Hence both formulations have the same optimal value and strong time consistency follows. If one only has information about the support $\mathcal{I}_t$, and hence takes $\mathfrak{M}_t$ to be the set of all probability measures supported on the interval $\mathcal{I}_t$, $t = 1, ..., T$, then both the multistage-static and distributionally robust DP formulations collapse to the so-called adjustable robust formulation (cf. Ben-Tal et al. (2004), Shapiro (2011)), which is purely deterministic. As a consequence, both formulations have the same optimal value and weak time consistency follows. However, the recent work of Shapiro and Xin (2017) (itself inspired in part by an earlier version of this paper, Xin, Goldberg and Shapiro (2013)), shows that even in that setting strong time consistency need not hold, and studies related phenomena in several particular problems with general moment constraints. However, we note that all inventory problems considered in Shapiro and Xin (2017) are weakly time consistent with both formulations having the same optimal value due to the rectangularity of the underlying set of measures. Shapiro and Xin (2017) also shows that there are interesting connections between the notions of weak and strong time-consistency and the concept of "strict monotonicity" for risk measures (e.g., Shapiro (2017)), and we leave further investigations of this connection as an interesting direction for future research. We also note that the existence of optimal time inconsistent policies was investigated earlier in several purely robust (i.e. deterministic) settings. In particular, Bertsimas, Iancu and Parrilo (2010) demonstrated the optimality of so-called affine policies in certain settings, and Delage and Iancu (2015) explicitly constructed optimal time inconsistent policies in an inventory control setting.

Connection to rectangularity. To contextualize our definitions within the broader literature, we here briefly review the relevant notion of rectangularity. Our definitions will closely follow those given in Shapiro (2016), although we note that many closely related definitions have appeared previously throughout the literature (see e.g. Iyengar (2005)). Consider the cost $Z^\pi = Z^\pi(D_{[T]})$ of a policy π, defined in (24). Let $\widehat{\mathfrak{M}}$ be a set of probability distributions for the demand vector $D_{[T]}$, and let $Q \in \widehat{\mathfrak{M}}$. At the moment we do not assume that Q is of the product form $Q = Q_1 \times \cdots \times Q_T$, we will discuss this later. We can write

$$\mathbb{E}_Q[Z^\pi] = \mathbb{E}_Q\left[\mathbb{E}_{Q|D_1}\left[\cdots \mathbb{E}_{Q|D_{[T-2]}}\left[\mathbb{E}_{Q|D_{[T-1]}}[Z^\pi]\right]\right]\right], \tag{27}$$

where $\mathbb{E}_{Q|D_{[t]}}[Z^\pi]$ is the conditional expectation, given $D_{[t]}$, with respect to the distribution Q of $D_{[T]}$. Of course, this conditional expectation is a function of $D_{[t]}$. Consequently,

$$\sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q[Z^\pi] \leq \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q\left[\sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_1}\left[\cdots \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_{[T-1]}}[Z^\pi]\right]\right]. \tag{28}$$

The right hand side of (28) leads to the nested formulation

$$\inf_{\pi \in \Pi} \left\{ \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q \left[\sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_1} \left[\cdots \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_{[T-1]}} [Z^\pi] \right] \right] \right\}. \tag{29}$$

In particular, if the set $\widehat{\mathfrak{M}}$ is defined in the form (21), i.e., consists of products of probability measures (with the t-th measure drawn from $\mathfrak{M}_t$), then formulation (29) simplifies to

$$\inf_{\pi \in \Pi} \left\{ \sup_{Q_1 \in \mathfrak{M}_1} \mathbb{E}_{Q_1} \left[\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2|D_1} \left[\cdots \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T|D_{[T-1]}} [Z^\pi] \right] \right] \right\}. \tag{30}$$

It follows from (28) that the optimal value of (29) is greater than or equal to the optimal value of the multistage-static Problem (23). Moreover, the optimal value of (29) can be strictly greater than the optimal value of (23). Let us demonstrate this through the following simple example.

Example 1 *Let $T = 2$ and the set $\widehat{\mathfrak{M}}$ be of the product form (21). Suppose further that $\mathcal{I}_1 = [0, 1]$, $\mu_1 = 1/2$ and $\sigma_1^2 = 1/4$. Then $\mathfrak{M}_1 = \{Q_1\}$ is a singleton with $Q_1 = p_1 \delta_0 + p_2 \delta_1$, $p_1 = p_2 = 1/2$, i.e., with probability $1/2$ the demand D_1 can be either zero or one. Let us fix some policy $\pi \in \Pi$ and let $Z^\pi = Z^\pi(D_1, D_2)$ be the corresponding objective function. Then the associated cost under formulation (23) equals*

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} [Z^\pi(D_1, D_2)] = \sup_{Q_2 \in \mathfrak{M}_2} \left(p_1 \mathbb{E}_{Q_2}[Z^\pi(0, D_2)] + p_2 \mathbb{E}_{Q_2}[Z^\pi(1, D_2)] \right), \tag{31}$$

while the associated cost under formulation (29) equals

$$\mathbb{E}_{Q_1} \left[\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2|D_1} [Z^\pi(D_1, D_2)] \right] = p_1 \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}[Z^\pi(0, D_2)] + p_2 \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}[Z^\pi(1, D_2)]. \tag{32}$$

Note that the worst case distribution $Q_2 \in \mathfrak{M}_2$ in (31) has to be the same for all possible realizations of the demand D_1. In contrast, the worst case distribution $Q_2 \in \mathfrak{M}_2$ in (32) is allowed to depend on realized D_1. Hence the right hand side of (32) can be strictly greater than the right hand side of (31).

In line with the definition given in Shapiro (2016), we say that the set $\widehat{\mathfrak{M}}$ of probability measures is rectangular if such strict inequality does not occur for any r.v. Z^π, i.e. the two formulations are equivalent in their optimal values. More formally, we make the following definition.

Definition 3.4 *Consistent with the definition given in Shapiro (2016), we say that the set $\widehat{\mathfrak{M}}$ of probability measures is rectangular if for every measurable and non-negative function f,*

$$\sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q[f(D_{[T]})] = \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q \left[\sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_1} \left[\cdots \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_{Q|D_{[T-1]}} [f(D_{[T]})] \right] \right]. \tag{33}$$

We note that under additional compactness assumptions on the support of $D_{[T]}$, Shapiro (2016) formally explores several related concepts and subtleties of this definition, e.g. proves that one can associate a rectangular set of probability measures to any given (possibly non-rectangular) set of probability measures, but as such a compactness condition does not hold in our setting we do not explore that further here.

For a rectangular set $\widehat{\mathfrak{M}}$ the static formulation

$$\inf_{\pi \in \Pi} \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q\big[Z^\pi\big], \tag{34}$$

is equivalent (in an appropriate sense, see e.g. Shapiro (2016, 2017)) to the formulation (29). Furthermore, the natural generalization of the distributionally robust DP equations (25) can be applied to (34), with both formulations having a common optimal policy and the same optimal value.

We note that the concept of rectangularity has been central to the past literature on time consistency (cf. Epstein and Schneider (2003), Grunwald and Halpern (2011), Iancu, Petrik and Subramanian (2015)), especially as it relates to optimization (cf. Iyengar (2005), Nilim and El Ghaoui (2005), Wiesemann, Kuhn and Rustem (2013)). In several of these works, connections were made between tractability of the associated robust MDP and various notions of rectangularity (e.g. (s,a)-rectangularity, s-rectangularity). We refer the interested reader to Wiesemann, Kuhn and Rustem (2013) and the references therein for details. Our definition of rectangularity is aimed directly at the decomposability property of the static formulation ensuring its equivalence to the corresponding dynamic formulation (see Shapiro (2016) for details).

In general the set of product measures $\mathfrak{M}$ we consider in this work is not rectangular, as certified by the possible lack of weak time-consistency which we will soon demonstrate. We note that a rectangular analogue of the set of measures $\mathfrak{M}$ defined in (20)-(21) would be the set of all joint distributions Q for $D_{[T]}$ such that

$$D_t \in \mathfrak{P}(\mathcal{I}_t) \quad , \quad \mathbb{E}_Q[D_t|D_{[t-1]}] = \mu_t \quad , \quad \mathbb{E}_Q[D_t^2|D_{[t-1]}] = \mu_t^2 + \sigma_t^2, \quad t = 1, \ldots, T. \tag{35}$$

Non-rectangular (and intractable) formulations for robust MDP are described in both Iyengar (2005) and Nilim and El Ghaoui (2005). In Iyengar (2005), it is referred to as the static formulation, while in Nilim and El Ghaoui (2005), it is referred to as the stationary formulation. In both of these settings, these non-rectangular formulations essentially equate to requiring nature to select the same transition kernel every time a given state (and action, depending on the formulation) is encountered, as opposed to being able to select a different kernel every time a given state is visited in the robust MDP, and we refer the reader to Iyengar (2005), Nilim and El Ghaoui (2005), and

Wiesemann, Kuhn and Rustem (2013) for details. Although our multistage-static formulation could similarly be phrased in terms of a particular kind of dependency between the choices of nature in a robust MDP framework, and would be significantly different from either of the aforementioned non-rectangular formulations, we do not pursue such an investigation here, and leave the formalization of such connections as a direction for future research.

4.   Time consistency : sufficient conditions and (counter) examples
4.1.   Sufficient conditions for weak time consistency

In this section, we provide simple sufficient conditions for the weak time consistency of Problem (23). Our condition is essentially equivalent to monotonicity of the associated base-stock constants. Intuitively, in this case the inventory manager can always order up to the optimal inventory level with which to enter the next time period, irregardless of previously realized demand. Thus any potential for the adversary to take advantage of previously realized demand information in the distributionally robust DP formulation is "masked" by the fact that the actual attained inventory level will always be this idealized level, under both formulations. We note that several previous works have identified monotonicity of base-stock levels as a condition which causes various inventory problems to become tractable, in a variety of settings (cf. Veinott (1965), Ignall and Veinott (1969), Jagannathan (1978), Zipkin (2000)). In particular, Jagannathan (1978) studied a similar distributionally robust inventory model with moment constraints and identified monotonicity of base-stock levels as a sufficient condition for a myopic base-stock policy to be optimal. For completeness of the paper (as well as use in the later proofs), we state a variant of the results of Jagannathan (1978) in this section and include a proof in the appendix.

We begin by providing a different (but equivalent) formulation for Problem (23), in which all relevant instances of y_t are rewritten in terms of the appropriate x_t functions, as this will clarify the precise structure of the relevant cost-to-go functions. As a notational convenience, let $c_{T+1} = 0$, in which case we define

$$\hat{\Psi}_t(x_t, d_t) := (c_t - \rho c_{t+1})x_t + b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+, \; t = 1, ..., T. \tag{36}$$

Let us define the problem

$$\inf_{\pi \in \Pi} \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q \left[\sum_{t=1}^{T} \rho^{t-1} \hat{\Psi}_t \big(x_t(y_t), D_t \big) \right] - c_1 y_1 + \sum_{t=1}^{T-1} \rho^t c_{t+1} \mu_t. \tag{37}$$

Then, using straightforward substitution we can make the following observation.

Observation 5 *Problem* (23) *and Problem* (37) *are equivalent, i.e. each policy* $\pi \in \Pi$ *has the same value under both formulations.*

We now derive a lower bound for any policy by allowing the policy maker to reselect her inventory at the start of each stage, at no cost. As it turns out, this bound is "attainable" when the set of base-stock levels is monotone increasing. For $x \in \mathbb{R}$, let us define

$$\eta_t(x) := \sup_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\hat{\Psi}_t(x, D_t)], \ \ \Gamma_t^x := \arg\max_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\hat{\Psi}_t(x, D_t)], \tag{38}$$

and let

$$\begin{aligned} \hat{\eta}_t &:= \inf_{x \in \mathbb{R}} \eta_t(x) = \inf_{x \in \mathbb{R}} \sup_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\hat{\Psi}_t(x, D_t)], \\ \hat{\Gamma}_t &:= \arg\min_{x \in \mathbb{R}} \eta_t(x) = \arg\min_{x \in \mathbb{R}} \sup_{Q_t \in \mathfrak{M}_t} \mathbb{E}_{Q_t}[\hat{\Psi}_t(x, D_t)]. \end{aligned} \tag{39}$$

For $j \geq 1$, and probability measures $Q_1, \ldots, Q_j$, let us define $\otimes_{t=1}^{j} Q_t := Q_1 \times \ldots \times Q_j$, i.e. the associated product measure with the corresponding marginals. Then we have the following.

Lemma 4.1 *Suppose that the sets $\Gamma_t^x, \hat{\Gamma}_t$ are non-empty for all $x \in \mathbb{R}$, $t = 1, ..., T$. Let us fix any $\pi = (x_1, \ldots, x_T) \in \Pi$, and $i \geq 0$. Then for any given $Q_1 \in \mathfrak{M}_1, \ldots, Q_i \in \mathfrak{M}_i$, there exist $Q_{i+1} \in \mathfrak{M}_{i+1}, \ldots, Q_T \in \mathfrak{M}_T$ such that*

$$\mathbb{E}_{\otimes_{j=1}^{T} Q_j}\left[\hat{\Psi}_t\big(x_t(y_t), D_t\big)\right] \geq \hat{\eta}_t \text{ for all } t \geq i+1. \tag{40}$$

Furthermore, the optimal value of Problem (23) is at least $\sum_{t=1}^{T} \rho^{t-1} \hat{\eta}_t - c_1 y_1 + \sum_{t=1}^{T-1} \rho^t c_{t+1} \mu_t$.

We defer the proof to the technical appendix (Section 6). We now show that the bound of Lemma 4.1 is "realizable" when the set of base-stock levels is monotone increasing, and that in this case the associated base-stock policy is optimal for both the multistage-static and distributionally robust DP formulations. In particular, in this setting, the associated base-stock policy is time consistent, and thus the multistage-static problem is weakly time consistent. Again we defer the proof to the appendix Section 6.

Theorem 3 *Suppose there exists a nondecreasing sequence x_t^*, $t = 1, ..., T$, such that $y_1 \leq x_1^*$, and $x_t^* \in \hat{\Gamma}_t$, $t = 1, ..., T$, where $\hat{\Gamma}_t$ is defined in (39). Also suppose $\mathcal{I}_t \subset \mathbb{R}_+$ for all $t = 1, ..., T$. Then the base-stock policy π for which $x_t(y) = \max\{y, x_t^*\}$ for all $y \in \mathbb{R}$, is an optimal policy for the multistage-static formulation, and a robust-w.p.1-optimal policy for the distributionally robust DP formulations, and attains value $\sum_{t=1}^{T} \rho^{t-1} \hat{\eta}_t - c_1 y_1 + \sum_{t=1}^{T-1} \rho^t c_{t+1} \mu_t$ under both formulations. Consequently, this base-stock policy is time consistent, and the multistage-static problem is weakly time consistent.*

We note that Theorem 3 implies that if the parameters $\mu_t, \sigma_t, c_t, b_t, h_t$ and $\mathcal{I}_t$ are the same for all $t = 1, ..., T$, and hence the sets $\mathfrak{M}_t$ are also the same for all t, then the multistage-static problem is weakly time consistent, and the multistage-static and distributionally robust DP formulations have the same optimal value.

4.2.   Sufficient conditions for strong time consistency

In this section, we show that under additional assumptions, which ensure that the variance in each stage is sufficiently large, the multistage-static problem is strongly time consistent. As we will see, in this case there is a unique optimal base-stock policy, and in this policy all base-stock constants equal zero, the intuition being that when the variance is sufficiently large, it becomes undesirable to give nature any additional "wiggle room". We further note that such a base-stock policy has been widely adopted in practice and the resulting inventory system is a so-called Make-To-Order (MTO) or "Pull" system. In such a system, no inventory is carried and the replenishment is based on actual demands instead of forecasts (cf. Williams (1984), Arreola-Risa and DeCroix (1998), Federgruen and Katalan (1999), Rajagopalan (2002), Kaminsky and Kaya (2009)). We will later see in Section 4.3.2 that deviating slightly from this setting may lead to a lack of strong time consistency. In particular, our results demonstrate that strong time consistency is a very fragile property. Our sufficient conditions are as follows.

Theorem 4 *Suppose that* $b'_t := b_t - c_t + \rho c_{t+1} > 0$, $h'_t := h_t + c_t - \rho c_{t+1} > 0$, $\sigma_t, \mu_t > 0$, $\mathcal{I}_t = \mathbb{R}_+, t = 1, \ldots, T$, $y_1 = 0$, *and*

$$\frac{\sigma_t^2}{\mu_t^2} > \frac{b'_t}{h'_t}, \;\; t = 1, ..., T. \tag{41}$$

Then the set of optimal policies for the multistage-static problem is exactly the set of policies

$$\Pi^0 := \left\{ \pi = (x_1, \ldots, x_T) \in \Pi : x_1(y_1) = 0, x_t(z) = 0 \, for \, all \, z \leq 0 \, and \, t \in [1, T] \right\},$$

and the multistage-static problem is strongly time consistent.

We defer the proof to the technical appendix (Section 6).

We note that under certain rectangularity-related assumptions, necessary and sufficient conditions for the existence of time-inconsistent optimal policies in an inventory setting with moment constraints was very recently provided in Shapiro and Xin (2017). However, those results are not applicable to the setting we consider, as the uncertainty sets we consider here are inherently non-rectangular, and thus (for example) our formulation allows for the non-existence of weak time-consistency, as well as the two formulations having different optimal values, and even the possibility that no policy of base-stock form is optimal for the multistage-static formulation (while the assumptions of Shapiro and Xin (2017) do not allow for such behavior). Furthermore, Shapiro and Xin (2017) considers only 2-period problems, while the sufficient conditions provided in this work hold in the general multi-period setting.

4.3. Further investigation of time (in)consistency

We now demonstrate that the question of time (in)consistency becomes quite delicate for inventory models with moment constraints, by considering a series of examples in which our model exhibits interesting (and sometimes counterintuitive) behavior. In particular: (i) the problem can fail to be weakly time consistent, (ii) the problem can be weakly but not strongly time consistent, and (iii) the problem can be strongly time consistent even if every associated optimal policy takes different values under the multistage-static and distributionally robust DP formulations. We also prove that, although the distributionally robust DP formulation always has an optimal policy of the base-stock form, there may be no such optimal policy for the multistage-static formulation. We note that (i) and (ii) are subtle phenomena which the simpler models discussed in several previous works (e.g. Shapiro (2012)) cannot exhibit. We also note that (iii) emphasizes an interesting and surprising feature of our model and definitions: (strong) time consistency can hold even when the underlying family of measures from which nature can select is non-rectangular. This stands in contrast to much of the related work on time consistency, where rectangularity is essentially taken as a prerequisite for time consistency. We also note that (iii) stands in contrast to some alternative, less policy-focused definitions of time consistency, e.g. those definitions appearing in the literature on risk measures (cf. Epstein and Schneider (2003)), under which time consistency could not hold if an optimal policy took different values under the two formulations. We view our results as a step towards understanding the subtleties which can arise when taking a policy-centric view of time consistency in an operations management setting. Throughout this section, we will let Π_s^{opt} denote the set of all optimal policies for the corresponding multistage-static problem, and Π_d^{opt} denote the set of all robust-w.p.1-optimal policies for the corresponding distributionally robust DP problem.

4.3.1. Example: a multistage-static problem that is not weakly time consistent. In this section, we explicitly provide an example for which the multistage-static problem is not weakly time consistent. Furthermore, for this example, the multistage-static and distributionally robust DP formulations have different optimal values.

Let us define $y_1 = 10$, $\rho = 1$,

$$\mathcal{I}_1 = [1, 3], \quad \mu_1 = 2, \quad \sigma_1 = 1, \quad c_1 = 0, \quad b_1 = 2, \quad h_1 = 2,$$

$$\mathcal{I}_2 = \mathbb{R}_+, \quad \mu_2 = 8, \quad \sigma_2 = 2, \quad c_2 = 0, \quad b_2 = 1, \quad h_2 = 1.$$

Let $\tilde{\Pi}_s$ denote the set of policies $\tilde{\pi} = (\tilde{x}_1, \tilde{x}_2)$ such that $\tilde{x}_1(10) = 10$, $\tilde{x}_2(9) = 9$, $\tilde{x}_2(7) = 7$, and $\tilde{\Pi}_d$ denote the set of policies $\tilde{\pi} = (\tilde{x}_1, \tilde{x}_2)$ such that $\tilde{x}_1(10) = 10$, $\tilde{x}_2(9) = 9$, $\tilde{x}_2(7) = 8$. Note that here (and in later statements) we have defined a set of policies by specifying a required behavior at only a few values, and allow the behavior at all other values to be arbitrary (subject to the overall policy belonging to Π).

Theorem 5 $\Pi_s^{opt} = \tilde{\Pi}_s$, *and the optimal value of the multistage-static problem is 18. On the other hand,* $\Pi_d^{opt} \subseteq \tilde{\Pi}_d$, *and the optimal value of the distributionally robust DP problem is* $17 + \frac{\sqrt{5}}{2} > 18$. *Consequently, the multistage-static problem is not weakly time consistent, and the multistage-static and distributionally robust DP problems have different optimal values.*

We defer the proof to the technical appendix (Section 6).

4.3.2. Example: a multistage-static problem that is weakly time consistent, but *not* strongly time consistent. In this section, we explicitly provide an example showing that it is possible for the multistage-static problem to be weakly time consistent, but not strongly time consistent. In particular, there is a base-stock policy π^*, with associated base-stock constants x_1^*, x_2^* satisfying the conditions of Theorem 3, which is optimal for the multi-stage static formulation and robust-w.p.1-optimal for the distributionally robust DP formulation, yet the multistage-static problem has other non-trivial optimal policies which are not robust-w.p.1-optimal for the distributionally robust DP formulation. The intuitive explanation is as follows. In the multistage-static formulation, one can leverage the randomness in the realization of D_1 to construct a policy π' such that with positive probability $x_2^{\pi'}(y_2)$ is slightly below x_2^*, and with the remaining probability is slightly above x_2^*. Since in the multistage-static formulation nature cannot observe the realized inventory in stage 2 before selecting a worst-case distribution, it turns out that such a policy incurs the same cost as π' under the multistage-static formulation. However, under the distributionally robust DP formulation, such a perturbation leads to sub-optimality. We note that such a lack of strong time consistency can also be interpreted as resulting from the fact that optimality of a policy for the static formulation does not require optimality for every possible measure which nature can select, analogous to the ideas explored (in the robust optimization setting) in Iancu and Trichakis (2014). We note that in this example, even though the multistage-static problem is not strongly time consistent, both formulations have the same optimal value, as dictated by Theorem 3.

Let us define $y_1 = 0$, $\rho = 1$,

$$\mathcal{I}_1 = [1,3], \quad \mu_1 = 2, \quad \sigma_1 = 1, \quad c_1 = 0, \quad b_1 = 1, \quad h_1 = 1,$$

$$\mathcal{I}_2 = \mathbb{R}_+, \quad \mu_2 = 10, \quad \sigma_2 = 1, \quad c_2 = 0, \quad b_2 = 1, \quad h_2 = 1.$$

Then we prove the following, whose proof we defer to the technical appendix (Section 6).

Theorem 6 *The multistage-static problem is weakly time consistent, but not strongly time consistent.*

4.3.3. Example: a multistage-static problem that is strongly time consistent, but the two formulations have a *different* optimal value. In this section, we explicitly provide an example showing that it is possible for the multistage-static problem to be strongly time consistent, yet for the two formulations to have different optimal values. We note that, although it is expected that there will be settings where the two formulations have different optimal values, it is somewhat surprising that this is possible even when the two formulations have the same set of optimal policies. As discussed previously, we note that this possibility stands in contrast to several related works which consider alternative, less policy-focused definitions of time consistency, e.g. those definitions appearing in the literature on risk measures.

Let us define $y_1 = 0$, $\rho = 1$,

$$\mathcal{I}_1 = [1,3], \quad \mu_1 = 2, \quad \sigma_1 = 1, \quad c_1 = 0, \quad b_1 = 0, \quad h_1 = 0,$$

$$\mathcal{I}_2 = \mathbb{R}_+, \quad \mu_2 = 100, \quad \sigma_2 = 5, \quad c_2 = 2, \quad b_2 = 1, \quad h_2 = 1.$$

Let $\tilde{\Pi}$ denote the set of policies $\tilde{\pi} = (\tilde{x}_1, \tilde{x}_2)$ such that $\tilde{x}_1(0) = 102$, $\tilde{x}_2(101) = 101$, $\tilde{x}_2(99) = 99$. Then we prove the following, whose proof we defer to the technical appendix (Section 6).

Theorem 7 $\Pi_s^{opt} = \tilde{\Pi}$, and the multistage-static problem is strongly time consistent. However, the optimal value of the multistage-static problem equals 5, while the optimal value of the distributionally robust DP problem equals $\sqrt{26} > 5$.

4.3.4. Example: a multistage-static problem that has no optimal policy of base-stock form. In this section, we explicitly provide an example showing that it is possible for the multistage-static problem to have no optimal base-stock policy, where we note that in all our previous examples the associated multistage-static problem did indeed have an optimal base-stock policy (possibly different from that of the associated distributionally robust DP problem). Note that this stands in contrast to the distributionally robust DP formulation, which always has an optimal base-stock policy by Observation 4. It remains an interesting open question to develop a deeper understanding of the set of optimal policies for the multistage-static problem, where we again note that some preliminary investigations of such distributionally robust problems with independence constraints can be found in Lam and Ghosh (2013). Both the results of Lam and Ghosh (2013), and our own result, indicate that the structure of optimal policies for the multistage-static problem may be very complicated.

To prove the desired result, it will be useful to consider a family of problems parameterized by a parameter ϵ. In particular, let $\epsilon \in \left(0, \frac{1}{2}(\sqrt{6} - 2)\right)$ be any sufficiently small strictly positive number. It may be easily verified that for any such ϵ, one has $\epsilon \in (0, \frac{1}{4})$, and

$$\frac{1}{2} - 2\epsilon - \epsilon^2 > 0. \tag{42}$$

Let us define $y_1 = 10 - \epsilon$, $\rho = 1$,

$$\mathcal{I}_1 = [1 - \epsilon, \ 3 + \epsilon], \quad \mu_1 = 2, \quad \sigma_1 = 1, \quad c_1 = 0, \quad b_1 = 2, \quad h_1 = 2,$$

$$\mathcal{I}_2 = \mathbb{R}_+, \quad \mu_2 = 8, \quad \sigma_2 = 3, \quad c_2 = 0, \quad b_2 = 1, \quad h_2 = 1.$$

Then we prove the following, whose proof we defer to the technical appendix (Section 6).

Theorem 8 *Suppose ϵ satisfies (42). Then any admissible policy $\tilde{\pi} = (\tilde{x}_1, \tilde{x}_2) \in \Pi$ satisfying $\tilde{x}_1(y_1) = y_1$, $\tilde{x}_2(D_1) = y_1 - D_1 + \epsilon$ belongs to Π_s^{opt}, and the corresponding optimal value equals $19 - 2\epsilon$. Moreover, no base-stock policy belongs to Π_s^{opt}.*

5.   Conclusion

In this paper, we analyzed the notion of time consistency in the context of managing an inventory under distributional uncertainty. In particular, we studied the associated multistage distributionally robust optimization problem, when only the mean, variance and distribution support are known for the demand at each stage. Our contributions were three-fold. First, we gave a refined policy-centric definition for time consistency in this setting, and put our definition in the broad context of prior work on time consistency and rectangularity. More precisely, we defined two natural formulations for the relevant optimization problem. In the multistage-static formulation, the policy-maker cannot recompute her policy after observing realized demand. In the distributionally robust DP formulation, she is allowed to reperform her minimax computations at each stage. If there exists a policy which is optimal for both formulations (w.p.1 under every joint distribution for demand belonging to the uncertainty set), we say that the policy is *time consistent*, and the problem is *weakly time consistent*. If every optimal policy for the multistage-static formulation is time consistent, we say that the problem is *strongly time consistent*.

Second, we gave sufficient conditions for weak and strong time consistency. Intuitively, our sufficient condition for weak time consistency coincides with the existence of an optimal base-stock policy in which the base-stock constants are monotone increasing. Our sufficient condition for strong time consistency can be interpreted in two ways. On the one hand, strong time consistency holds if the unique optimal base-stock policy for the distributionally robust DP formulation is to order-up to 0 at each stage, i.e., the well-known Make-To-Order policy. Alternatively, we saw that this condition also has an interpretation in terms of requiring that the demand variances are sufficiently large relative to their respective means. Third, we gave a series of examples of two-stage problems exhibiting interesting and counterintuitive time (in)consistency properties, showing that the question of time consistency can be quite subtle in this setting. In particular: (i) the problem

can fail to be weakly time consistent, (ii) the problem can be weakly but not strongly time consistent, and (iii) the problem can be strongly time consistent even if every associated optimal policy takes different values under the multistage-static and distributionally robust DP formulations. We also proved that, although the distributionally robust DP formulation always has an optimal policy of base-stock form, there may be no such optimal policy for the multistage-static formulation. This stands in contrast to the analogous setting, analyzed in Shapiro (2012), in which only the mean and support of the demand distribution is known at each stage, for which it is known that such phenomena cannot occur (as the problem is always weakly time consistent).

We departed from much of the past literature by demonstrating both negative *and positive* results regarding time consistency when the underlying family of distributions from which nature can select is non-rectangular, a setting in which most of the literature focuses on demonstrating hardness of the underlying optimization problems and other negative results. Furthermore, our example demonstrating that it is possible for the multistage-static problem to be strongly time consistent, but with a different optimal value than the distributionally robust DP formulation, stands in contrast to the definition of time consistency typically used in the theory of risk measures, i.e. the notion of dynamic consistency coming from Epstein and Schneider (2003), under which a problem may be deemed time inconsistent based on the values that a given optimal policy takes under the different formulations, and even the values taken by suboptimal policies. Indeed, our definitions are motivated by the fact that in an optimization setting, one may be primarily concerned only with the implementability of optimal policies, irregardless of their values and the values of suboptimal policies, building on the more optimization-oriented definitions provided in Carpentier et al. (2012) and the recent work Homem-de-Mello and Pagnoncelli (2016).

Our work leaves many interesting directions for future research. The general question of time consistency remains poorly understood. Furthermore, our work has shown that this question can be quite subtle. For the particular model we consider here, it would be interesting to develop a better understanding of precisely when time consistency holds. It is also an intriguing question to understand how much our two formulations can differ in optimal value and policy, even when time inconsistency occurs, along the lines of Huang et al. (2011), Asamov and Ruszczyński (2015), and Iancu, Petrik and Subramanian (2015). On a related note, it is largely open to develop a broader understanding of the optimal solution to the multistage-static problem, or even approximately optimal solutions, as well as related algorithms, where we note that preliminary investigations along these lines were recently carried out in Lam and Ghosh (2013). Of course, it is also an open challenge to understand the question of time consistency more broadly, how precisely the various definitions of time consistency presented throughout the literature relate to one-another, and more generally to understand the relationship between different ways to model multistage optimization under uncertainty.

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6. Appendix
6.1. Proof of Theorem 1

Proof of Theorem 1 : We first compute the value of $\psi(x)$ for all $x \in \mathbb{R}$, and proceed by a case analysis. First, suppose $x < 0$. In this case, $\mathbb{E}_Q[\Psi(x, D)] = cx + b(\mu - x)$ for all $Q \in \mathfrak{M}$, and thus

$$\psi(x) = cx + b(\mu - x). \tag{43}$$

Now, suppose $x \geq 0$. Then it is easily verified that

$$\psi(x) = cx + \frac{(h-b)(x-\mu)}{2} + \frac{b+h}{2} \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q\left[|x - D|\right]. \tag{44}$$

Hence to compute $\psi(x)$, it suffices to solve $\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q\left[|x - D|\right]$, and we proceed by a case analysis. Recall that $\mathfrak{f}(z) := \left((z - \mu)^2 + \sigma^2\right)^{\frac{1}{2}}$ for all $z \in \mathbb{R}$, and $\mathfrak{f}^{-1}(z)$ denotes the reciprocal of $\mathfrak{f}(z)$.

First, suppose $x \geq \frac{\mu^2 + \sigma^2}{2\mu}$. Let us define $\bar{\lambda} = (\bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2)$ such that

$$\bar{\lambda}_0 := \frac{1}{2}\left(x^2 \mathfrak{f}^{-1}(x) + \mathfrak{f}(x)\right), \quad \bar{\lambda}_1 := -x\mathfrak{f}^{-1}(x), \quad \bar{\lambda}_2 := \frac{1}{2}\mathfrak{f}^{-1}(x),$$

and let $\bar{g}(d) := \bar{\lambda}_0 + \bar{\lambda}_1 d + \bar{\lambda}_2 d^2$ for all $d \in \mathbb{R}$. Then it follows from a straightforward calculation that $\bar{g}(d)$ and $|x - d|$ are tangent at $\bar{d}_1 := x - \mathfrak{f}(x)$ and $\bar{d}_2 := x + \mathfrak{f}(x)$, and consequently $\bar{g}(d) \geq |x - d|$ for all $d \in \mathbb{R}_+$. Hence $\bar{\lambda}$ is feasible for the dual Problem (8). Also, as $x \geq \frac{\mu^2 + \sigma^2}{2\mu}$ implies $\bar{d}_1 \geq 0$, it is easily verified that the probability measure $\bar{Q}$ such that

$$\bar{Q}(\bar{d}_1) = \sigma^2\left(\sigma^2 + \left(x - \mathfrak{f}(x) - \mu\right)^2\right)^{-1}, \quad \bar{Q}(\bar{d}_2) = 1 - \sigma^2\left(\sigma^2 + \left(x - \mathfrak{f}(x) - \mu\right)^2\right)^{-1}$$

is feasible for the primal Problem (7). It follows from Proposition 2.1 that $\bar{Q}$ is an optimal primal solution. Combining the above and simplifying the relevant algebra, we conclude that in this case

$$\psi(x) = \psi_1(x) := c\mu + \frac{b+h}{2}\mathfrak{f}(x) - \frac{b-h-2c}{2}(x - \mu). \tag{45}$$

Alternatively, suppose $x \in [0, \frac{\mu^2 + \sigma^2}{2\mu})$. Let us define $\hat{\lambda} = (\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2)$ such that

$$\hat{\lambda}_0 := x \quad, \quad \hat{\lambda}_1 := 1 - 4x\mu(\mu^2 + \sigma^2)^{-1} \quad, \quad \hat{\lambda}_2 := 2x\left(\mu(\mu^2 + \sigma^2)^{-1}\right)^2,$$

and let $\hat{g}(d) := \hat{\lambda}_0 + \hat{\lambda}_1 d + \hat{\lambda}_2 d^2$ for all $d \in \mathbb{R}$. Then it follows from a straightforward calculation that $\hat{g}(d)$ and $|x - d|$ are tangent at $\hat{d}_1 := \mu^{-1}(\mu^2 + \sigma^2)$, and intersect at $\hat{d}_2 := 0$, with $\hat{g}'(0) \geq -1$. It follows that $\hat{g}(d) \geq |x - d|$ for all $d \in \mathbb{R}_+$. Hence $\hat{\lambda}$ is feasible for the dual Problem (8). Also, it is easily verified that the probability measure $\hat{Q}$ such that

$$\hat{Q}(\hat{d}_1) = \mu^2(\mu^2 + \sigma^2)^{-1}, \quad \hat{Q}(\hat{d}_2) = 1 - \mu^2(\mu^2 + \sigma^2)^{-1}$$

is feasible for the primal Problem (7). It follows from Proposition 2.1 that $\hat{Q}$ is an optimal primal solution. Combining the above and simplifying the relevant algebra, we conclude that in this case

$$\psi(x) = \psi_2(x) := \frac{(h+c)\sigma^2 - (b-c)\mu^2}{\mu^2 + \sigma^2} x + b\mu. \tag{46}$$

We now use the above to complete the proof of the theorem. Note that since by assumption $b > c$, it follows from (43) that $\arg\min_{x \in \mathbb{R}} \psi(x) \subseteq \mathbb{R}_+$. Recall that $\kappa = \frac{b-h-2c}{b+h}$. Furthermore, our assumptions, i.e. $b > c, h + c > 0$, imply that $|\kappa| < 1$. Let $\chi := \mu + \kappa\sigma(1-\kappa^2)^{-\frac{1}{2}}$. It follows from a straightforward calculation that ψ_1 is a strictly convex function on $\mathbb{R}$, and $\psi_1(\chi) = 0$, i.e. ψ_1 is strictly decreasing on $(-\infty, \chi)$, and strictly increasing on (χ, ∞). Furthermore, it follows from a similar calculation that

$$\frac{\sigma^2}{\mu^2} - \frac{b-c}{h+c} \quad \text{is the same sign as} \quad \frac{\mu^2 + \sigma^2}{2\mu} - \chi. \tag{47}$$

We now proceed by a case analysis. First, suppose $\frac{\sigma^2}{\mu^2} > \frac{b-c}{h+c}$. In this case, ψ_2 is a linear function with strictly positive slope, and thus $\arg\min_{x \in [0, \frac{\mu^2+\sigma^2}{2\mu}]} \psi(x) = \{0\}$. Furthermore, it follows from (47) that $\chi < \frac{\mu^2+\sigma^2}{2\mu}$, which implies that ψ_1 is strictly increasing on $[\frac{\mu^2+\sigma^2}{2\mu}, \infty)$. It follows from the continuity of ψ that $\arg\min_{x \geq \frac{\mu^2+\sigma^2}{2\mu}} \psi(x) = \{\frac{\mu^2+\sigma^2}{2\mu}\}$. Combining the above, we conclude that $\arg\min_{x \in \mathbb{R}} \psi(x) = \{0\}$.

Next, suppose $\frac{\sigma^2}{\mu^2} < \frac{b-c}{h+c}$. In this case, ψ_2 is a linear function with strictly negative slope, and thus $\arg\min_{x \in [0, \frac{\mu^2+\sigma^2}{2\mu}]} \psi(x) = \{\frac{\mu^2+\sigma^2}{2\mu}\}$. Furthermore, it follows from (47) that $\chi > \frac{\mu^2+\sigma^2}{2\mu}$, which implies that $\arg\min_{x \geq \frac{\mu^2+\sigma^2}{2\mu}} \psi(x) = \{\chi\}$. Combining the above, we conclude that $\arg\min_{x \in \mathbb{R}} \psi(x) = \{\chi\}$.

Finally, suppose that $\frac{\sigma^2}{\mu^2} = \frac{b-c}{h+c}$. In this case, ψ_2 is a constant function, and thus $\arg\min_{x \in [0, \frac{\mu^2+\sigma^2}{2\mu}]} \psi(x) = [0, \frac{\mu^2+\sigma^2}{2\mu}]$. Furthermore, it follows from (47) that $\chi = \frac{\mu^2+\sigma^2}{2\mu}$, which implies that $\arg\min_{x \geq \frac{\mu^2+\sigma^2}{2\mu}} \psi(x) = \{\frac{\mu^2+\sigma^2}{2\mu}\}$. Combining the above, we conclude that $\arg\min_{x \in \mathbb{R}} \psi(x) = [0, \frac{\mu^2+\sigma^2}{2\mu}]$.

Combining all of the above with another straightforward calculation completes the proof of the theorem. $\square$

6.2. Proof of Proposition 2.2

Proof of Proposition 2.2 : Let $\delta := \frac{\sigma^2}{\mu^2+\sigma^2}, \tau := \frac{\mu^2+\sigma^2}{\mu}$. Let Q_2^* be the probability measure such that

$$Q_2^*(0) = \delta, \quad Q_2^*(\tau) = 1 - \delta.$$

Recall that $b - c > 0$, and $(h+c)\sigma^2 > (b-c)\mu^2$, which we denote by assumption A1. Note that the value of any feasible solution Q_1 to Problem (11) is at least $\mathbb{E}_{Q_1 \times Q_2^*}\Big[\Psi(D_1, D_2)\Big]$, which itself equals the sum of $c\mu$ and

$$
\mathbb{E}_{Q_1}\Big[\Big(\delta\big((b-c)[0-D_1]_+ + (h+c)[D_1-0]_+\big) + (1-\delta)\big((b-c)[\tau-D_1]_+ + (h+c)[D_1-\tau]_+\big)\Big)I(D_1 > 0)\Big]
$$
(48)

$$
+ \mathbb{E}_{Q_1}\Big[\Big(\delta\big((b-c)[0-D_1]_+ + (h+c)[D_1-0]_+\big) + (1-\delta)\big((b-c)[\tau-D_1]_+ + (h+c)[D_1-\tau]_+\big)\Big)I(D_1 < 0)\Big]
$$
(49)

$$
+ \mathbb{E}_{Q_1}\Big[\Big(\delta\big((b-c)[0-D_1]_+ + (h+c)[D_1-0]_+\big) + (1-\delta)\big((b-c)[\tau-D_1]_+ + (h+c)[D_1-\tau]_+\big)\Big)I(D_1 = 0)\Big]
$$
(50)

Note that if $P(D_1 > 0) > 0$, then (48) is at least

$$
\begin{aligned}
\mathbb{E}\Big[\frac{\sigma^2}{\mu^2+\sigma^2}(h+c)D_1 + \frac{\mu^2}{\mu^2+\sigma^2}(b-c)(\frac{\mu^2+\sigma^2}{\mu} - D_1)\Big|D_1 > 0\Big]&P(D_1 > 0) \\
> \quad \mathbb{E}\Big[\frac{\mu^2}{\mu^2+\sigma^2}(b-c)D_1 + \frac{\mu^2}{\mu^2+\sigma^2}(b-c)(\frac{\mu^2+\sigma^2}{\mu} - D_1)\Big|D_1 > 0\Big]&P(D_1 > 0) \quad \text{by A1} \\
= \quad (b-c)\mu P(D_1 > 0).&
\end{aligned}
$$
(51)

Similarly, if $P(D_1 < 0) > 0$, then (49) is at least

$$
\begin{aligned}
\mathbb{E}\Big[-\frac{\sigma^2}{\mu^2+\sigma^2}(b-c)D_1 + \frac{\mu^2}{\mu^2+\sigma^2}(b-c)(\frac{\mu^2+\sigma^2}{\mu} - D_1)\Big|D_1 < 0\Big]&P(D_1 < 0) \\
= \quad \mathbb{E}\Big[(b-c)(\mu - D_1)\Big|D_1 < 0\Big]P(D_1 < 0) \quad > \quad (b-c)\mu P(D_1 < 0).&
\end{aligned}
$$
(52)

Furthermore, if $P(D_1 = 0) > 0$, then (50) equals $(b-c)\mu P(D_1 = 0)$. Combining with (51), (52), and the fact that the measure δ_0 attains value $b\mu$ (by Theorem 1), completes the proof. $\square$

6.3. Proof of Theorem 2

Proof of Theorem 2 : Recall that $\eta := \frac{1}{2}(c_1 + c_2)$, and $\mathfrak{f}(z) := \big((z-\mu)^2 + \sigma^2\big)^{\frac{1}{2}}$ for all $z \in \mathbb{R}$. Also, letting $h_1(d) := -d + c_1, h_2(d) := d - c_2$ for all $d \in \mathbb{R}$, we have that $\zeta(d) = \max\{h_1(d), 0, h_2(d)\}$ for all $d \in \mathbb{R}$. Let Q be the probability measure described in (12), and $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ the vector described in (13). Let $g(d) := \lambda_0 + \lambda_1 d + \lambda_2 d^2$. We now prove that $g(d) \geq \zeta(d)$ for all $d \in \mathbb{R}$. It follows from a straightforward calculation that $g(d)$ is tangent to $h_1(d)$ at $d_1 := \eta - \mathfrak{f}(\eta)$, and $g(d)$ is tangent to $h_2(d)$ at $d_2 := \eta + \mathfrak{f}(\eta)$. Thus $g(d) \geq \max\big(h_1(d), h_2(d)\big)$ for all $d \in \mathbb{R}$, and to prove the desired claim it suffices to demonstrate that $g(d) \geq 0$ for all $d \geq 0$. It is easily verified that for all $d \in \mathbb{R}$,

$$
g(d) = \tfrac{1}{2}\mathfrak{f}^{-1}(\eta)(d-\eta)^2 + \tfrac{1}{2}\big(\mathfrak{f}(\eta) + c_1 - c_2\big).
$$
(53)

Recall that

$$
\frac{1}{4}(2\mu - 3c_1 + c_2)(3c_2 - c_1 - 2\mu) \leq \sigma^2,
$$

which we denote by assumption A2. It follows from another straightforward calculation that assumption A2 is equivalent to requiring that $\frac{1}{2}\big(\mathfrak{f}(\eta) + c_1 - c_2\big) \geq 0$. Combining with (53), we conclude that A2 implies $g(d) \geq 0$ for all $d \in \mathbb{R}$, completing the proof that $g(d) \geq \zeta(d)$ for all $d \in \mathbb{R}$. Hence λ is feasible for the dual Problem (8). Also, it is easily verified that Q is feasible for the primal Problem (7). It follows from Proposition 2.1 that Q is an optimal primal solution, and λ is an optimal dual solution. That these optimal solutions are unique then follows from the second part of Proposition 2.1 and a straightforward contradiction argument. Combining the above and simplifying the relevant algebra completes the proof. $\quad\square$

6.4. Proof of Lemma 4.1

Proof of Lemma 4.1 :   Suppose $i \in \{0, ..., T\}$ and $Q_1, \ldots, Q_i$ are fixed. As a notational convenience, for $k \in [1, T]$, let $\mathbb{E}_k[\cdot]$ denote $\mathbb{E}_{\otimes_{j=1}^k Q_j}[\cdot]$. We now prove that (40) holds for all $t \geq i+1$, and proceed by induction. Our particular induction hypothesis will be that there exist $Q_{i+1}, \ldots, Q_{i+n}$ such that

$$\mathbb{E}_{i+n}\big[\hat{\Psi}_t\big(x_t(y_t), D_t\big)\big] \geq \hat{\eta}_t \text{ for all } t \in [i+1, i+n]. \tag{54}$$

We first treat the base case $n = 1$. It follows from Jensen's inequality, and the independence structure of the measures in $\mathfrak{M}$, that for any $Q_{i+1} \in \mathfrak{M}_{i+1}$,

$$\mathbb{E}_{i+1}\big[\hat{\Psi}_{i+1}\big(x_{i+1}(y_{i+1}), D_{i+1}\big)\big] \geq \mathbb{E}_{Q_{i+1}}\big[\hat{\Psi}_{i+1}\big(\mathbb{E}_i[x_{i+1}(y_{i+1})], D_{i+1}\big)\big].$$

Taking Q_{i+1} to be any element of $\Gamma_{i+1}^{\mathbb{E}_i[x_{i+1}(y_{i+1})]}$ ($\Gamma_1^{x_1(y_1)}$ if $i = 0$) completes the proof for $n = 1$.

Now, suppose the induction holds for some n. It again follows from Jensen's inequality, and the independence structure of the measures in $\mathfrak{M}$, that for any $Q_{i+n+1} \in \mathfrak{M}_{i+n+1}$,

$$\mathbb{E}_{i+n+1}\big[\hat{\Psi}_{i+n+1}\big(x_{i+n+1}(y_{i+n+1}), D_{i+n+1}\big)\big] \geq \mathbb{E}_{Q_{i+n+1}}\big[\hat{\Psi}_{i+n+1}\big(\mathbb{E}_{i+n}[x_{i+n+1}(y_{i+n+1})], D_{i+n+1}\big)\big].$$

Taking Q_{i+n+1} to be any element of $\Gamma_{i+n+1}^{\mathbb{E}_{i+n}[x_{i+n+1}(y_{i+n+1})]}$ completes the induction, and the proof, where the second part of the lemma follows by letting $i = 0$. $\quad\square$

6.5. Proof of Theorem 3

Proof of Theorem 3 :   Note that under these assumptions, for any measure $Q \in \mathcal{M}$ (and in fact any non-negative joint distribution for demand), for any such base-stock policy π, w.p.1 $x_t^\pi(y_t) = x_t^*$ for all $t = 1, ..., T$. It then follows from a straightforward induction that π is a robust-w.p.1-optimal policy for the distributionally robust DP formulation, and furthermore for all $t = 1, ..., T$ and $y \leq x_t^*$,

$$V_t(y) = \hat{\eta}_t - c_t x_{t-1}^* + c_t D_{t-1} + \sum_{s=t+1}^{T} \rho^{s-t}(\hat{\eta}_s + c_s \mu_{s-1}),$$

and

$$V_1(y) = \sum_{t=1}^{T} \rho^{t-1}\hat{\eta}_t - c_1 y + \sum_{t=1}^{T-1} \rho^t c_{t+1}\mu_t.$$

Combining with Lemma 4.1 and Observation 4 completes the proof. $\quad\square$

6.6. Proof of Theorem 4

Proof of Theorem 4 : Let Π^{opt} denote the set of optimal policies for the multistage-static problem. It follows from Theorem 1.(i) and Theorem 3 that $\Pi^0 \subseteq \Pi^{opt}$, and every policy $\pi \in \Pi^0$ is time consistent. Thus to prove the theorem, it suffices to demonstrate that $\Pi^0 = \Pi^{opt}$, and we begin by showing that $\bar{\pi} = (\bar{x}_1, \ldots, \bar{x}_T) \in \Pi^{opt}$ implies $\bar{x}_1(y_1) = 0$. Indeed, it follows from Lemma 4.1 that $\bar{\pi} \in \Pi^{opt}$ implies

$$\sup_{Q \in \mathfrak{M}_1} \mathbb{E}_Q \big[\hat{\Psi}_1 \big(\bar{x}_1(y_1), D_1 \big) \big] = \hat{\eta}_1 = b_1 \mu_1.$$

That $\bar{x}_1(y_1)$ must equal 0 then follows from Theorem 1.

We now show that $\bar{\pi} \in \Pi^{opt}$ implies $\bar{x}_2(z) = 0$ for all $z \leq 0$. We proceed by contradiction. Suppose that there exists $z' \leq 0$ such that $\bar{x}_2(z') \neq 0$. It is easily verified that there exists $Q_1 \in \mathfrak{M}_1$ such that $Q_1(-z') > 0$, and consequently for this choice of Q_1, $\bar{x}_2(y_2)$ is not a.s. equal to 0. We conclude from Proposition 2.2 that there exists $Q_2 \in \mathfrak{M}_2$ such that

$$\mathbb{E}_{Q_1 \times Q_2} \big[\hat{\Psi}_2 \big(\bar{x}_2(y_2), D_2 \big) \big] > \hat{\eta}_2 = b_2 \mu_2.$$

As we have already demonstrated that $\bar{x}_1(y_1) = 0$, and $Q_1 \in \mathfrak{M}_1$, we conclude that

$$\mathbb{E}_{Q_1} \big[\hat{\Psi}_1 \big(\bar{x}_1(y_1, D_1) \big) \big] = \hat{\eta}_1 = b_1 \mu_1.$$

Combining with Lemma 4.1 then yields a contradiction. The proof that $\bar{x}_t(z) = 0$ for all $z \leq 0$ and $t \geq 3$ follows from a nearly identical argument, and we omit the details. $\square$

6.7. Proof of Theorem 5

We first characterize the set of optimal policies for the multistage-static problem.

Lemma 6.1 $\Pi_s^{opt} = \tilde{\Pi}_s$, *and the multistage-static problem has optimal value 18.*

Proof : It follows from Observation 1 that $\mathfrak{M}_1$ consists of the single probability measure Q_1 such that $Q_1(1) = Q_1(3) = \frac{1}{2}$. Let D_1 denote a random variable distributed as Q_1. Note that for any policy $\pi = (x_1, x_2) \in \Pi$, one has that $x_1(y_1) = x_1(10) \geq 10$. Consequently, $\mathsf{Pr}(x_1(y_1) \geq D_1) = 1$, and $|x_1(y_1) - D_1| = x_1(y_1) - D_1$ w.p.1. It then follows from a straightforward calculation that the cost of any policy $\pi = (x_1, x_2) \in \Pi$ under the multistage-static formulation equals

$$2x_1(10) - 4 + \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \Big[\tfrac{1}{2} \big(\big| x_2 \big(x_1(10) - 1 \big) - D_2 \big| + \big| x_2 \big(x_1(10) - 3 \big) - D_2 \big| \big) \Big]. \tag{55}$$

Let $\bar{\pi} = (\bar{x}_1, \bar{x}_2)$ denote any optimal policy for the multistage-static problem, i.e. $\bar{\pi} \in \Pi_s^{opt}$. Then it follows from (55) and a straightforward contradiction argument that

$$\bar{x}_1(10) = 10. \tag{56}$$

Combining (55) and (56), we conclude that

$$\big(\bar{x}_2(9), \bar{x}_2(7)\big) \in \underset{(x,y):x\geq 9,y\geq 7}{\arg\min} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\Big[\tfrac{1}{2}\big(|x-D_2|+|y-D_2|\big)\Big]. \tag{57}$$

Furthermore, it follows from Lemma 4.1 and Theorem 1 that

$$\inf_{(x,y):x\geq 9,y\geq 7} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\Big[\tfrac{1}{2}\big(|x-D_2|+|y-D_2|\big)\Big] \ \geq \ \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\big[|8-D_2|\big] \ = \ 2. \tag{58}$$

Noting that

$$\tfrac{1}{2}\big(|9-D_2|+|7-D_2|\big) = 1 + \max(-D_2+7, 0, D_2-9),$$

it then follows from a straightforward calculation and Theorem 2 that

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\Big[\tfrac{1}{2}\big(|9-D_2|+|7-D_2|\big)\Big] = 2. \tag{59}$$

Combining the above, we conclude that $\tilde{\Pi}_s \subseteq \Pi_s^{opt}$. Also, it then follows from a straightforward calculation that the multistage-static problem has optimal value 18.

We now prove that $\tilde{\Pi}_s = \Pi_s^{opt}$. Indeed, suppose for contradiction that there exists some optimal policy $\hat{\pi} = (\hat{x}_1, \hat{x}_2) \notin \tilde{\Pi}_s$. In that case, it follows from (56) and (57) that $\tfrac{1}{2}\big(\hat{x}_2(9) + \hat{x}_2(7)\big) > 8$. However, it then follows from Jensen's inequality, Theorem 1, and (58) that

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\Big[\tfrac{1}{2}\big(|\hat{x}_2(9)-D_2|+|\hat{x}_2(7)-D_2|\big)\Big] \ \geq \ \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\Big[\big|\tfrac{1}{2}\big(\hat{x}_2(9)+\hat{x}_2(7)\big)-D_2\big|\Big] \ > \ 2.$$

Combining with (58) and (59) yields a contradiction, completing the proof $\quad\square$

We now (partially) characterize the set of robust-w.p.1-optimal policies for the distributionally robust DP problem.

Lemma 6.2 $\Pi_d^{opt} \subseteq \tilde{\Pi}_d$, and the distributionally robust DP problem has optimal value $17 + \frac{\sqrt{5}}{2}$.

Proof : Let $\bar{\pi} = (\bar{x}_1, \bar{x}_2)$ denote any robust-w.p.1-optimal policy for the distributionally robust DP problem, i.e. $\bar{\pi} \in \Pi_d^{opt}$. Then it again follows from a straightforward contradiction argument that

$$\bar{x}_1(10) = 10. \tag{60}$$

It then follows from (26) that

$$\bar{x}_2(9) \in \underset{x\geq 9}{\arg\min} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}[|x-D_2|],$$

and

$$\bar{x}_2(7) \in \underset{x\geq 7}{\arg\min} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}[|x-D_2|].$$

The lemma then follows from Theorem 1 and a straightforward calculation. $\quad\square$

Combining Lemmas 6.1 and 6.2 completes the proof of Theorem 5.

6.8. Proof of Theorem 6

We first prove that the multistage-static problem is weakly time consistent.

Lemma 6.3 *The multistage-static problem is weakly time consistent, and both the multistage-static and distributionally robust DP problems have optimal value 2.*

Proof :   Note that

$$\hat{\Psi}_1(x_1, d_1) = |x_1 - d_1|, \quad \hat{\Psi}_2(x_2, d_2) = |x_2 - d_2|.$$

It follows from Observation 1 that $\mathfrak{M}_1$ consists of the single probability measure Q_1 such that $Q_1(1) = Q_1(3) = \frac{1}{2}$. It follows from Theorem 1 and a straightforward calculation that

$$\hat{\Gamma}_1 = [1, 3] \quad , \quad \hat{\Gamma}_2 = 10 \quad , \quad \hat{\eta}_2 = 1.$$

Combining the above with Theorem 3, we conclude that the base-stock policy π such that $x_1(y) = \max\{3, y\}$, and $x_2(y) = \max\{10, y\}$ for all $y \in \mathbb{R}$, is optimal for both the multistage-static formulation and robust-w.p.1-optimal for the distributionally robust DP formulation, which have common optimal value 2. $\square$

We now prove that the multistage-static problem is not strongly time consistent. In particular, consider the policy $\pi' = (x_1', x_2')$ such that

$$x_1'(y) = \max\{3, y\}, \quad \text{and} \quad x_2'(y) = \begin{cases} 9.9, & \text{if } y \le 0, \\ \max\{10.1, y\}, & \text{otherwise.} \end{cases} \tag{61}$$

Lemma 6.4 *The policy $\pi' \in \Pi_s^{opt}$, but $\pi' \notin \Pi_d^{opt}$. Consequently, the multistage-static problem is not strongly time consistent.*

Proof :   We first show that $\pi' \in \Pi_s^{opt}$. It follows from a straightforward calculation that the cost of π' under the multistage-static formulation equals

$$\mathbb{E}_{Q_1}|3 - D_1| + 0.1 + \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \max\{9.9 - D_2, 0, D_2 - 10.1\}. \tag{62}$$

It is easily verified that the conditions of Theorem 2 are met, and we may apply Theorem 2 to conclude that $\arg\max_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \max\{9.9 - D_2, 0, D_2 - 10.1\}$ is the probability measure Q_2 such that $Q_2(9) = \frac{1}{2}, Q_2(11) = \frac{1}{2}$. It follows that the value of (62) equals 2, and we conclude that $\pi' \in \Pi_s^{opt}$, completing the proof.

We now show that $\pi' \notin \Pi_d^{opt}$. Suppose, for contradiction, that $\pi' \in \Pi_d^{opt}$. It then follows from a straightforward calculation (and considering the measure $Q_1 \in \mathfrak{M}_1$ such that $Q_1(1) = Q_1(3) = \frac{1}{2}$) that

$$9.9 \in \arg\min_{x \ge 0} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}[|x - D_2|]. \tag{63}$$

However, it follows from Theorem 1 that the right-hand side of (63) is the singleton $\{10\}$, completing the proof. $\square$

Combining Lemmas 6.3 and 6.4 completes the proof of Theorem 6.

6.9. Proof of Theorem 7

We first characterize the set of optimal policies for the multistage-static problem.

Lemma 6.5 $\Pi_s^{opt} = \tilde{\Pi}$, *and the multistage-static problem has optimal value 5.*

Proof :   It follows from Observation 1 that $\mathfrak{M}_1$ consists of the single probability measure Q_1 such that $Q_1(1) = Q_1(3) = \frac{1}{2}$. In this case, the cost of any policy $\pi = (x_1, x_2) \in \Pi$ under the multistage-static formulation equals

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[\mathbb{E}_{Q_1}\left[2\big(x_2(x_1(0) - D_1) - (x_1(0) - D_1)\big) + \big|x_2(x_1(0) - D_1) - D_2\big| \right] \right]. \tag{64}$$

We now prove that for any policy $\bar{\pi} = (\bar{x}_1, \bar{x}_2) \in \Pi_s^{opt}$, one has that

$$\bar{x}_2\big(\bar{x}_1(0) - 1\big) = \bar{x}_1(0) - 1 \quad \text{and} \quad \bar{x}_2\big(\bar{x}_1(0) - 3\big) = \bar{x}_1(0) - 3. \tag{65}$$

Indeed, note that w.p.1, it follows from the triangle inequality that

$$
\begin{aligned}
&2\big(x_2(x_1(0) - D_1) - (x_1(0) - D_1)\big) + \big|x_2(x_1(0) - D_1) - D_2\big| \\
&= 2\big(x_2(x_1(0) - D_1) - (x_1(0) - D_1)\big) + \big|x_2(x_1(0) - D_1) - (x_1(0) - D_1) + (x_1(0) - D_1) - D_2\big| \\
&\geq 2\big(x_2(x_1(0) - D_1) - (x_1(0) - D_1)\big) + \big|(x_1(0) - D_1) - D_2\big| - \big|x_2(x_1(0) - D_1) - (x_1(0) - D_1)\big| \\
&= x_2(x_1(0) - D_1) - (x_1(0) - D_1) + \big|x_1(0) - D_1 - D_2\big|.
\end{aligned}
\tag{66}
$$

Now, suppose for contradiction that (65) does not hold. It follows that

$$\mathbb{E}_{Q_1}\left[x_2\big(x_1(0) - D_1\big) - \big(x_1(0) - D_1\big) \right] > 0,$$

and combining with (66), we conclude that (64) is strictly greater than

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[\mathbb{E}_{Q_1}\left[\big|x_1(0) - D_1 - D_2\big| \right] \right]. \tag{67}$$

Noting that (67) is the cost incurred by some policy satisfying (65) completes the proof.

We now complete the proof of the lemma. It suffices from the above to prove that

$$\arg\min_{x_1 \in \mathbb{R}_+} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[\tfrac{1}{2}\big(|x_1 - 1 - D_2| + |x_1 - 3 - D_2|\big) \right] = \{102\}. \tag{68}$$

It follows from a straightforward calculation that as long as $x_1 \geq 3$, $(x_1 - 100)(104 - x_1) \leq 25$ and $x_1 - 2 - ((x_1 - 2 - 100)^2 + 25)^{\frac{1}{2}} \geq 0$, which holds for all $x_1 \in [100, 104]$, the conditions of Theorem 2 are met. We may thus apply Theorem 2 to conclude that for all $x_1 \in [100, 104]$,

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[\tfrac{1}{2}\left(|x_1 - 1 - D_2| + |x_1 - 3 - D_2|\right)\right] \tag{69}$$

has the unique optimal solution $\hat{Q}_2$ such that

$$\hat{Q}_2\left(x_1 - 2 - ((x_1 - 2 - 100)^2 + 25)^{\frac{1}{2}}\right) = 25\left(25 + \left(x_1 - 2 - ((x_1 - 2 - 100)^2 + 25)^{\frac{1}{2}} - 100\right)^2\right)^{-1},$$

and

$$\hat{Q}_2\left(x_1 - 2 + ((x_1 - 2 - 100)^2 + 25)^{\frac{1}{2}}\right) = 1 - 25\left(25 + \left(x_1 - 2 - ((x_1 - 2 - 100)^2 + 25)^{\frac{1}{2}} - 100\right)^2\right)^{-1}.$$

It then follows from a straightforward calculation that for $x_1 \in [100, 104]$, (69) has the value

$$g(x_1) := \left(x_1^2 - 204x_1 + 10429\right)^{\frac{1}{2}}.$$

It is easily verified that g is a strictly convex function on $[100, 104]$, g has its unique minimum on that interval at the point 102, and $g(102) = 5$. The desired result then follows from the fact that (69) is a convex function of x_1 on $\mathbb{R}$. $\square$

We now prove that the multistage-static problem is strongly time consistent.

Lemma 6.6 *The multistage-static problem is strongly time consistent, and the optimal value of the distributionally robust DP problem equals $\sqrt{26}$.*

Proof : First, we note that as in the multistage-static setting, any policy $\bar{\pi} = (\bar{x}_1, \bar{x}_2) \in \Pi_d^{opt}$ also satisfies (65). The proof is very similar to that used for the multistage-static case, and we omit the details. To prove the lemma, it thus suffices to prove that

$$\arg\min_{x_1 \in \mathbb{R}_+} \left(\tfrac{1}{2} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 1 - D_2|\right] + \tfrac{1}{2} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 3 - D_2|\right]\right) = \{102\}. \tag{70}$$

It is easily verified that for all $x_1 \in [100, 104]$, we may apply Theorem 1 to conclude that

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 1 - D_2|\right] = \left((x_1 - 101)^2 + 25\right)^{\frac{1}{2}},$$

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 3 - D_2|\right] = \left((x_1 - 103)^2 + 25\right)^{\frac{1}{2}}.$$

We conclude that for all $x_1 \in [100, 104]$,

$$\tfrac{1}{2} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 1 - D_2|\right] + \tfrac{1}{2} \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[|x_1 - 3 - D_2|\right] \tag{71}$$

equals

$$g(x_1) := \frac{1}{2} \left(\left((x_1 - 101)^2 + 25 \right)^{\frac{1}{2}} + \left((x_1 - 103)^2 + 25 \right)^{\frac{1}{2}} \right). \tag{72}$$

It is easily verified that $g(x)$ is a strictly convex function of x on $[100, 104]$, g has its unique minimum on that interval at the point 102, and $g(102) = \sqrt{26}$. The desired result then follows from the fact that (71) is a convex function of x_1 on $\mathbb{R}$. $\quad\square$

Combining Lemmas 6.5 and 6.6 completes the proof of Theorem 7.

6.10. Proof of Theorem 8

Let $\tilde{Q}_2$ denote the probability measure such that $\tilde{Q}_2(5) = \tilde{Q}_2(11) = \frac{1}{2}$. It may be easily verified that $\tilde{Q}_2 \in \mathfrak{M}_2$. We begin by proving the following auxiliary lemma.

Lemma 6.7

$$\sup_{Q_1 \in \mathfrak{M}_1, \ Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \left[|10 - D_1 - D_2| \right] = 3.$$

Proof : Note that

$$\mathbb{E}_{Q_1 \times Q_2} \left[|10 - D_1 - D_2| \right] = \mathbb{E}_{Q_2} \left[\mathbb{E}_{Q_1} \left[|10 - D_1 - D_2| \,\Big|\, D_2 \right] \right].$$

Let us define

$$\phi_{Q_1}(d) \stackrel{\Delta}{=} \mathbb{E}_{Q_1} \left[|10 - D_1 - D_2| \,\Big|\, \{D_2 = d\} \right],$$

and

$$q(d) \stackrel{\Delta}{=} \frac{1}{6} (d - 8)^2 + \frac{3}{2} = \frac{73}{6} - \frac{8}{3}d + \frac{1}{6}d^2.$$

As $\tilde{Q}_2 \in \mathfrak{M}_2$, to prove the lemma, it follows from Proposition 2.1 that it suffices to demonstrate that for all $Q_1 \in \mathfrak{M}_1$, $q(5) = \phi_{Q_1}(5), q(11) = \phi_{Q_1}(11)$, and $q(d) \geq \phi_{Q_1}(d)$ for all $d \in \mathbb{R}$, as in this case for any $Q_1 \in \mathfrak{M}_1$, $\sup_{Q_2 \in \mathfrak{M}_2} E_{Q_2}[\phi_{Q_1}(D_2)] = E_{Q_2}[q(D_2)] = 3$. We now prove that $q(d) \geq \phi_{Q_1}(d)$ for all $d \in \mathbb{R}$. For any $Q_1 \in \mathfrak{M}_1$, since $10 - D_1 \in [7 - \epsilon, \ 9 + \epsilon]$ w.p.1, it follows that $\phi_{Q_1}(d) = 10 - \mu_1 - d = 8 - d$ if $d \in [0, 7 - \epsilon]$, and $\phi_{Q_1}(d) = d + \mu_1 - 10 = d - 8$ if $d \in [9 + \epsilon, \infty)$. It is easily verified that $q(d) - (8 - d) \geq 0$, and $q(d) - (d - 8) \geq 0$, for all $d \in \mathbb{R}$. It follows that $q(d) \geq \phi_{Q_1}(d)$ for all $d \in (-\infty, 7 - \epsilon] \bigcup [9 + \epsilon, \infty)$. Noting that $\phi_{Q_1}(d)$ is a convex function of d on $(-\infty, \infty)$, we conclude that $\phi_{Q_1}(d) \leq \max \left(\phi_{Q_1}(7 - \epsilon), \phi_{Q_1}(9 + \epsilon) \right)$ for all $d \in [7 - \epsilon, 9 + \epsilon]$. As it is easily verified that $\inf_{d \in \mathbb{R}} q(d) = \frac{3}{2}$, to prove that $q(d) \geq \phi_{Q_1}(d)$ for $d \in [7 - \epsilon, 9 + \epsilon]$, it suffices to show that $\max \left(\phi_{Q_1}(7 - \epsilon), \phi_{Q_1}(9 + \epsilon) \right) \leq \frac{3}{2}$. As $\phi_{Q_1}(7 - \epsilon) = 8 - (7 - \epsilon) = 1 + \epsilon < \frac{3}{2}$, and $\phi_{Q_1}(9 + \epsilon) = (9 + \epsilon) - 8 = 1 + \epsilon < \frac{3}{2}$, combining the above we conclude that $q(d) \geq \phi(d)$ for all $d \in \mathbb{R}$. As it is easily verified that $q(5) = \phi_{Q_1}(5) = 3$ and $q(11) = \phi_{Q_1}(11) = 3$, combining the above completes the proof. $\quad\square$

Proof of Theorem 8 : Note that the cost under any policy $\pi = (x_1, x_2) \in \Pi$ under the multistage-static formulation equals

$$\sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[2|x_1(y_1) - D_1| + |x_2(D_1) - D_2| \Big].$$

As $D_1 \le 3 + \epsilon \le 10 - \epsilon$ w.p.1, and $x_1(y_1) \ge y_1 = 10 - \epsilon$, we conclude that w.p.1

$$|x_1(y_1) - D_1| = x_1(y_1) - D_1 \ge 10 - \epsilon - D_1.$$

Combining with the fact that $\mu_1 = 2$, we conclude that

$$\mathbb{E}_{Q_1 \times Q_2} \Big[2|x_1(y_1) - D_1| \Big] \ge 2 \big(10 - \epsilon - 2 \big) = 2(8 - \epsilon).$$

As $\frac{\sigma_2^2}{\mu_2^2} = \frac{9}{64} < \frac{b_2}{h_2} = 1$, and $\big(h_2 b_2 \big)^{\frac{1}{2}} \sigma_2 = 3$, it follows from Lemma 4.1 and Theorem 1 that

$$\mathbb{E}_{Q_1 \times Q_2} \Big[|x_2(D_1) - D_2| \Big] \ge 3.$$

Combining the above, we conclude that the cost incurred under any policy π is at least $19 - 2\epsilon$.

We now show that the cost incurred under any such policy $\tilde{\pi}$ achieves this bound, and is thus optimal. In particular,

$$\sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[2|\tilde{x}_1(y_1) - D_1| + |\tilde{x}_2(D_1) - D_2| \Big]$$

equals

$$\sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[2|10 - \epsilon - D_1| + |10 - D_1 - D_2| \Big]$$

$$= \sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[2(10 - \epsilon - D_1) + |10 - D_1 - D_2| \Big]$$

$$= 2(10 - \epsilon - \mu_1) + \sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[|10 - D_1 - D_2| \Big] \quad = \quad 19 - 2\epsilon,$$

where the final equality follows from Lemma 6.7.

Next we show that there is no optimal base-stock policy, i.e. no base-stock policy belongs to Π_s^{opt}. Indeed, let us suppose for contradiction that $\hat{\pi}$ is a base-stock policy with constants $\hat{x}_1$, $\hat{x}_2$. The cost incurred under such a policy $\hat{\pi}$ equals

$$\sup_{Q_1 \in \mathfrak{M}_1, \; Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \Big[2| \max(\hat{x}_1, y_1) - D_1| + \big| \max \big(\max(\hat{x}_1, y_1) - D_1, \hat{x}_2 \big) - D_2 \big| \Big].$$

It follows from the fact that $D_1 \le 3 + \epsilon < 10 - \epsilon$ w.p.1 for all $Q_1 \in \mathfrak{M}_1$, and a straightforward contradiction argument (the details of which we omit), that $\hat{\pi}$ cannot be optimal unless $\hat{x}_1 \le 10 - \epsilon$, in which case repeating our earlier arguments, we conclude that $\max(\hat{x}_1, y_1) = 10 - \epsilon$, and for any $Q_1 \in \mathfrak{M}_1, Q_2 \in \mathfrak{M}_2$,

$$\mathbb{E}_{Q_1 \times Q_2} \Big[2| \max(\hat{x}_1, y_1) - D_1| \Big] = 2(8 - \epsilon).$$

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Thus to prove the desired claim, it suffices to demonstrate that

$$\inf_{\hat{x}_2 \in \mathbb{R}} \sup_{Q_1 \in \mathfrak{M}_1, Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \left[\left| \max\{10 - \epsilon - D_1, \hat{x}_2\} - D_2 \right| \right] > 3. \tag{73}$$

We treat two different cases: $\hat{x}_2 \in (-\infty, 7 + \frac{1}{2}\epsilon]$ and $\hat{x}_2 \in [7 + \frac{1}{2}\epsilon, \infty)$. If $\hat{x}_2 \leq 7 + \frac{1}{2}\epsilon$, let the probability measure $\tilde{Q}_1$ be such that $\tilde{Q}_1(1) = \tilde{Q}_1(3) = \frac{1}{2}$, where it is easily verified that $\tilde{Q}_1 \in \mathfrak{M}_1$. In this case,

$$\sup_{Q_1 \in \mathfrak{M}_1, Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_1 \times Q_2} \left[\left| \max\{10 - \epsilon - D_1, \hat{x}_2\} - D_2 \right| \right] \tag{74}$$

is at least

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{\tilde{Q}_1 \times Q_2} \left[\left| \max\{10 - \epsilon - D_1, \hat{x}_2\} - D_2 \right| \right]$$
$$= \sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \left[\frac{1}{2} \left| \max\{7 - \epsilon, \hat{x}_2\} - D_2 \right| + \frac{1}{2} \left| 9 - \epsilon - D_2 \right| \right], \tag{75}$$

where the final equality follows from the fact that $\hat{x}_2 \leq 7 + \frac{1}{2}\epsilon$ implies $\max\{9 - \epsilon, \hat{x}_2\} = 9 - \epsilon$. It follows from convexity of the absolute value function that (75) is at least

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \left[\left| \frac{1}{2} \max\{7 - \epsilon, \hat{x}_2\} + \frac{1}{2}(9 - \epsilon) - D_2 \right| \right]. \tag{76}$$

Note that

$$\frac{1}{2} \max\{7 - \epsilon, \hat{x}_2\} + \frac{1}{2}(9 - \epsilon) \geq \frac{1}{2}(7 - \epsilon) + \frac{1}{2}(9 - \epsilon)$$
$$= 8 - \epsilon. \tag{77}$$

Letting $z \overset{\Delta}{=} \frac{1}{2} \max\{7 - \epsilon, \hat{x}_2\} + \frac{1}{2}(9 - \epsilon)$, note that (76) equals $\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \left[(z - D_2)^+ + (D_2 - z)^+ \right]$. Applying Theorem 1 with $c = 0, b = h = 1$, and noting that $\frac{\mu_2^2 + \sigma_2^2}{2\mu_2} = \frac{73}{16} < 8 - \epsilon = z$, we conclude that (76) equals

$$\left(\left(\frac{1}{2} \max\{7 - \epsilon, \hat{x}_2\} + \frac{1}{2}(9 - \epsilon) - 8 \right)^2 + 9 \right)^{\frac{1}{2}}. \tag{78}$$

Combining (77) with the fact that

$$\frac{1}{2} \max\{7 - \epsilon, \hat{x}_2\} + \frac{1}{2}(9 - \epsilon) \leq \frac{1}{2}\left(7 + \frac{1}{2}\epsilon\right) + \frac{1}{2}(9 - \epsilon)$$
$$= 8 - \frac{1}{4}\epsilon,$$

we conclude that (78) is strictly greater than 3, completing the proof of (73) for the case $\hat{x}_2 \leq 7 + \frac{1}{2}\epsilon$.

Alternatively, if $\hat{x}_2 \geq 7 + \frac{1}{2}\epsilon$, let the probability measure $\tilde{Q}_1$ be such that $\tilde{Q}_1(\frac{1+2\epsilon}{1+\epsilon}) = \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}$ and $\tilde{Q}_1(3 + \epsilon) = \frac{1}{(1+\epsilon)^2+1}$. Again, it is easily verified that $\tilde{Q}_1 \in \mathfrak{M}_1$. In this case, (74) is at least

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2} \left[\frac{1}{(1+\epsilon)^2 + 1} |\hat{x}_2 - D_2| + \frac{(1+\epsilon)^2}{(1+\epsilon)^2 + 1} \left| \max\left\{ 10 - \epsilon - \frac{1+2\epsilon}{1+\epsilon}, \hat{x}_2 \right\} - D_2 \right| \right]. \tag{79}$$

It follows from convexity of the absolute value function that (79) is at least

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[\left|\frac{1}{(1+\epsilon)^2+1}\hat{x}_2 + \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}\max\left\{10-\epsilon-\frac{1+2\epsilon}{1+\epsilon}, \hat{x}_2\right\} - D_2\right|\right]. \tag{80}$$

Letting $z \triangleq \frac{1}{(1+\epsilon)^2+1}\hat{x}_2 + \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}\max\left\{10-\epsilon-\frac{1+2\epsilon}{1+\epsilon}, \hat{x}_2\right\}$, note that (80) equals

$$\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2}\left[(z-D_2)^+ + (D_2-z)^+\right].$$

Furthermore,

$$\frac{1}{(1+\epsilon)^2+1}\hat{x}_2 + \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}\max\left\{10-\epsilon-\frac{1+2\epsilon}{1+\epsilon}, \hat{x}_2\right\}$$
$$\geq \frac{1}{(1+\epsilon)^2+1}\left(7+\frac{1}{2}\epsilon\right) + \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}\left(10-\epsilon-\frac{1+2\epsilon}{1+\epsilon}\right)$$
$$= 8 + \frac{\frac{1}{2}-2\epsilon-\epsilon^2}{(1+\epsilon)^2+1}\epsilon. \tag{81}$$

Applying Theorem 1 with $c=0, b=h=1$, and noting that $\frac{\mu_2^2+\sigma_2^2}{2\mu_2} = \frac{73}{16} < 8 + \frac{\frac{1}{2}-2\epsilon-\epsilon^2}{(1+\epsilon)^2+1}\epsilon = z$ (having applied (42)), we conclude that (80) equals

$$\left(\left(\frac{1}{(1+\epsilon)^2+1}\hat{x}_2 + \frac{(1+\epsilon)^2}{(1+\epsilon)^2+1}\max\left\{10-\epsilon-\frac{1+2\epsilon}{1+\epsilon}, \hat{x}_2\right\} - 8\right)^2 + 9\right)^{\frac{1}{2}}. \tag{82}$$

Combining with (81) and (42), we conclude that (82) is strictly greater than 3, completing the proof of (73) for the case $\hat{x}_2 \leq 7 + \frac{1}{2}\epsilon$, which completes the proof. $\quad\square$

Acknowledgments