Simple and explicit bounds for multi-server queues with universal $\frac{1}{1-\rho}$ (and better) scaling

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We consider the FCFS $GI/GI/n$ queue, and prove the first simple and explicit bounds that scale gracefully and universally as $\frac{1}{1-\rho}$ (and better), with ρ the corresponding traffic intensity. In particular, we prove the first multi-server analogue of Kingman's bound, which has been an open problem for over fifty years. Our main results are bounds for the tail of the steady-state queue length and the steady-state probability of delay, where the strength of our bounds (e.g. in the form of tail decay rate) is a function of how many moments of the inter-arrival and service distributions are assumed finite. Supposing that the inter-arrival and service times, distributed as random variables A and S, have finite rth moment for some $r > 2$, and letting $\mu_A(\mu_S)$ denote $\frac{1}{\mathbb{E}[A]}(\frac{1}{\mathbb{E}[S]})$, our bounds are simple and explicit functions of only $\mathbb{E}\big[(A\mu_A)^r\big], \mathbb{E}\big[(S\mu_S)^r\big], r$, and $\frac{1}{1-\rho}$. Our simple and explicit bounds scale gracefully even when the number of servers grows large and the traffic intensity converges to unity simultaneously, as in the Halfin-Whitt scaling regime.

Our proofs proceed by explicitly analyzing the bounding process which arises in the stochastic comparison bounds of [30, 35] for multi-server queues. Along the way we derive several novel results for suprema of random walks with stationary increments and negative drift, pooled renewal processes, and various other quantities which may be of independent interest.

Key words: many-server queues, stochastic comparison, Kingman's bound, renewal process, Halfin-Whitt

1. Introduction.

The multi-server queue with independent and identically distributed (i.i.d.) inter-arrival and service times, and first-come-first-serve (FCFS) service discipline, is a fundamental object of study in Operations Research and Applied Probability. Its study was originally motivated by the design of telecommunication networks in the early 20th century, and pioneered by engineers such as Erlang [47]. Since that time the model has found many additional applications across a wide range of domains [98]. In the pioneering work of Erlang, it was realized that when both inter-arrival and service times are exponentially distributed (i.e. $M/M/n$) all relevant quantities can be computed in (essentially) closed form. These insights were extended to the case of multi-server queues with exponentially distributed service times and general inter-arrival times (i.e. $GI/M/n$) in [51] by considering the relevant embedded Markov chain. For the setting of non-Markovian service times, early progress was made in the analysis of single-server queues. In the early 20th century Pollaczek and Khintchine derived an explicit formula for the expected number in queue in such systems when inter-arrival times are Markovian (i.e. $M/GI/1$). Lindley, Spitzer, and others developed the theoretical foundation for analyzing general single-server models (i.e. $GI/GI/1$) as the suprema of one-dimensional random walks with i.i.d. increments.

1.1. Kingman's bound.

Another key result for $GI/GI/1$ queues came in the seminal 1962 paper of John Kingman [54], in which a simple and explicit upper bound was given for the steady-state expected waiting time $\mathbb{E}[W]$. We note that by Little's law [9], any such bound for $\mathbb{E}[W]$ yields a corresponding bound for the steady-state expected number of jobs waiting in queue (excluding those jobs in service) $\mathbb{E}[L]$. This bound, now referred to as Kingman's bound, states that

$$\mathbb{E}[W] \leq \frac{\sigma_A^2 + \sigma_S^2}{2\mathbb{E}[A]} \times \frac{1}{1-\rho} \quad , \quad \mathbb{E}[L] \leq \frac{\sigma_A^2 + \sigma_S^2}{2(\mathbb{E}[A])^2} \times \frac{1}{1-\rho}, \tag{1}$$

with $\sigma_A^2(\sigma_S^2)$ the variance of the inter-arrival (service) distribution, $\mathbb{E}[A]$ the mean inter-arrival time, and ρ the **traffic intensity**, defined (for the $GI/GI/1$ queue) as the ratio of the mean service time $\mathbb{E}[S]$ to the mean inter-arrival time $\mathbb{E}[A]$. Note that Kingman's bound for $\mathbb{E}[L]$ may be equivalently written as follows:

$$\mathbb{E}[L] \leq \frac{1}{2}\big(Var[A\mu_A] + \rho^2 Var[S\mu_S]\big) \times \frac{1}{1-\rho}, \tag{2}$$

with $\mu_A = \frac{1}{\mathbb{E}[A]}, \mu_S = \frac{1}{\mathbb{E}[S]}$, and Var[X] denoting the variance of X for a general random variable (r.v.) X. We also note that $Var[A\mu_A] = c_A^2, Var[S\mu_S] = c_S^2$, the corresponding squared coefficients of variation. Importantly, note that since irregardless of how A and S are scaled, it will always be true that $\mathbb{E}[A\mu_A] = \mathbb{E}[S\mu_S] = 1$, the quantity $\frac{1}{2} \times \big(Var[A\mu_A] + \rho^2 Var[S\mu_S]\big)$ is in some sense insensitive / unchanged / scale-free as one passes to heavy-traffic, with the term $\frac{1}{1-\rho}$ dictating how $\mathbb{E}[L]$ scales as $\rho \uparrow 1$ in a broad sense. Note that the following slight weakening of Kingman's bound (which is essentially equivalent as $\rho \uparrow 1$),

$$\mathbb{E}[L] \leq \frac{1}{2}\big(Var[A\mu_A] + Var[S\mu_S]\big) \times \frac{1}{1-\rho}, \tag{3}$$

captures the fundamental essence of Kingman's bound in heavy traffic. As most performance metrics of the general $GI/GI/1$ queue have no simple closed-form solution, this combination of **simplicity, accuracy, and scalability** has made Kingman's bound very attractive over the years, from the perspective of both real queueing applications and as a theoretical tool. In the same paper, Kingman established that (under appropriate technical conditions) this bound becomes tight as

$\rho \uparrow 1$, i.e. if one considers an appropriate sequence of $GI/GI/1$ queues in heavy-traffic. In particular, let us consider a sequence of $GI/GI/1$ queues indexed by their traffic intensity ρ, with $W_\rho(L_\rho)$ the steady-state waiting time (number waiting in queue) in the system with traffic intensity ρ. Let E_1 denote an exponential distribution with mean 1. For a sequence of r.v.s $\{X_n, n \geq 1\}$ and a limiting r.v. X_∞, let $\{X_n, n \geq 1\} \Rightarrow X_\infty$ denote weak convergence of the sequence to X_∞, where (with a slight abuse of notation) we also use $\Rightarrow$ to denote weak convergence when the corresponding collection of r.v.s is parametrized by a more general index set. Then in [54], Kingman proves that

$$\{(1-\rho)W_\rho, \rho \uparrow 1\} \Rightarrow \frac{\mathbb{E}[A]}{2} \times \big(Var[A\mu_A] + Var[S\mu_S]\big) \times E_1. \tag{4}$$

Upon applying the distributional Little's Law [9] and a straightforward weak convergence argument, it follows that

$$\{(1-\rho)L_\rho, \rho \uparrow 1\} \Rightarrow \frac{1}{2} \times \big(Var[A\mu_A] + Var[S\mu_S]\big) \times E_1, \tag{5}$$

where in both cases convergence of moments holds under appropriate technical conditions. We note that several slight strengthenings of Kingman's bound, which behave the same as $\rho \uparrow 1$, have been proven for single-server queues over the years, and we refer the reader to [88, 89, 25] for details.

1.2. The quest for a multi-server analogue of Kingman's bound.

Soon after Kingman's seminal work, other authors had begun using the tools of weak convergence to attempt to extend this analysis to the more complicated multi-server queue. Indeed, Kingman himself conjectured in 1965 that the weak-conergence result (4) should also hold for a sequence of $GI/GI/n$ queues which approaches heavy-traffic (with n held fixed) as the traffic intensity ρ (defined as $\frac{\mu_A}{n\mu_S}$ for $GI/GI/n$ queues) approaches 1 [55]. Such a weak convergence result was proven in [57]. Namely, it was proven in [57] that if one considers an appropriate sequence of $GI/GI/n$ queues in heavy-traffic (again indexed by their traffic intensity ρ), with n held fixed as $\rho \uparrow 1$, then

$$\{(1-\rho)W_\rho, \rho \uparrow 1\} \Rightarrow \frac{\mathbb{E}[A]}{2} \times \big(Var[A\mu_A] + Var[S\mu_S]\big) \times E_1, \tag{6}$$

from which it follows that

$$\{(1-\rho)L_\rho, \rho \uparrow 1\} \Rightarrow \frac{1}{2}\big(Var[A\mu_A] + Var[S\mu_S]\big) \times E_1, \tag{7}$$

where again in both cases convergence of moments holds under appropriate technical conditions. We note that closely related finite-horizon (i.e. not steady-state) analogues of these results were proven (using the framework of weak convergence) earlier in [12, 46], and an analogous weak-convergence result for steady-state work-in-system (i.e. sum of remaining work of all jobs in the system, both in service and in queue) was proven under the assumption of bounded service times in [60]. All of these results (i.e. [57, 12, 46, 60]) were proven by formalizing the fact that for such a sequence of $GI/GI/n$ queues (i.e. n held fixed, $\rho \uparrow 1$), the work-in-system is well-approximated by that in a single-server queue with the same inter-arrival and service-time distributions, but whose server operates at rate n (as opposed to the default of rate 1). The most powerful of these results proceeded through a careful analysis of the so-called Kiefer-Wolfowitz (K-W) vector, introduced in [53], the sorted version of the n-dimensional vector whose ith component corresponds to the workload on server i, where each server has its own queue, and upon arriving to the system a job immediately joins the queue of the server that it will eventually be served on under FCFS (i.e. the queue of minimal workload). Certain recursions governing the evolution of this vector were used to analyze the generating function of the work-in-system in [57], and the more fine-grained analysis needed for analyzing e.g. waiting times was carried out by bounding the gap between different components of the K-W vector as $\rho \uparrow 1$ [57, 53].

En route to proving (6), several authors proved explicit bounds which characterized the error of the associated single-server approximation [60, 58]. However, these bounds were premised on keeping the number of servers fixed while $\rho \uparrow 1$, i.e. their accuracy relied on the validity of (6), and in all cases involved bounding the rate of weak-convergence in (6), a line of research also pursued in [52, 65, 66]. **As such, it was recognized that these approaches seemed incapable of proving an absolute, universal, non-asymptotic inequality scaling correctly across a broad range of n and ρ, i.e. a multi-server analogue of Kingman's bound.**

Around the same time, a different line of literature was focused more on trying to prove such a general explicit bound. Indeed, in [56], Kingman himself notes that if instead of FCFS one considers a modified system in which jobs are routed to individual servers cyclically, one can (by analyzing any one server, and reasoning that waiting times in such a system must be stochastically larger than in the original system) prove the following (such bounds were later made rigorous by other authors, see e.g. [97] and the discussion therein). For a general FCFS $GI/GI/n$ queue,

$$\mathbb{E}[L] \leq \frac{1}{2}\big(Var[A\mu_A] + nVar[S\mu_S]\big) \times \frac{1}{1-\rho}. \tag{8}$$

Comparing (8) to (3), one sees the crucial difference: the price one pays to get an explicit result holding universally is the extra factor of n in front of $Var[S\mu_S]$. Under the so-called Halfin-Whitt type scaling (in which one considers a sequence of queues with n diverging and ρ scaling like $1 - Bn^{-\frac{1}{2}}$ for some fixed $B > 0$), it is known that such a bound scales in a fundamentally incorrect manner (i.e. the relative error is unbounded) [41]. An essentially equivalent bound was proven in [18], and a further slight strengthening was proven in [63]. Much later, but by considering related approaches, [78, 79, 74] derived additional bounds which, although decisive for understanding which moments of the waiting time distribution are finite, suffer from the same incorrect scaling.

The fundamental problem with these bounds is that they rely on bounding systems in which the workload on different servers are "overly decoupled", deviating too far conceptually from the single-server approximations which had successfully proven (7).

Thus another line of literature was born, which attempted to develop new techniques for relating multi-server queues to single-server queues in more subtle ways. By that time, much more was known about lower-bounding multi-server queues by appropriate sped-up single-server queues, as such bounds appeared even in the seminal 1955 paper of Kiefer and Wolfowitz (see [53] p. 13, Equation 7.4). As regards the prospects for deriving comparable upper-bounds, the hope was that although such a sped-up single-server yields a lower bound for the total work-in-system, it could still yield an upper-bound for the work (or number of jobs) in queue, the first component of the K-W vector, or the waiting time.

The prospects for such a (perhaps counter-intuitive) result were significantly bolstered by the results of [61], which proved exactly such a result for $GI/M/n$ queues (similar results were also proven in [19]), from which the inequality (3) follows for multi-server queues with Markovian service times. In [84], such a bound is proven for general inter-arrival and service times, but only for the very restrictive setting that $\rho \leq n^{-1}$ (i.e. very light traffic). In [4] (p. 214, Corollaries 3 and 4), for the family of service distributions which are so-called NBU (which includes those distributions with increasing failure rate), it is shown that the waiting time in a $GI/GI/n$ queue can be bounded by the waiting time in an appropriate single-server queue with the same arrival process, but for which each service time is distributed as the minimum of n independent services in the original system. For these distributions, one could again apply Kingman's bound to the resulting single-server system to derive a variant of (3), the strength of which would depend on the behavior of the minimum. This analysis was preceded by that of [68], which proved related (albeit in a sense weaker) bounds (see p. 794) for the special case of Erlang service times. [26] also provides conditions under which such a bound holds for deterministic processing times. For the special case of two servers, and again under the NBU and related assumptions on service times, it was proven in [80] that one can bound the waiting time by that in a single-server system working at rate 2.

Other approaches that various authors have taken in an attempt to derive simple bounds for multi-server queues include those based on convexity (e.g. [76, 87, 90, 26]), other modified service disciplines such as that in which jobs of different sizes have their own dedicated server pools (e.g. [21, 81, 42]), large deviations theory (e.g. [77]), and robust optimization (e.g. [91, 92, 93, 7]). However, to date, none of those approaches have been able to make substantial progress towards proving a bound such as (3) in the general case.

We note that more recently, with the rise in interest in queues in the Halfin-Whitt regime, there has been a new line of work (in the spirit of e.g. [58, 52, 65, 66]) proving bounds on the error of heavy-traffic approximations in the Halfin-Whitt regime [23, 16, 17, 15, 38, 39, 62, 48, 49]. However, outside the case of Markovian service times, all of these results suffer from the presence of non-explicit constants, which may depend on the underlying service distribution in a very complicated and unspecified way. Furthermore, in the Halfin-Whitt setting, the relevant limiting quantities themselves generally have no explicit representation (in contrast to the setting of classical heavy traffic). Namely, the analogue of (7) is weak convergence to a very complicated limiting r.v. depending on the entire service distribution [73, 2, 32]. Lyapunov function arguments have also been used to yield bounds [34, 32, 33, 23], but again suffer from the presence of non-explicit constants. In addition, many of these results require that the service time distribution comes from a restrictive class (such as phase-type), and/or makes other technical assumptions (e.g. the presence of a strictly positive abandonment rate), where we note that extending these bounds beyond these classes would in some cases require developing a far deeper understanding of the potentially complex interaction between the error in distributional approximation with the error in heavy-traffic approximation. Also, essentially all of the aforementioned heavy-traffic corrections require that one restrict to a specific type of heavy-traffic scaling, e.g. either classical heavy-traffic or Halfin-Whitt scaling, and do not hold universally (i.e. irregardless of how one approaches heavy-traffic). Recently, some progress has been made towards developing such universal bounds for single-server systems [45] and in multi-server systems with Markovian service times [15], but such universal bounds have remained elusive for general multi-server systems.

For further discussion regarding the massive literature on this problem, we refer the interested reader to the surveys [28, 69, 95], and especially point the reader to [95] for an overview of the many heuristic approximations available in the literature. We note that this body of work is further complicated by several mistakes in the literature, as discussed by Daley in [25, 26, 27] and Wolff in [97]. Indeed, even the popular textbook [36] seems to incorrectly claim the bound (3) for general multi-server queues in its Section 7.1.3.

1.2.1. Why $\frac{1}{1-\rho}$? In the above discussion, we several times referenced the fact that certain bounds "scaled as $\frac{1}{1-\rho}$", or "did not scale correctly" because they did not scale as $\frac{1}{1-\rho}$. It is of course reasonable to ask why, and in what precise sense, $\frac{1}{1-\rho}$ should be the bar. There are at least two fundamental justifications here. First, it follows from well-known results for the M/M/n queue [41] that for any fixed $B>0$, there exist $\zeta_B \in (0,\infty)$, independent of n and ρ, such that any $M/M/n$ queue for which $\rho \in (1-Bn^{-\frac{1}{2}},1)$ satisfies $\zeta_B \times \frac{1}{1-\rho} \leq \mathbb{E}[L] \leq \frac{1}{1-\rho}$. Thus, in a fairly general sense, this is the correct scaling for the $M/M/n$ queue in heavy-traffic. Second, this is the correct scaling in both classical heavy-traffic and the Halfin-Whitt scaling. In particular, if $\{L_n, n\geq 1\}$ is a sequence of r.v.s corresponding to the steady-state queue-lengths of a sequence of multi-server queues being scaled according to either classical heavy-traffic, Halfin-Whitt heavy-traffic, or even non-degenerate slow-down heavy-traffic [6], then letting ρ_n denote the traffic intensity of the nth queue in the sequence, one has (under appropriate technical conditions) that $\{(1-\rho_n)L_n, n \geq 1\}$ is tight and has a non-degenerate limit [57, 30, 2, 6]. Indeed, the $\frac{1}{1-\rho}$ scaling is a guiding meta-principle throughout the entire literature on multi-server queues. We note that different works often present / realize this phenomena from slightly different angles, e.g. one may get a slightly different result if analyzing waiting times (as opposed to queue lengths) as one must apply Little's law to appropriately "translate" the $\frac{1}{1-\rho}$ scaling, and many papers may formally prove a weak-convergence type result without formally proving that the corresponding sequence of expected values scales in the same way.

Of course, it is possible to come up with sequences of queues where $\frac{1}{1-\rho}$ is not the appropriate scaling. For example, it follows from well-known results for the M/M/n queue [41] that, letting Q denote the steady-state number-in-system (number in service + number waiting in queue), $\mathbb{E}[L] = \mathbb{P}(Q \geq n) \times \frac{\rho}{1-\rho}$. Thus if one lets $\rho \uparrow 1$ in such a manner that $\mathbb{P}(Q \geq n)$ approaches 0, $\frac{1}{1-\rho}$ will no longer be asymptotically correct. For example, suppose one has an $M/M/n$ queue with n large and $\rho = 1 - n^{-\frac{1}{4}}$. In that case, it follows from [41] that $\mathbb{P}(Q \geq n) \to 0$ as $n \to \infty$, and hence $\frac{1}{1-\rho}$ would significantly overestimate the expected queue length. None-the-less, as explained above, the literature certainly supports the general rule-of-thumb that for a multi-server queue in heavy-traffic, L should scale as $\frac{1}{1-\rho}$.

1.2.2. Summary of state-of-the-art. In summary, the quest for a multi-server analogue of Kingman's bound remains an open problem despite over 50 years of research. This includes not just whether the exact bound (3) holds, but whether any bound scaling gracefully and universally as $\frac{1}{1-\rho}$, i.e. any bound representable as a simple function of a few normalized moments times $\frac{1}{1-\rho}$, holds. Over the years, Daley has several times lamented on this state of affairs, and we refer the reader to [25, 26, 27, 3] for some directly related discussion, in which Daley conjectures that such a bound should indeed be possible (and even that (3) should hold). However, in spite of Daley's optimism, other works bring this into question. For example, the results of [37] prove that two queues whose inter-arrival and service times have the same first two moments can still have very different mean waiting times. Similarly, the known results for queues in the Halfin-Whitt regime (e.g. [2, 32, 23]) suggest that, in contrast to the classical heavy-traffic setting in which only one simple limiting behavior is possible (as dictated by (6)), under the Halfin-Whitt scaling the limiting behavior of the steady-state waiting time may depend in a very complex way on the underlying service distribution. In light of such results, the existence of a simple bound, which scales universally as $\frac{1}{1-\rho}$ across different notions of heavy-traffic and depends only on a few normalized moments, is entirely unclear.

1.2.3. Other approaches to analyzing queueing systems. Although having a simple, explicit, and accurate bound for general multi-server queues would be highly desirable, there are many other approaches one can take to analyzing queueing systems. For example, there is a vast literature on numerical approaches to understanding $GI/GI/n$ queues, including simulation [13, 67], the computation and subsequent inversion of transforms [8, 1], matrix-analytic methods [72], numerical analysis of diffusions [24], convex optimization [10], and robust optimization [7]. These methods have their own pros and cons in different settings, but generally have aims tangential to the formulation of simple and explicit bounds (which will be the subject of our own investigations). As such, we will not discuss those methods and the associated literatures in any depth, nor will we discuss the vast literature on more complicated queueing models (e.g. queueing networks, systems with heavy tails, etc.), or heuristic approaches which are not rigorously justified.

1.2.4. Other metrics in queueing systems. Steady-state queue-length and waiting time are natural quantities to consider when analyzing a queueing system, and have been the central quantities of interest in much of the queueing literature. Far less is known as regards simple, explicit, and universal bounds for other metrics in multi-server queues. For example, as regards the steady-state probability of delay (s.s.p.d.), i.e. $\mathbb{P}(Q \geq n)$, one of the few works along these lines seems to be [82].

1.3. Stochastic comparison results of [30] and [35].

As they will be especially pertinent to our own investigations, we now briefly review the results of [30] and [35], in which stochastic comparison arguments were used to bound the steady-state number of jobs waiting in queue in a $GI/GI/n$ queue by the all-time supremum of a one-dimensional random walk (with negative drift) involving the original arrival process and an independent pooled renewal process (with renewal intervals corresponding to the service distribution). The weak limit of this supremum (under the Halfin-Whitt scaling) was analyzed in [30], and used to show that the steady-state queue length scales like $n^{\frac{1}{2}}$, and to bound the large deviations behavior of the associated limiting process. These results were extended by considering a closely related stochastic comparison argument in [35], which was able to yield bounds on the s.s.p.d. when framed as a double limit within the Halfin-Whitt regime (letting $n \to \infty$ for each fixed excess parameter $B > 0$, and analyzing the resulting limit as B itself approaches either 0 or ∞). Although providing several qualitative insights into how various quantities behave asymptotically in the Halfin-Whitt regime, these results did not provide any explicit bounds for any fixed $GI/GI/n$ system, as the aforementioned supremum was only analyzed asymptotically (within the Halfin-Whitt regime), and even then in many regards the associated weak limit was itself only analyzed as certain parameters approached either 0 or ∞. We note that the analysis in [30] relied heavily on non-explicit versions of certain powerful maximal inequalities from probability, which convert bounds on the increments of a process into bounds for associated suprema, and have often been used to prove tightness results for sequences of stochastic processes, including queues [11]. More recently, generalizations of such bounds (i.e. the method of chaining) have also been used in the study of measure-valued processes associated with infinite-server queues [70] and multi-server queues with dependencies in the Halfin-Whitt regime [71].

1.4. Our contribution.

In this paper, we prove the first such Kingman-like bound for multi-server queues. Our bounds are simple, explicit, and scale universally as a simple function of a few normalized moments (of the inter-arrival and service distributions) times $\frac{1}{1-\rho}$, across all notions of heavy traffic (including both the classical and Halfin-Whitt scalings). In certain cases, we are actually able to prove bounds which scale asymptotically better than $\frac{1}{1-\rho}$. We also develop analogous bounds for the s.s.p.d. with the same desirable properties.

1.5. Outline of paper.

The remainder of our paper proceeds as follows. In Section 2, we state our main results, along with several corollaries and applications (e.g. to the Halfin-Whitt regime). In Section 3, we review the stochastic comparison results of [30] and [35], upon which our analysis will build. In Section 4, we provide very general conditional bounds on the supremum which arises in [30, 35], where these conditional bounds are of the form (for example) "If the moments of certain processes satisfy ..., then the supremum of interest satisfies ...". In Section 5, we provide an in-depth analysis of the processes arising in the aforementioned conditional bounds, notably certain pooled renewal processes, under the assumption that inter-arrival and service times have finite rth moment for some $r > 2$. Combined with our previous conditional results, we then complete the proof of our main results. We provide a summary of our results, concluding remarks, and some directions for future research in Section 6. Finally, we include a technical appendix in Section 7, which contains several technical arguments from throughout the paper.

2. Main Results.

2.1. Notation.

Let us fix an arbitrary FCFS $GI/GI/n$ queue with inter-arrival distribution A and service time distribution S, and denote this queueing system by $\mathcal{Q}^n$. Let $\mathcal{N}_o(\mathcal{A}_o)$ denote an ordinary renewal process with renewal distribution $S(A)$, and $N_o(t)\big(A_o(t)\big)$ the corresponding counting processes. Let $\{\mathcal{N}_i, i \geq 1\}\Bigg(\{\mathcal{N}_{o,i}, i \geq 1\}\Bigg)$ denote a mutually independent collection of equilibrium (ordinary) renewal processes with renewal distribution S; $\mathcal{A}$ an independent equilibrium renewal process with renewal distribution A; and $\{N_i(t), i \geq 1\}\bigg(\{N_{o,i}(t), i \geq 1\}\bigg)$, $A(t)$ the corresponding counting processes. Here we recall that an equilibrium renewal process is one in which the first renewal interval is distributed as the equilibrium distribution, i.e. if the renewal distribution is X, letting $R(X)$ denote a r.v. such that $\mathbb{P}(R(X) > y) = \frac{1}{\mathbb{E}[X]}\int_y^{\infty}\mathbb{P}(X > z)dz$, the first renewal interval is distributed as $R(X)$. Also, let $\mu_A(\mu_S)$ denote $\frac{1}{\mathbb{E}[A]}(\frac{1}{\mathbb{E}[S]})$; $\sigma_A(\sigma_S)$ denote $\big(Var[A]\big)^{\frac{1}{2}}\Big(\big(Var[S]\big)^{\frac{1}{2}}\Big)$; and $c_A(c_S)$ denote $\mu_A\sigma_A(\mu_S\sigma_S)$. Also, let $\{A_i, i \geq 1\}$ $(\{S_i, i \geq 1\})$ denote the sequence of inter-event times in $\mathcal{A}_o(\mathcal{N}_o)$. Let us evaluate all empty summations to zero, and all empty products to unity; and as a convention take $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. For an event $\mathcal{E}$, let $I(\mathcal{E})$ denote the corresponding indicator function. Unless stated otherwise, all processes should be assumed right-continuous with left limits (r.c.l.l.), as is standard in the literature. Also, for our results involving steady-state queue lengths, we will generally require that for any given initial condition, the total number of jobs in $\mathcal{Q}^n$ (number in service + number waiting in queue) converges in distribution (as time goes to infinity, independent of the particular initial condition) to a steady-state r.v. $Q^n(\infty)$. As a shorthand, we will denote this assumption by saying "$Q^n(\infty)$ exists", and refer the interested reader to [5] for a discussion of technical conditions ensuring this property holds. We will adopt a parallel convention when talking about the steady-state waiting time of an arriving job. Namely, for our results on waiting times, we will generally require that for any given initial condition, the distribution of the waiting time (in queue, not counting time in service) of the kth arrival to the system converges in distrubition (as $k \to \infty$, independent of the particular initial condition) to a steady-state r.v. $W^n(\infty)$. As a shorthand, we will denote this assumption by saying "$W^n(\infty)$ exists". Also, supposing that $Q^n(\infty)$ exists, let $L^n(\infty)$ denote the steady-state number of jobs waiting in queue, i.e. $L^n(\infty)$ is distributed as $\max\big(0, Q^n(\infty) - n\big)$. For a general r.v. X, let X^+ denote $\max(0, X)$. For $k \geq 1$, let $\rho_k \triangleq \frac{\mu_A}{k\mu_S}$. Whenever there is no ambiguity as regards a particular $GI/GI/n$ system, we will let $L(\infty), W(\infty), Q(\infty), \rho$ denote $L^n(\infty), W^n(\infty), Q^n(\infty), \rho_n$. Note that for any $GI/GI/n$ queue, one can always rescale both the service and inter-arrival times so that $\mathbb{E}[S] = \mu_S = 1$, without changing either ρ or the distribution of $Q^n(\infty)$. As doing so will simplify (notationally) several arguments and statements, sometimes we impose the additional assumption that $\mathbb{E}[S] = \mu_S = 1$, and will point out whenever this is the case.

2.2. Main results.

Our main results are the following novel, explicit bounds for general multi-server queues whose inter-arrival and service time distributions have finite $2+\epsilon$ moments, which scale universally as $\frac{1}{1-\rho}$ (and sometimes better). Our bounds depend only on a single normalized moment of the service and inter-arrival time distributions, as well as $\frac{1}{1-\rho}$, and are the first such simple and explicit bounds for general multi-server queues. Our bounds can be thought of as the first multi-server analogue of

Kingman's bound, which (although conjectured to exist for several decades) had remained elusive in spite of over fifty years of active research on multi-server queues. For $r > 2$, let

$$C_{r,1} \triangleq \left(10^{120} r^{32} (r-2)^{-12}\right)^r \quad , \quad C_{r,2} \triangleq \left(\frac{10r}{r-2}\right)^r C_{r,1};$$

where we note that $C_{r,1}, C_{r,2}$ are absolute constants depending only on r (and not on any parameters or distributions of any particular queueing systems).

THEOREM 1. *Suppose that for a GI/GI/n queue with inter-arrival distribution A and service time distribution S, there exists $r > 2$ s.t. $\mathbb{E}[S^r] < \infty, E[A^r] < \infty$. Suppose also that $0 < \mu_A < n\mu_S < \infty$, and that $Q(\infty)$ exists. Then for all $x > 0$, $\mathbb{P}\big(L(\infty) \geq \frac{x}{1-\rho}\big)$ is at most*

$$C_{r,1}\left(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\right)^3 x^{-\frac{r}{2}}; \tag{9}$$

and the steady-state probability of delay (s.s.p.d.), $\mathbb{P}\big(Q(\infty) \geq n\big)$, is at most

$$C_{r,1}\left(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\right)^3 \left(n(1-\rho)^2\right)^{-\frac{r}{2}}. \tag{10}$$

Note that since finiteness of higher moments implies finiteness of lower moments, the fact that $C_{r,1}$ diverges as $r \to \infty$ does not by any means imply that the strength of the bounds degrades as one assumes that more moments are finite. Instead, as one assumes finiteness of higher moments, one may trade-off between a larger prefactor (in the form of $C_{r,1}$) and the associated faster tail decay rate. To clarify this point, we note the following immediate corollary of Theorem 1.

COROLLARY 1. *Suppose that for a GI/GI/n queue with inter-arrival distribution A and service time distribution S, $\mathbb{E}[S^3] < \infty, E[A^3] < \infty$ (higher moments may or may not be finite). Suppose also that $0 < \mu_A < n\mu_S < \infty$, and that $Q(\infty)$ exists. Then for all $x > 0$, $\mathbb{P}\big(L(\infty) \geq \frac{x}{1-\rho}\big)$ is at most*

$$10^{450}\left(\mathbb{E}\big[(S\mu_S)^3\big]\mathbb{E}\big[(A\mu_A)^3\big]\right)^3 x^{-\frac{3}{2}};$$

and the s.s.p.d. is at most

$$10^{450}\left(\mathbb{E}\big[(S\mu_S)^3\big]\mathbb{E}\big[(A\mu_A)^3\big]\right)^3 \left(n(1-\rho)^2\right)^{-\frac{3}{2}}.$$

Of course, there was nothing special about the number 3 in Corollary 1, and an analogous result can be easily derived for any number strictly greater than 2. We note that if one assumes only finite $2+\epsilon$ moments for some very small ϵ, then the prefactor $C_{r,1}$ (with $r = 2+\epsilon$) diverges as $\epsilon \downarrow 0$. This of course stands in contrast to the classical Kingman's bound for a single server, which required only finite second moment. Whether our own results can be extended to the setting in which one only assumes a finite second moment, and whether the associated prefactor must diverge as $r \downarrow 2$

or $r \uparrow \infty$, remains an interesting open question. We also note that the prefactor (e.g. the 10^{450} appearing in Corollary 1) implies that for any given r taken in Theorem 1, there will exist a range of x for which the given bound simply evaluates to unity and hence is not meaningful. However, the point is that for any given assumed r, the bound is very meaningful for all sufficiently large x, with "sufficiently large" depending only on r and the normalized moments.

2.3. Further implications of our main results.

We now state several direct implications of our main results, further emphasizing their utility. In all cases these results follow directly from our main results, straightforward algebra/calculus, and Little's Law.

2.3.1. Mean waiting time and the multi-server Kingman's bound. We now state several implications for the mean steady-state waiting time and number in queue, including our multi-server analogue of Kingman's bound. In all cases, our bounds follow from Theorem 1 by a straightforward integration of the minimum of (9) and unity.

COROLLARY 2 **(Multi-server analogue of Kingman's bound)**. *Under the same assumptions as Theorem 1,*

$$\mathbb{E}\big[L(\infty)\big] \leq C_{r,2}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 \times \frac{1}{1-\rho}; \tag{11}$$

and supposing in addition that $W(\infty)$ exists,

$$\mathbb{E}\big[W(\infty)\big] \leq \mathbb{E}[A]C_{r,2}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 \times \frac{1}{1-\rho}.$$

We conclude that Daley was indeed correct that for general multi-server queues, $\mathbb{E}\big[L(\infty)\big]$ can be bounded by a simple function of a few normalized moments of A and S times $\frac{1}{1-\rho}$. Namely, Corollary 2 confirms the existence of a multi-server analogue of Kingman's bound. As before, we remind the reader that since finiteness of higher moments implies finiteness of lower moments, the fact that $C_{r,2}$ diverges as $r \to \infty$ does not imply that the strength of the bounds degrades as one assumes that more moments are finite. To clarify this point, we note the following immediate consequence of Corollary 2.

COROLLARY 3. *Suppose that for a $GI/GI/n$ queue with inter-arrival distribution A and service time distribution S, $\mathbb{E}[S^3] < \infty, E[A^3] < \infty$ (higher moments may or may not be finite). Suppose also that $0 < \mu_A < n\mu_S < \infty$, and that $Q(\infty)$ exists. Then*

$$\mathbb{E}\big[L(\infty)\big] \leq 10^{450}\bigg(\mathbb{E}\big[(S\mu_S)^3\big]\mathbb{E}\big[(A\mu_A)^3\big]\bigg)^3 \times \frac{1}{1-\rho}.$$

2.3.2. Better than $\frac{1}{1-\rho}$ scaling. We now observe that by integrating the minimum of (9) and (10), which also yields a bound for $\mathbb{P}\big(L(\infty) \geq \frac{x}{1-\rho}\big)$, we can (for a broad range of parameters) obtain a scaling better than $\frac{1}{1-\rho}$.

COROLLARY 4 **(Better than $\frac{1}{1-\rho}$ scaling)**. *Under the same assumptions as Theorem 1,*

$$\mathbb{E}\big[L(\infty)\big] \leq C_{r,2}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 \times \Big(n(1-\rho)^2\Big)^{-(\frac{r}{2}-1)} \times \frac{1}{1-\rho}.$$

Corollary 4 goes beyond the $\frac{1}{1-\rho}$ scaling, with the additional correction term $\Big(n(1-\rho)^2\Big)^{-(\frac{r}{2}-1)}$ capturing the fact that the relevant expectations should grow more slowly than $\frac{1}{1-\rho}$ when the s.s.p.d. goes to zero. Indeed, these bounds provide a vast generalization of the fact that in an M/M/n queue, $\mathbb{E}[L(\infty)] = \mathbb{P}\big(Q(\infty) \geq n\big) \times \frac{\rho}{1-\rho}$, with $\Big(n(1-\rho)^2\Big)^{-(\frac{r}{2}-1)}$ acting as a proxy for $\mathbb{P}\big(Q(\infty) \geq n\big)$. We note that $\Big(n(1-\rho)^2\Big)$ becomes large exactly when considering queues for which $\rho < 1 - n^{-\frac{1}{2}}$, i.e. when considering queues in the Halfin-Whitt regime as one transitions to the setting in which the s.s.p.d. goes to zero.

2.3.3. Higher order moments. We now state additional implications for higher moments. We note in addition to being the first such bounds for multi-server queues which scale universally as $\frac{1}{1-\rho}$, it seems that our bounds even shed new light on the single-server queue, for which the past literature on explicit bounds for higher moments seems to be largely restricted to non-explicit recursive formulas (e.g. [83]).

COROLLARY 5. *Under the same assumptions as Theorem 1, for all $z \in [1, \frac{r}{2})$,*

$$\mathbb{E}\Big[\big(L(\infty)\big)^z\Big] \leq \Big(\frac{10rz}{r-2z}\Big)^r C_{r,1}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 \times \big(\frac{1}{1-\rho}\big)^z.$$

We note that other moment relations for various additional quantities could also be obtained using e.g. the generalized Little's Law [96] and distributional Little's Law [9], although we do not pursue that here. We also note that one could easily use the logic behind Corollary 4 to derive bounds for $\mathbb{E}\big[\big(L(\infty)\big)^z\big]$ which scale better than $\big(\frac{1}{1-\rho}\big)^z$, although we do not formally state such a result here.

2.3.4. Halfin-Whitt regime. We now state several implications for queues in the Halfin-Whitt regime, in which ρ scales as $1 - Bn^{-\frac{1}{2}}$, where we note that this scaling regime has attracted a lot of attention recently due to its use in modeling utilization-efficiency trade-offs in service systems. We state several of our results with a normalization of $n^{-\frac{1}{2}}$, as this is standard in the literature (e.g. [41, 30]), although a normalization of $1 - \rho = Bn^{-\frac{1}{2}}$ would in some sense be more natural (in light of Theorem 1). In all cases, our results follow immediately from our previous main results and some straightforward algebra, and are the first simple, explicit, and general bounds which scale correctly in this setting. We state two types of results. First, we state results for general queues which are not explicitly in the "Halfin-Whitt" scaling as traditionally defined (i.e. for an appropriate sequence of queues), but which satisfy as an inequality the defining traffic intensity condition of the Halfin-Whitt regime.

COROLLARY 6. *Under the same assumptions as Theorem 1, and supposing in addition that* $\rho \leq 1 - Bn^{-\frac{1}{2}}$ *for some* $B > 0$*, the following holds. For all* $x > 0$*,* $\mathbb{P}\big(L(\infty) \geq xn^{\frac{1}{2}}\big)$ *is at most*

$$C_{r,1}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 B^{-\frac{r}{2}} x^{-\frac{r}{2}};$$

the steady-state probability of delay (s.s.p.d.), $\mathbb{P}\big(Q(\infty) \geq n\big)$*, is at most*

$$C_{r,1}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 B^{-r};$$

the normalized expected number in queue, $\mathbb{E}\big[\big(n^{-\frac{1}{2}}L(\infty)\big)\big]$*, satisfies*

$$\mathbb{E}\big[\big(n^{-\frac{1}{2}}L(\infty)\big)\big] \leq C_{r,2}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 B^{-(r-1)};$$

and for all $z \in [1, \frac{r}{2})$*, the higher order moments satisfy*

$$\mathbb{E}\big[\big(n^{-\frac{1}{2}}L(\infty)\big)^z\big] \leq \Big(\frac{10rz}{r-2z}\Big)^r C_{r,1}\bigg(\mathbb{E}\big[(S\mu_S)^r\big]\mathbb{E}\big[(A\mu_A)^r\big]\bigg)^3 B^{-\frac{r}{2}}.$$

Next, we state the corresponding results for a sequence of queues explicitly in the "Halfin-Whitt" scaling as traditionally defined. Namely, let us fix non-negative unit mean r.v.s $\hat{A}$ and $\hat{S}$, and an excess parameter $B > 0$. For $n > B^2$, let $\hat{Q}^n_B(\infty)$ denote a r.v. distributed as the steady-state number in system in the $GI/GI/n$ queue with inter-arrival distribution $\frac{\hat{A}}{n - Bn^{\frac{1}{2}}}$ and service time distribution $\hat{S}$ (supposing that all relevant steady-state quantities exist). Let $\hat{L}^n_B(\infty) = \max\big(0, \hat{Q}^n_B(\infty) - n\big)$. Then our results imply the following.

COROLLARY 7. *Suppose that for some* $r > 2$*, it holds that* $\mathbb{E}[\hat{A}^r] < \infty, \mathbb{E}[\hat{S}^r] < \infty$*, and that* $n > B^2$*. Then for all* $x > 0$*,* $\mathbb{P}\big(n^{-\frac{1}{2}}\hat{L}^n_B(\infty) \geq x\big)$ *is at most*

$$C_{r,1}\bigg(\mathbb{E}\big[\hat{S}^r\big]\mathbb{E}\big[\hat{A}^r\big]\bigg)^3 B^{-\frac{r}{2}} x^{-\frac{r}{2}};$$

the steady-state probability of delay (s.s.p.d.), $\mathbb{P}\big(\hat{Q}^n_B(\infty) \geq n\big)$*, is at most*

$$C_{r,1}\bigg(\mathbb{E}\big[\hat{S}^r\big]\mathbb{E}\big[\hat{A}^r\big]\bigg)^3 B^{-r};$$

the normalized expected number in queue, $\mathbb{E}\big[\big(n^{-\frac{1}{2}}\hat{L}^n_B(\infty)\big)\big]$*, satisfies*

$$\mathbb{E}\big[\big(n^{-\frac{1}{2}}\hat{L}^n_B(\infty)\big)\big] \leq C_{r,2}\bigg(\mathbb{E}\big[\hat{S}^r\big]\mathbb{E}\big[\hat{A}^r\big]\bigg)^3 B^{-(r-1)};$$

and for all $z \in [1, \frac{r}{2})$, the higher order moments satisfy

$$\mathbb{E}\big[\big(n^{-\frac{1}{2}}\hat{L}^n_B(\infty)\big)^z\big] \le \Big(\frac{10rz}{r-2z}\Big)^r C_{r,1}\bigg(\mathbb{E}\big[\hat{S}^r\big]\mathbb{E}\big[\hat{A}^r\big]\bigg)^3 B^{-\frac{r}{2}}.$$

These results give explicit and universal bounds on the steady-state queue length, for queues in the Halfin-Whitt regime, in terms of only a single moment of $\hat{A}$ and $\hat{S}$, and the excess parameter B. These results are the first of their kind for queues in this regime, for which (as discussed earlier) all previous explicit results were known only for the case of Markovian service times. These results also have important implications for the s.s.p.d. Namely, they give simple, explicit, non-asymptotic bounds on how the s.s.p.d. decays with B. Indeed, although the Halfin-Whitt regime is (in a sense) defined by the s.s.p.d. having a non-trivial value in $(0,1)$ even for very large numbers of servers, no simple, explicit, non-asymptotic bounds on this quantity were previously known.

We note that our results regarding the s.s.p.d. partially resolve an open question posed in [21] regarding bounds for this quantity under the FCFS discipline, although [21] actually conjectures that (under stronger assumptions) the s.s.p.d. should have a Gaussian decay (in B). On a related note, the results of [35] imply that, in the Halfin-Whitt regime, if one is willing to settle for a purely asymptotic result, then for large B the s.s.p.d. indeed has a Gaussian decay (in B). The results of [30] similarly imply that, again if one is willing to settle for a purely asymptotic and non-explicit result, then for large x the probability that the rescaled queue-length exceeds x should have an exponential decay (in x). Also, the results of [78, 79] imply the existence of more finite moments (for e.g. the queue length) than is implied by our own bounds. In all cases, bridging these gaps remain interesting open questions, and we refer the reader to [33] for some further relevant discussion as regards bridging asymptotic and non-asymptotic results in the Halfin-Whitt regime. Of course, as mentioned earlier, acheiving a simple and explicit bound which not only scales correctly, but is actually exact in heavy-traffic (a la Kingman's bound) may actually be impossible in the Halfin-Whitt regime, as the underlying limit processes seem to be inherently complicated.

2.3.5. Prefactors and the elephant in the room. Let us now briefly take a moment to address the proverbial "elephant in the room" - namely, the massive prefactors in these results. One important point is that in all proofs, simplicity was opted for over tightening these constants. Thus, presumably a more careful analysis using essentially the same exact ideas would lead to a significantly reduced prefactor. Furthermore, we view our results as a significant "proof-of-concept" as regards simple and explicit bounds for multi-server queues, and believe that future work, building on our own, will ultimately lead to the formulation of more practical bounds.

3. Review of upper bounds from [30] and [35].

In [30], the authors proved that $Q^n(\infty)$ can be bounded from above (in distribution) by the supremum of a certain one-dimensional random walk, which comes from analyzing a modified queueing system in which an artificial arrival is added to the system whenever a server would otherwise go idle. We note that although to simplify notations the authors of [30, 35] imposed the restriction that $\mathbb{P}(A=0)=\mathbb{P}(S=0)=0$ (to preclude having to deal with simultaneous events), this restriction is unnecessary and the proofs of [30, 35] can be trivially modified to accomodate this setting. As such, we state the relevant stochastic-comparison results of [30, 35] without that unnecessary assumption.

THEOREM 2 (**[30]**). *Suppose that for a $GI/GI/n$ queue with inter-arrival distribution A and service time distribution S, it holds that $0<\mu_A<n\mu_S<\infty$, and that $Q^n(\infty)$ exists. Then for all $x\geq 0$,*

$$\mathbb{P}\big(Q^n(\infty)-n\geq x\big)\leq \mathbb{P}\Bigg(\sup_{t\geq 0}\Big(A(t)-\sum_{i=1}^{n}N_i(t)\Big)\geq x\Bigg). \tag{12}$$

In [35], the author extends the framework of [30] considerably and derives analogous bounds for the s.s.p.d., for which Theorem 2 only provides trivial bounds. In particular, Theorem 4 of [35] implies the following bound. We note that as [35] actually states a more general result, for completeness we explicitly provide the derivation of this bound from Theorem 4 of [35] in the appendix.

THEOREM 3 (**[35]**). *Under the same assumptions as Theorem 2, it holds that*

$$\mathbb{P}\big(Q^n(\infty)\geq n\big)\leq \mathbb{P}\Bigg(\sup_{t\geq 0}\Big(A(t)-\sum_{i=1}^{n-\lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor}N_i(t)\Big)\geq \lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor\Bigg). \tag{13}$$

4. Bounds for $\sup_{t\geq 0}\big(A(t)-\sum_{i=1}^{n}N_i(t)\big)$.

In this section we prove explicit bounds for $\sup_{t\geq 0}\big(A(t)-\sum_{i=1}^{n}N_i(t)\big)$ under minimal assumptions. In light of Theorems 2 and 3, such bounds will be key to deriving bounds for the $GI/GI/n$ queue. First, we introduce some additional notation, and note that many results will not be stated in terms of the number of servers n, but will instead be stated in terms of a (potentially different) number n', to allow for the application of both Theorems 2 and 3 (which require considering pooled renewal processes with different numbers of components). For $n'\geq 1$, let $\mathcal{A}_{o,n'}$ denote the ordinary renewal process with renewal distribution $\min(2\rho_{n'},1)A$, where we take $\mathcal{A}_{o,n'}$ independent of $\{\mathcal{N}_i, i\geq 1\}$. Also, let $A_{o,n'}(t)$ denote the corresponding counting process. Note that we may construct $\{A(t),t\geq 0\},\{A_o(t),t\geq 0\},\{A_{0,n'}(t),t\geq 0\}$ on the same probability space s.t. w.p.1,

$$A(t)\leq 1+A_o(t)\leq 1+A_{o,n'}(t)\text{ for all }t\geq 0, \tag{14}$$

the final inequality following from the fact that since the renewal distribution of $\mathcal{A}_{o,n'}$ is a constant (at most one) multiple of the renewal distribution of $\mathcal{A}$, both may be constructed on the same probability space s.t. w.p.1 $A_{o,n'}(t) = A_o\big(\frac{t}{\min(2\rho_{n'},1)}\big) \geq A_o(t)$ for all $t \geq 0$. For $n' \geq 1$, let $\mu_{A,n'} \triangleq \frac{\mu_A}{\min(2\rho_{n'},1)}$, where we note that $\mu_{A,n'} = \max(\frac{1}{2}n'\mu_S, \mu_A)$. Also, let $\{A_{i,n'}, i \geq 1\}$ denote the sequence of inter-event times in $\mathcal{A}_{o,n'}$. We note that $\mathbb{E}[A_{1,n'}] = \frac{1}{\mu_{A,n'}}$, and that $\mu_A < n'\mu_S$ implies $\mu_{A,n'} < n'\mu_S$. By working with $\mathcal{A}_{o,n'}$, we will preclude certain technicalities which arise when considering queues with many servers and low traffic intensity.

Our explicit bounds for $\sup_{t\geq 0}\big(A(t) - \sum_{i=1}^n N_i(t)\big)$ will be the following conditional result, which translates bounds on the moments of $|\sum_{i=1}^n N_i(t) - n\mu_S t|$ and $|k - \mu_A \sum_{i=1}^k A_i|$ into bounds on the tail of $\sup_{t\geq 0}\big(A(t) - \sum_{i=1}^n N_i(t)\big)$.

Theorem 4. *Suppose that $\mathbb{E}[S] = 1$, and that for some positive integer n' and some fixed $C_1, C_2, C_3 > 0; r_1 > s_1 > 1$; $r_3 > s_3 > 1$; and $r_2 > 2$, the following conditions hold:*

(i) $0 < \mu_A < n' < \infty$.

(ii) For all $t \geq 1$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_1}\big] \leq C_1 n'^{\frac{r_1}{2}} t^{s_1}.$$

(iii) For all $t \in [0,1]$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_2}\big] \leq C_2 \max\big(n't, (n't)^{\frac{r_2}{2}}\big).$$

(iv) For all $k \geq 1$,

$$\mathbb{E}\big[|k - \mu_A \sum_{i=1}^{k} A_i|^{r_3}\big] \leq C_3 k^{s_3}.$$

Then for all $x \geq 16$, $\mathbb{P}\Bigg(\sup_{t\geq 0}\Bigg(A(t) - \sum_{i=1}^{n'} N_i(t)\Bigg) \geq x\Bigg)$ is at most

$$\left(\frac{10^6(r_1+r_2+r_3)^5}{(s_1-1)(s_3-1)(r_1-s_1)(r_3-s_3)(r_2-2)}\right)^{r_1+r_2+r_3+1}(1+C_1)(1+C_2)(1+C_3)$$
$$\times \Bigg(n'^{\frac{r_1}{2}}(n'-\mu_{A,n'})^{-s_1}x^{-(r_1-s_1)} + n'^{\frac{r_2}{2}}(n'-\mu_{A,n'})^{-\frac{r_2}{2}}x^{-\frac{r_2}{2}} + (n'-\mu_{A,n'})^{-s_3}(n')^{r_3}\mu_{A,n'}^{-(r_3-s_3)}x^{-(r_3-s_3)}\Bigg).$$

We will prove Theorem 4 in several parts. First, we implement two straightforward technical simplifications. In particular, (1) we reduce the general setting to the setting in which $\rho_n \geq \frac{1}{2}$ by working with the process $\mathcal{A}_{o,n'}$, and (2) we reduce the problem to bounding two separate suprema, one for the arrival process and one for the departure process. We proceed by applying a simple union bound to the right-hand-side of (12), in which case we derive the following result by adding and subtracting $\frac{1}{2}(n'\mu_S + \mu_{A,n'})t = \mu_{A,n'}t + \frac{1}{2}(n'\mu_S - \mu_{A,n'})t = n'\mu_S t - \frac{1}{2}(n'\mu_S - \mu_{A,n'})t$ in (12), and applying (14).

LEMMA 1. *Suppose that $\mathbb{E}[S] = 1$, and for some strictly positive integer n', it holds that $0 < \mu_A < n' < \infty$. Then for all $x > 2$, it holds that $\mathbb{P}\Bigg(\sup_{t\ge 0} \big(A(t) - \sum_{i=1}^{n'} N_i(t)\big) \ge x \Bigg)$ is at most*

$$\mathbb{P}\Bigg(\sup_{t\ge 0} \bigg(A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t \bigg) \ge \frac{1}{2}x - 1 \Bigg) \tag{15}$$

$$+ \quad \mathbb{P}\Bigg(\sup_{t\ge 0} \bigg(n't - \sum_{i=1}^{n'} N_i(t) - \frac{1}{2}(n' - \mu_{A,n'})t \bigg) \ge \frac{1}{2}x \Bigg). \tag{16}$$

The remainder of the proof of Theorem 4 proceeds roughly as follows.

(i) Bound the supremum of $n't - \sum_{i=1}^{n'} N_i(t)$ over sets of consecutive integers.

(ii) Bound the supremum of $n't - \sum_{i=1}^{n'} N_i(t)$ over intervals of length at most 1.

(iii) Combine (i) and (ii) to bound $\sup_{t\ge 0} \Bigg(n't - \sum_{i=1}^{n'} N_i(t) - \frac{1}{2}(n' - \mu_{A,n'})t \Bigg)$.

(iv) Bound the supremum of $k - \mu_{A,n'} \sum_{i=1}^{k} A_{i,n'}$ over sets of consecutive integers.

(v) Use (iv) to bound $\sup_{t\ge 0} \Bigg(A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t \Bigg)$.

We note that a similar logic was used to bound the same supremum non-explicitly for queues restricted to the Halfin-Whitt regime in [30]. Indeed, several of our arguments will be much more general, explicit, and tight versions of related arguments used in [30]. Before embarking on (i) - (v), we begin by reviewing a maximal inequality of Billlingsley, which will be critical for converting the moment bounds for partial sums posited in the assumptions of Theorem 4 into the bounds for suprema appearing in (i) - (v).

4.1. Review of a maximal inequality of Billingsley.

We begin by reviewing a particular maximal inequality of Billingsley. Such inequalities give general results for converting bounds on the difference between any two partial sums of a sequence of r.v.s into bound for the supremum of the partial sums of the given sequence, and are a common tool in proving tightness of stochastic processes. In particular, the following maximal inequality follows immediately from [59] Theorem 2 by setting the function they call $g(i,j)$ (defined for every pair of non-negative integers $i \le j$) equal to $C \times (j - i + 1)$ for any given $C > 0$. As the authors there note, the result also follows almost immediately from certain maximal inequalities present in an earlier edition of Billingsley's celebrated book on weak convergence [11]. We note that [59] actually proves a tighter bound, with related results also appearing in [64] and several follow-up works. For ease of exposition we present the following simpler bound which follows directly from [59]. For completeness, we include a proof that our bound follows from those of [59] in the appendix.

Lemma 2 (**[59] Theorem 2**). *Let $\{X_l, 1 \le l \le L\}$ be a completely general sequence of r.v.s. Suppose that for some fixed $\gamma > 1$, $\nu \ge \gamma$, and $C > 0$ the following condition holds:*

(i) For all $\lambda > 0$ and non-negative integers $1 \le i \le j \le L$,

$$\mathbb{P}\big(|\sum_{k=i}^{j} X_k| \ge \lambda\big) \le \big(C(j-i+1)\big)^{\gamma}\lambda^{-\nu}.$$

Then it must also hold that

$$\mathbb{P}\big(\max_{i\in[1,L]} |\sum_{k=1}^{i} X_k| \ge \lambda\big) \le \big(6\frac{\nu+1}{\gamma-1}\big)^{\nu+1}(CL)^{\gamma}\lambda^{-\nu}.$$

4.2. Bound the supremum of $n't - \sum_{i=1}^{n'} N_i(t)$ over sets of consecutive integers.

In this subsection we prove a bound for the supremum term associated with $\sum_{i=1}^{n'} N_i(t)$, when evaluated at finite subsets of consecutive integer times. In particular, we will prove the following result.

Lemma 3. *Suppose that $\mathbb{E}[S] = 1$, and that for some fixed $n' \ge 1, C_1 > 0, s_1 > 1$, and $r_1 \ge s_1$, the following condition holds:*

(i) For all $t \ge 1$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_1}\big] \le C_1 n'^{\frac{r_1}{2}} t^{s_1}.$$

Then it also holds that for all non-negative integers k and $\lambda > 0$,

$$\mathbb{P}\big(\max_{j\in[1,k]} \big|n'j - \sum_{i=1}^{n'} N_i(j)\big| \ge \lambda\big)$$

is at most

$$\big(6\frac{r_1+1}{s_1-1}\big)^{r_1+1} C_1 n'^{\frac{r_1}{2}} k^{s_1} \lambda^{-r_1}.$$

Proof We proceed by verifying that for each fixed $k \ge 1$, the conditions of Lemma 2 hold for $\left\{n' - \sum_{i=1}^{n'}\big(N_i(j) - N_i(j-1)\big), j = 1, \dots, k\right\}$. Let us fix some $k \ge 1$, and non-negative integers $l \le m \le k$. Then for any $\lambda > 0$, it follows from the fact that the given sequence of r.v.s is centered and stationary, the independence of $\{N_i(t), i \ge 1\}$, our assumptions, and Markov's inequality (after raising both sides to the r_1 power), that for all $1 \le l \le m \le k$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{j=l}^{m}\left(n' - \sum_{i=1}^{n'}\big(N_i(j) - N_i(j-1)\big)\right)\right| \ge \lambda\right) \\
\le\ & \mathbb{E}\left[\left|\sum_{j=l}^{m}\left(n' - \sum_{i=1}^{n'}\big(N_i(j) - N_i(j-1)\big)\right)\right|^{r_1}\right]\lambda^{-r_1} \\
=\ & \mathbb{E}\left[\left|n'(m-l+1) - \sum_{i=1}^{n'}\big(N_i(m) - N_i(l-1)\big)\right|^{r_1}\right]\lambda^{-r_1}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\Big|\sum_{i=1}^{n'} N_i(m-l+1) - n'(m-l+1)\Big|^{r_1}\Big]\lambda^{-r_1} \\
&\leq C_1 n'^{\frac{r_1}{2}}(m-l+1)^{s_1}\lambda^{-r_1}.
\end{aligned}
$$

Thus we find that the conditions of Lemma 2 are met with $L = k$, $\{X_l, 1 \leq l \leq L\} = \Big\{n' - \sum_{i=1}^{n'}\big(N_i(l) - N_i(l-1)\big), l = 1,\ldots,k\Big\}$, $C = (C_1 n'^{\frac{r_1}{2}})^{\frac{1}{s_1}}$, $\nu = r_1$, $\gamma = s_1$, and the desired result follows.

4.3. Bound the supremum of $n't - \sum_{i=1}^{n'} N_i(t)$ over intervals of length at most 1.

In this subsection we prove a bound for the supremum term associated with $\sum_{i=1}^{n'} N_i(t)$, when evaluated over intervals of length at most 1. In particular, we will prove the following result.

LEMMA 4. *Suppose that $\mathbb{E}[S] = 1$, and that for some fixed $n' \geq 1, C_2 > 0$ and $r_2 > 2$, the following condition holds:*

(i) For all $t \in [0,1]$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_2}\big] \leq C_2 \max\big(n't, (n't)^{\frac{r_2}{2}}\big).$$

Then it also holds that for all $t_0 \in [0,1]$ and $\lambda \geq 2$,

$$\mathbb{P}\Big(\sup_{t\in[0,t_0]}\big(n't - \sum_{i=1}^{n'} N_i(t)\big) \geq \lambda\Big) \tag{17}$$

is at most

$$(24\frac{r_2+1}{r_2-2})^{r_2+1} C_2 (n't_0)^{\frac{r_2}{2}}\lambda^{-r_2}.$$

Proof We begin by noting that it suffices to bound the supremum of interest over a suitable mesh, which follows immediately from the fact that w.p.1 $n't - \sum_{i=1}^{n'} N_i(t)$ can increase by at most 1 over any interval of length at most $(n')^{-1}$. In particular, (17) is at most

$$\mathbb{P}\Big(1 + \max_{k\in[0,\lfloor n't_0\rfloor]}\big(k - \sum_{i=1}^{n'} N_i(\frac{k}{n'})\big) \geq \lambda\Big). \tag{18}$$

We now verify that the conditions of Lemma 2 hold for $\Big\{1 - \sum_{i=1}^{n'}\big(N_i(\frac{k}{n'}) - N_i(\frac{k-1}{n'})\big), k = 1,\ldots,\lfloor n't_0\rfloor\Big\}$. Let us fix some non-negative integers $m \leq j \leq \lfloor n't_0\rfloor$ (note that if $\lfloor n't_0\rfloor < 1$ the result is trivial). Then for any $\lambda > 0$, it follows from stationary increments, centeredness, and Markov's inequality (after raising both sides to the r_2 power) that

$$\mathbb{P}\Big(\Big|\sum_{l=m}^{j}\Big(1 - \sum_{i=1}^{n'}\big(N_i(\frac{l}{n'}) - N_i(\frac{l-1}{n'})\big)\Big)\Big| \geq \lambda\Big) \tag{19}$$

is at most

$$\mathbb{E}\Big[\Big|\sum_{l=m}^{j}\Big(1-\sum_{i=1}^{n'}\big(N_i(\frac{l}{n'})-N_i(\frac{l-1}{n'})\big)\Big)\Big|^{r_2}\Big]\lambda^{-r_2}$$

$$= \quad \mathbb{E}\Big[\big|(j-m+1)-\sum_{i=1}^{n'}N_i(\frac{j-m+1}{n'})\big|^{r_2}\Big]\lambda^{-r_2},$$

which by our assumptions (and noting that in this case the $n't$ appearing in our assumptions equals $j-m+1$) is at most $C_2\max\big(j-m+1,(j-m+1)^{\frac{r_2}{2}}\big)$. Since $j-m+1$ is a non-negative integer and $\frac{r_2}{2}\geq 1$, it follows that (19) is at most $C_2(j-m+1)^{\frac{r_2}{2}}\lambda^{-r_2}$. We thus find that the conditions of Lemma 2 are met with $L=\lfloor n't_0\rfloor$, $\{X_l, 1\leq l\leq L\}=\Big\{1-\sum_{i=1}^{n'}\big(N_i(\frac{l}{n'})-N_i(\frac{l-1}{n'})\big), l=1,\ldots,\lfloor n't_0\rfloor\Big\}$, $C=(C_2)^{\frac{2}{r_2}}$, $\nu=r_2$, $\gamma=\frac{r_2}{2}$. Thus for all $z>0$,

$$\mathbb{P}\Big(\max_{k\in[0,\lfloor n't_0\rfloor]}\big(k-\sum_{i=1}^{n'}N_i(\frac{k}{n'})\big)\geq z\Big)\leq \big(6\frac{r_2+1}{\frac{r_2}{2}-1}\big)^{r_2+1}C_2(n't_0)^{\frac{r_2}{2}}z^{-r_2}.$$

It then follows from (18), and the fact that $\lambda\geq 2$ implies $(\lambda-1)^{-r_2}\leq 2^{r_2}\lambda^{-r_2}$, that (17) is at most

$$2^{r_2}\times\big(6\frac{r_2+1}{\frac{r_2}{2}-1}\big)^{r_2+1}\times C_2\times(n't_0)^{\frac{r_2}{2}}\lambda^{-r_2}$$

$$\leq \quad \big(24\frac{r_2+1}{r_2-2}\big)^{r_2+1}C_2(n't_0)^{\frac{r_2}{2}}\lambda^{-r_2},$$

completing the proof.

4.4. Bound $\sup_{t\geq 0}\Big(n't-\sum_{i=1}^{n'}N_i(t)-\frac{1}{2}(n'-\mu_{A,n'})t\Big)$ by combining Lemmas 3 and 4.

We now combine our two bounds for the supremum associated with $n't-\sum_{i=1}^{n'}N_i(t)$, namely Lemma 3 which provides bounds over sets of integer times, and Lemma 4 which provides bounds over intervals of length at most 1. We proceed by proving a very general result for the all-time supremum of continuous-time random walks with stationary increments and negative drift, which exactly converts appropriate bounds for the supremum over consecutive integers and over intervals of length at most 1 to bounds for the all-time supremum. We note that similar arguments have been used to bound all-time suprema of stochastic processes (cf. [85]), also in the heavy-tailed setting (cf. [86]). To allow for minimal assumptions, and to allow for the fully range of applicability to our setting of interest (and for completeness), we include a self-contained exposition and proof. We will rely on the following result, whose proof we defer to the appendix.

Lemma 5. *Let $\{\phi(t),t\geq 0\}$ be a stochastic process with stationary increments such that $\phi(0)=0$. Here, stationary increments means that for all $s_0\geq 0$, $\{\phi(s+s_0)-\phi(s_0),s\geq 0\}$ has the same distribution (on the process level) as $\{\phi(s),s\geq 0\}$. Suppose there exist strictly positive finite constants H_1,H_2,s,r_1,r_2 and $Z\geq 0$ such that $r_1>s>1$ and $r_2>2$, and the following two conditions hold:*

(i) For all integers $m \geq 1$ and real numbers $\lambda \geq Z$,

$$\mathbb{P}(\max_{j \in \{0,\dots,m\}} \phi(j) \geq \lambda) \leq H_1 m^s \lambda^{-r_1}.$$

(ii) For all $t_0 \in (0,1]$ and $\lambda \geq Z$,

$$\mathbb{P}(\sup_{0 \leq t \leq t_0} \phi(t) \geq \lambda) \leq H_2 t_0^{\frac{r_2}{2}} \lambda^{-r_2}.$$

Then for any drift parameter $\nu > 0$, and all $\lambda \geq 4Z$, $\mathbb{P}\bigg(\sup_{t \geq 0}(\phi(t) - \nu t) \geq \lambda\bigg)$ is at most

$$(1 + \frac{1}{r_1 - s}) 4^{r_1 + r_2 + 2} \bigg(H_1 \nu^{-s} \lambda^{-(r_1 - s)} + H_2 (\lambda \nu)^{-\frac{r_2}{2}} \bigg)$$

With Lemma 5 in hand, we now combine with Lemmas 3 and 4 to prove the following bound for $\sup_{t \geq 0}\bigg(n't - \sum_{i=1}^{n'} N_i(t) - \nu t\bigg)$. We note that ultimately we will take $\nu = \frac{1}{2}(n' - \mu_{A,n'})$, but here we prove the result for general drift.

LEMMA 6. *Suppose that $\mathbb{E}[S] = 1$, and that for some fixed $n' \geq 1, C_1, C_2 > 0; r_1 > s_1 > 1$; and $r_2 > 2$:*

(i) For all $t \geq 1$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_1}\big] \leq C_1 n'^{\frac{r_1}{2}} t^{s_1}.$$

(ii) For all $t \in [0,1]$,

$$\mathbb{E}\big[|\sum_{i=1}^{n'} N_i(t) - n't|^{r_2}\big] \leq C_2 \max\big(n't, (n't)^{\frac{r_2}{2}}\big).$$

Then for all $\nu > 0$ and $\lambda \geq 8$,

$$\mathbb{P}\Bigg(\sup_{t \geq 0}\bigg(n't - \sum_{i=1}^{n'} N_i(t) - \nu t\bigg) \geq \lambda\Bigg)$$

is at most

$$\bigg(\frac{100(r_1 + r_2)^3}{(s_1 - 1)(r_1 - s_1)(r_2 - 2)}\bigg)^{r_1 + r_2 + 2} \bigg(C_1 n'^{\frac{r_1}{2}} \nu^{-s_1} \lambda^{-(r_1 - s_1)} + C_2 n'^{\frac{r_2}{2}} (\lambda \nu)^{-\frac{r_2}{2}} \bigg).$$

Proof By our assumptions and Lemma 3, for all non-negative integers k and $\lambda > 0$,

$$\mathbb{P}\big(\max_{j \in [1,k]} \big|n'j - \sum_{i=1}^{n'} N_i(j)\big| \geq \lambda\big) \leq \big(6 \frac{r_1 + 1}{s_1 - 1}\big)^{r_1 + 1} C_1 n'^{\frac{r_1}{2}} k^{s_1} \lambda^{-r_1}.$$

Next, by our assumptions and Lemma 4, for all $t_0 \in [0,1]$ and $\lambda \geq 2$,

$$\mathbb{P}\bigg(\sup_{t \in [0,t_0]} \big(n't - \sum_{i=1}^{n'} N_i(t)\big) \geq \lambda\bigg) \leq \big(24 \frac{r_2 + 1}{r_2 - 2}\big)^{r_2 + 1} C_2 n'^{\frac{r_2}{2}} t_0^{\frac{r_2}{2}} \lambda^{-r_2}.$$

It then follows from our assumptions that the conditions of Lemma 5 are met with $\phi(t) = n't - \sum_{i=1}^{n'} N_i(t)$, $s = s_1$, r_1, r_2, ν their given values, $Z = 2$,

$$H_1 = (6\frac{r_1+1}{s_1-1})^{r_1+1} C_1 n'^{\frac{r_1}{2}} \quad , \quad H_2 = (24\frac{r_2+1}{r_2-2})^{r_2+1} C_2 n'^{\frac{r_2}{2}}.$$

Combining the above with the implications of Lemma 5 and some straightforward algebra completes the proof.

4.5. Bound the supremum of $k - \mu_{A,n'} \sum_{i=1}^{k} A_{i,n'}$ over sets of consecutive integers.

In this subsection we prove a bound for the supremum of $k - \mu_{A,n'} \sum_{i=1}^{k} A_{i,n'}$, as an intermediate step towards bounding the supremum of $A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t$. In particular, we will prove the following result.

LEMMA 7. *Suppose that $0 < \mu_A < \infty$, and for some fixed $C_3 > 0, s_3 > 1$, and $r_3 \geq s_3$, the following condition holds:*

(i) For all $k \geq 1$,

$$\mathbb{E}\big[|k - \sum_{i=1}^{k}(\mu_A A_i)|^{r_3}\big] \leq C_3 k^{s_3}.$$

Then for all $n' \geq 1$ and non-negative integers k and $\lambda > 0$,

$$\mathbb{P}\big(\max_{j\in[1,k]} \big|j - \sum_{i=1}^{j}(\mu_{A,n'} A_{i,n'})\big| \geq \lambda\big)$$

is at most

$$(6\frac{r_3+1}{s_3-1})^{r_3+1} C_3 k^{s_3} \lambda^{-r_3}.$$

Proof We proceed by verifying that for each fixed $k \geq 1$, the conditions of Lemma 2 hold for $\Big\{1 - \mu_{A,n'} A_{j,n'}, j = 1, \ldots, k\Big\}$. First, note that

$$\mathbb{E}\big[|k - \sum_{i=1}^{k}(\mu_{A,n'} A_{i,n'})|^{r_3}\big] = \mathbb{E}\big[|k - \sum_{i=1}^{k}(\mu_A A_i)|^{r_3}\big]. \tag{20}$$

Let us fix some $k \geq 1$, and non-negative integers $l \leq m \leq k$. Then for any $\lambda > 0$, it follows from the fact that the given sequence of r.v.s is centered and stationary, the independence of $\{A_{i,n'}, i \geq 1\}$, (20), and Markov's inequality (after raising both sides to the r_3 power), that

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\sum_{j=l}^{m}\big(1 - \mu_{A,n'} A_{j,n'}\big)\Big| \geq \lambda\Big) \\
\leq\ & \mathbb{E}\Big[\Big|\sum_{j=l}^{m}\big(1 - \mu_{A,n'} A_{j,n'}\big)\Big|^{r_3}\Big]\lambda^{-r_3} \\
=\ & \mathbb{E}\Big[\Big|\sum_{i=1}^{m-l+1}(\mu_{A,n'} A_{i,n'}) - (m-l+1)\Big|^{r_3}\Big]\lambda^{-r_3} \\
\leq\ & C_3 (m-l+1)^{s_3} \lambda^{-r_3}.
\end{aligned}$$

Thus we find that the conditions of Lemma 2 are met with $L = k$, $\{X_l, 1 \le l \le L\} = \Big\{1 - \mu_{A,n'} A_{l,n'}, l = 1,\dots,k\Big\}$, $C = (C_3)^{\frac{1}{s_3}}$, $\nu = r_3$, $\gamma = s_3$, and the desired result follows.

4.6. Bound $\sup_{t\ge 0}\Big(A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t\Big)$ using Lemma 7.

We now use Lemma 7 to bound $\sup_{t\ge 0}\Big(A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t\Big)$. Here we prove the result for general linear drift, but will later connect back to the desired drift $\frac{1}{2}(n' - \mu_{A,n'})$. We proceed in three steps. First, we relate the desired supremum to a discrete-time supremum associated with $k - \mu_A \sum_{i=1}^k A_i$. In particular, we begin with the following lemma.

LEMMA 8. *Suppose that* $0 < \mu_A < \infty$. *Then for all* $n' \ge 1, \nu > 0$ *and* $\lambda > 0$,

$$\mathbb{P}\Big(\sup_{t\ge 0}\big(A_{o,n'}(t) - \mu_{A,n'}t - \nu t\big) \ge \lambda\Big) \tag{21}$$

equals

$$\mathbb{P}\Bigg(\sup_{k\ge 0}\Big(k - \mu_{A,n'}\sum_{i=1}^k A_{i,n'} - \frac{\nu}{\mu_{A,n'} + \nu}k\Big) \ge \lambda\big(1 + \frac{\nu}{\mu_{A,n'}}\big)^{-1}\Bigg). \tag{22}$$

Proof As $\{A_{o,n'}(t) - \mu_{A,n'}t - \nu t, t \ge 0\}$ jumps up only at times $\{\sum_{i=1}^k A_{i,n'}, k \ge 1\}$ and at all other times drifts downward at linear rate $-(\mu_{A,n'} + \nu)$, we conclude that we may examine the relevant supremum only at times $\{\sum_{i=1}^k A_{i,n'}, k \ge 0\}$, from which it follows that (21) equals

$$\mathbb{P}\Big(\sup_{k\ge 0}\big(k - (\mu_{A,n'} + \nu)\sum_{i=1}^k A_{i,n'}\big) \ge \lambda\Big). \tag{23}$$

Further observing that

$$\begin{aligned} k - (\mu_{A,n'} + \nu)\sum_{i=1}^k A_{i,n'} &= \big(1 + \frac{\nu}{\mu_{A,n'}}\big)k - (\mu_{A,n'} + \nu)\sum_{i=1}^k A_{i,n'} - \frac{\nu}{\mu_{A,n'}}k \\ &= \big(1 + \frac{\nu}{\mu_{A,n'}}\big)\big(k - \mu_{A,n'}\sum_{i=1}^k A_{i,n'} - \frac{\nu}{\mu_{A,n'} + \nu}k\big) \end{aligned}$$

completes the proof.

Second, we prove a general result for the all-time supremum of discrete-time random walks with stationary increments and negative drift, in analogy with Lemma 5, which we will use to analyze (22). In particular, we prove the following result, whose proof we defer to the appendix.

LEMMA 9. *Let* $\{\phi(k), k \ge 0\}$ *be a discrete-time stochastic process with stationary increments such that* $\phi(0) = 0$. *Here, stationary increments means that for all integers* $k_0 \ge 0$, $\{\phi(k + k_0) - \phi(k_0), k \ge 0\}$ *has the same distribution (on the process level) as* $\{\phi(k), k \ge 0\}$. *Suppose there exist strictly positive finite constants* H_3, s_3, r_3 *such that* $r_3 > s_3 \ge 1$, *and the following condition holds:*

(i) For all integers $m \geq 1$ and $\lambda > 0$,

$$\mathbb{P}(\max_{j\in\{0,\ldots,m\}} \phi(j) \geq \lambda) \leq H_3 m^{s_3} \lambda^{-r_3}.$$

Then for any drift parameter $\nu > 0$, and all $\lambda > 0$, $\mathbb{P}\bigg(\sup_{k\geq 0}(\phi(k) - \nu k) \geq \lambda\bigg)$ is at most

$$16 H_3 4^{r_3}(1 + \frac{1}{r_3 - s_3})\nu^{-s_3}\lambda^{-(r_3-s_3)}.$$

Finally, we combine Lemmas 8 and 9 to bound $\sup_{t\geq 0}\bigg(A_{o,n'}(t) - \mu_{A,n'}t - \frac{1}{2}(n' - \mu_{A,n'})t\bigg)$, proving the following.

Lemma 10. *Suppose that $0 < \mu_A < \infty$, and that for some fixed $C_3 > 0$ and $r_3 > s_3 > 1$, the following condition holds:*

(i) For all $k \geq 1$,

$$\mathbb{E}\big[|k - \mu_A \sum_{i=1}^{k} A_i|^{r_3}\big] \leq C_3 k^{s_3}.$$

Then for all $\nu > 0$ and $\lambda > 0$,

$$\mathbb{P}\bigg(\sup_{t\geq 0}\big(A_{o,n'}(t) - \mu_{A,n'}t - \nu t\big) \geq \lambda\bigg)$$

is at most

$$\bigg(\frac{10^3(r_3+1)^2}{(s_3-1)(r_3-s_3)}\bigg)^{r_3+1} C_3 \nu^{-s_3}(\mu_{A,n'} + \nu)^{r_3}\mu_{A,n'}^{-(r_3-s_3)}\lambda^{-(r_3-s_3)}.$$

Proof By Lemma 8, it suffices to bound

$$\mathbb{P}\Bigg(\sup_{k\geq 0}\bigg(k - \mu_{A,n'}\sum_{i=1}^{k} A_{i,n'} - \frac{\nu}{\mu_{A,n'} + \nu}k\bigg) \geq \lambda(1 + \frac{\nu}{\mu_{A,n'}})^{-1}\Bigg) \tag{24}$$

Our assumptions, (20), and Lemma 7 ensure that for all non-negative integers k and $\lambda' > 0$,

$$\mathbb{P}(\max_{j\in[1,k]} \big|j - \sum_{i=1}^{j}(\mu_{A,n'}A_{i,n'})\big| \geq \lambda') \tag{25}$$

is at most

$$(6\frac{r_3+1}{s_3-1})^{r_3+1} C_3 k^{s_3}(\lambda')^{-r_3}.$$

It then follows from our assumptions that the conditions of Lemma 9 are met with $\phi(k) = k - \mu_{A,n'}\sum_{i=1}^{k} A_{i,n'}$, s_3, r_3 their given values,

$$H_3 = (6\frac{r_3+1}{s_3-1})^{r_3+1} C_3.$$

We conclude (after setting the drift parameter equal to $\frac{\nu}{\mu_{A,n'}+\nu}$ and the target level equal to $\lambda(1 + \frac{\nu}{\mu_{A,n'}})^{-1}$), that for all $\lambda > 0$, (24) is at most

$$16\big(6\frac{r_3+1}{s_3-1}\big)^{r_3+1} C_3 4^{r_3}(1 + \frac{1}{r_3 - s_3})(\frac{\nu}{\mu_{A,n'} + \nu})^{-s_3}\big(\lambda(1 + \frac{\nu}{\mu_{A,n'}})^{-1}\big)^{-(r_3-s_3)}.$$

Combining with some straightforward algebra completes the proof.

4.7. Proof of Theorem 4.

With Lemmas 6 and 10 in hand, we now complete the proof of Theorem 4.

Proof of Theorem 4 Using Lemma 6 to bound (16), and Lemma 10 to bound (15), combined with Lemma 1 and some straightforward algebra, we conclude that for all $x \geq 16$, $\mathbb{P}\Bigg(\sup_{t \geq 0} \bigg(A(t) - \sum_{i=1}^{n'} N_i(t) \bigg) \geq x \Bigg)$ is at most

$$\left(\frac{100(r_1 + r_2)^3}{(s_1 - 1)(r_1 - s_1)(r_2 - 2)} \right)^{r_1 + r_2 + 2} \times \left(C_1 {n'}^{\frac{r_1}{2}} \big(\frac{1}{2}(n' - \mu_{A,n'})\big)^{-s_1} \big(\frac{x}{2}\big)^{-(r_1 - s_1)} + C_2 {n'}^{\frac{r_2}{2}} \big(\frac{x}{2} \times \frac{1}{2}(n' - \mu_{A,n'})\big)^{-\frac{r_2}{2}} \right) \tag{26}$$

$+$

$$\left(\frac{10^3 (r_3 + 1)^2}{(s_3 - 1)(r_3 - s_3)} \right)^{r_3 + 1} \times C_3 \big(\frac{1}{2}(n' - \mu_{A,n'})\big)^{-s_3} \big(\frac{1}{2}(n' + \mu_{A,n'})\big)^{r_3} \mu_{A,n'}^{-(r_3 - s_3)} \big(\frac{x}{2} - 1\big)^{-(r_3 - s_3)}. \tag{27}$$

It follows from some straightforward algebra, and the assumption that $n' > \mu_{A,n'}$, that (26) is at most

$$\left(\frac{16 \times 10^3 (r_1 + r_2 + r_3)^5}{(s_1 - 1)(s_3 - 1)(r_1 - s_1)(r_3 - s_3)(r_2 - 2)} \right)^{r_1 + r_2 + r_3 + 1} (1 + C_1)(1 + C_2)(1 + C_3) \times \left({n'}^{\frac{r_1}{2}} (n' - \mu_{A,n'})^{-s_1} x^{-(r_1 - s_1)} + {n'}^{\frac{r_2}{2}} (n' - \mu_{A,n'})^{-\frac{r_2}{2}} x^{-\frac{r_2}{2}} \right),$$

and (27) is at most

$$\left(\frac{64 \times 10^3 (r_1 + r_2 + r_3)^5}{(s_1 - 1)(s_3 - 1)(r_1 - s_1)(r_3 - s_3)(r_2 - 2)} \right)^{r_1 + r_2 + r_3 + 1} (1 + C_1)(1 + C_2)(1 + C_3) \times (n' - \mu_{A,n'})^{-s_3} (n')^{r_3} \mu_{A,n'}^{-(r_3 - s_3)} x^{-(r_3 - s_3)}.$$

Combining the above with some straightforward algebra completes the proof.

5. Making Theorem 4 completely explicit when second moments exist, and proof of Theorem 1.

In this section we show that the relevant (pooled) renewal processes satisfy the conditions of Theorem 4 for certain explicit constants (assuming finite second moment), and use the corresponding explicit result of Theorem 4, combined with the stochastic comparison results Theorems 2 and 3, to complete the proof of Theorem 1.

5.1. Bounding the central moments of $\sum_{i=1}^k N_i(t)$ for $t \geq 1$.

In this subsection we bound the central moments of $\sum_{i=1}^k N_i(t)$ for $t \geq 1$. In particular, we will prove the following.

LEMMA 11. *Suppose that $\mathbb{E}[S]=1$, and that $\mathbb{E}[S^r]<\infty$ for some $r\geq 2$. Then for all $k\geq 1$, $t\geq 1$, and $\theta>0$,*

$$\mathbb{E}\big[\big|\sum_{i=1}^k N_i(t)-kt\big|^r\big] \leq \mathbb{E}[S^r]\exp(\theta)\big(\frac{10^8 r^3}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+2} k^{\frac{r}{2}} t^{\frac{r}{2}}.$$

Our proof of Lemma 11 proceeds in several steps. First, we bound the rth central moment of $N_o(t)$, showing that this moment scales (with t) like $t^{\frac{r}{2}}$ and providing a completely explicit bound along these lines. Our proof can essentially be viewed as "making completely explicit", e.g. all constants explicitly worked out, the approach to bounding the central moments of a renewal process sketched in [40]. As noted in [40] (and used in [30]), a non-explicit bound proving that the rth central moment indeed scales asymptotically (with t) like $t^{\frac{r}{2}}$ was first proven in [20]. To our knowledge such a completely explicit bound is new, and may prove useful in other settings. In particular, we begin by proving the following.

LEMMA 12. *Suppose that $\mathbb{E}[S]=1$, and that $\mathbb{E}[S^r]<\infty$ for some $r\geq 2$. Then for all $t\geq 1$ and $\theta>0$,*

$$\mathbb{E}\left[\left|N_o(t)-t\right|^r\right] \leq \exp(\theta)\mathbb{E}[S^r]\big(\frac{10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1} t^{\frac{r}{2}}.$$

We begin with some preliminary technical results. First, we recall the the celebrated Burkholder-Rosenthal Inequality for bounding the moments of a martingale. We state a particular variant (chosen largely for simplicity, although tighter bounds are known) given in [44].

LEMMA 13 **(Burkholder-Rosenthal Inequality, [44])**. *Let $\{X_i, i\geq 1\}$ be a martingale difference sequence w.r.t. the filtration $\{\mathcal{F}_i, i\geq 0\}$. Namely, we have that $\{X_i, i\geq 1\}$ is adapted to $\{\mathcal{F}_i, i\geq 0\}$; $\mathbb{E}[|X_i|]<\infty$ for all $i\geq 1$; and $\mathbb{E}[X_i|\mathcal{F}_{i-1}]=0$ for all $i\geq 1$. Suppose also that $\{\sum_{i=1}^n X_i, n\geq 1\}$ converges a.s. to a limiting r.v. which we denote $\sum_{i=1}^\infty X_i$. Then for all $r\geq 2$,*

$$\left(\mathbb{E}\big[\big|\sum_{i=1}^\infty X_i\big|^r\big]\right)^{\frac{1}{r}}$$

is at most

$$10r\left(\mathbb{E}\left[\left(\sum_{i=1}^\infty \mathbb{E}[X_i^2|\mathcal{F}_{i-1}]\right)^{\frac{r}{2}}\right]\right)^{\frac{1}{r}} + 10r\left(\mathbb{E}\big[\sup_{i\geq 1}|X_i^r|\big]\right)^{\frac{1}{r}}.$$

Since for any sequence of r.v.s $\{Z_i, i=1,\ldots,n\}$ and $r\geq 1$ it follows from convexity that w.p.1

$$\big|\sum_{i=1}^n Z_i\big|^r \leq n^{r-1}\sum_{i=1}^n |Z_i|^r, \tag{28}$$

we deduce the following corollary.

COROLLARY 8. *Under the same definitions and assumptions as Lemma 13, for all $r \geq 2$, $\mathbb{E}\big[\big|\sum_{i=1}^{\infty} X_i\big|^r\big]$ is at most*

$$(20r)^r \mathbb{E}\Bigg[\bigg(\sum_{i=1}^{\infty} \mathbb{E}[X_i^2|\mathcal{F}_{i-1}]\bigg)^{\frac{r}{2}}\Bigg] + (20r)^r \mathbb{E}\big[\sup_{i\geq 1} |X_i^r|\big].$$

We next recall a certain inequality for the non-central moments of $N_o(t)$, proven in [40] Equation 5.11.

LEMMA 14 (**[40] Equation 5.11**). *Suppose that $0 < \mu_S < \infty$. Then for all $p \geq 1$ and $t \geq 1$,*

$$\mathbb{E}\Big[\big(N_o(t)+1\big)^p\Big] \leq (2t)^p \mathbb{E}\Big[\big(N_o(1)+1\big)^p\Big].$$

We also prove the following bounds for moments of $N_o(1)$, whose proof we defer to the appendix.

LEMMA 15. *Suppose that $0 < \mu_S < \infty$. Then for all $p \geq 1$ and $\theta > 0$,*

$$\mathbb{E}\big[\big(N_o(1)\big)^p\big] \leq \exp(\theta)\big(\frac{24p}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{p+2}.$$

Combining Lemmas 14 and 15 with (28) and some straightforward algebra, we come to the following corollary.

COROLLARY 9. *Suppose that $0 < \mu_S < \infty$. Then for all $p \geq 1$, $t \geq 1$, and $\theta > 0$,*

$$\mathbb{E}\Big[\big(N_o(t)+1\big)^p\Big] \leq \exp(\theta)\big(\frac{100p}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{p+2} t^p.$$

With Lemmas 13 - 15 and Corollaries 8 - 9 in hand, we now complete the proof of Lemma 12.

Proof of Lemma 12 By definition (as is well-known), $N_o(t)+1 = \min\{n \geq 1 : \sum_{i=1}^n S_i > t\}$ is a stopping time w.r.t. the natural filtration generated by $\{S_i, i \geq 1\}$. By the triangle inequality, w.p.1

$$\begin{aligned}\big|N_o(t)-t\big| &= \big|\big(N_o(t)+1\big)-t-1\big| \\ &\leq \big|\big(N_o(t)+1\big)-t\big|+1 \\ &\leq \Bigg|\sum_{i=1}^{N_o(t)+1} S_i - \big(N_o(t)+1\big)\Bigg| + \Bigg|\sum_{i=1}^{N_o(t)+1} S_i - t\Bigg| + 1. \qquad (29)\end{aligned}$$

It then follows from (28) and (29) that $\mathbb{E}\Bigg[\bigg|N_o(t)-t\bigg|^r\Bigg]$ is at most

$$3^{r-1}\mathbb{E}\Bigg[\Bigg|\sum_{i=1}^{N_o(t)+1} S_i - \big(N_o(t)+1\big)\Bigg|^r\Bigg] \qquad (30)$$

$$+ \quad 3^{r-1}\mathbb{E}\Bigg[\Bigg|\sum_{i=1}^{N_o(t)+1} S_i - t\Bigg|^r\Bigg] \qquad (31)$$

$$+ \quad 3^{r-1}. \qquad (32)$$

We next bound

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{N_o(t)+1} S_i - \big(N_o(t)+1\big)\Big|^r\Big], \tag{33}$$

and proceed by applying the celebrated Burkholder-Rosenthal Inequality. In particular, we will use Corollary 8 to bound (33). First, we rewrite (33) in terms of an appropriate martingale difference sequence. Namely, note that (33) equals

$$\begin{aligned}&\mathbb{E}\Big[\Big|\sum_{i=1}^{\infty} S_i I\big(N_o(t)+1\geq i\big) - \sum_{i=1}^{\infty} I\big(N_o(t)+1\geq i\big)\Big|^r\Big] \\ =\ &\mathbb{E}\Big[\Big|\sum_{i=1}^{\infty}(S_i-1) I\big(N_o(t)+1\geq i\big)\Big|^r\Big]. \end{aligned}\tag{34}$$

We now prove that $\{(S_i-1)I\big(N_o(t)+1\geq i\big), i\geq 1\}$ is a martingale difference sequence w.r.t. the filtration $\{\sigma(S_1,\ldots,S_i), i\geq 1\}$. Finite expectations and measurability are trivial. Furthermore, since $I\big(N_o(t)+1\geq i\big)$ is $\sigma(S_1,\ldots,S_{i-1})$-measurable (due to the greater than or equal to sign), it follows from independence and the basic properties of conditional expectation that w.p.1

$$\begin{aligned}&\mathbb{E}\Big[(S_i-1)I\big(N_o(t)+1\geq i\big)|\sigma(S_1,\ldots,S_{i-1})\Big] \\ =\ & I\big(N_o(t)+1\geq i\big)\mathbb{E}\Big[(S_i-1)|\sigma(S_1,\ldots,S_{i-1})\Big] \\ =\ & I\big(N_o(t)+1\geq i\big)\mathbb{E}\big[S_i-1\big] \quad = \quad 0.\end{aligned}$$

Thus we find that the conditions of Corollary 8 are satisfied with $X_i=(S_i-1)I\big(N_o(t)+1\geq i\big), \mathcal{F}_i = \sigma(S_1,\ldots,S_i)$. Before stating the given implication, we first show that several resulting terms can be simplified. First, note that

$$\begin{aligned}&\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{\infty}\mathbb{E}\Big[\Big((S_i-1)I\big(N_o(t)+1\geq i\big)\Big)^2\Big|\sigma(S_1,\ldots,S_{i-1})\Big]\Bigg)^{\frac{r}{2}}\Bigg] \\ =\ &\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{\infty}\mathbb{E}\Big[(S_i-1)^2 I\big(N_o(t)+1\geq i\big)\Big|\sigma(S_1,\ldots,S_{i-1})\Big]\Bigg)^{\frac{r}{2}}\Bigg] \\ =\ &\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{\infty} I\big(N_o(t)+1\geq i\big)\mathbb{E}\Big[(S_i-1)^2\Big|\sigma(S_1,\ldots,S_{i-1})\Big]\Bigg)^{\frac{r}{2}}\Bigg] \\ =\ &\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{\infty} I\big(N_o(t)+1\geq i\big)\mathbb{E}\big[(S-1)^2\big]\Bigg)^{\frac{r}{2}}\Bigg] \\ =\ &\Big(\mathbb{E}\big[(S-1)^2\big]\Big)^{\frac{r}{2}}\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{\infty} I\big(N_o(t)+1\geq i\big)\Bigg)^{\frac{r}{2}}\Bigg] \\ =\ &\Big(\mathbb{E}\big[(S-1)^2\big]\Big)^{\frac{r}{2}}\mathbb{E}\Big[\big(N_o(t)+1\big)^{\frac{r}{2}}\Big] \\ \leq\ &\mathbb{E}\big[|S-1|^r\big]\mathbb{E}\Big[\big(N_o(t)+1\big)^{\frac{r}{2}}\Big],\end{aligned}\tag{35}$$

the final inequality following from Jensen's inequality (applicable since $r \geq 2$). Second, note that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{i\geq 1}\bigg|\Big((S_i-1)I\big(N_o(t)+1\geq i\big)\Big)^r\bigg|\Bigg] \\
\leq\ & \mathbb{E}\Bigg[\sum_{i=1}^{\infty} I\big(N_o(t)+1\geq i\big)\big|S_i-1\big|^r\Bigg] \\
=\ & \mathbb{E}\Bigg[\sum_{i=1}^{N_o(t)+1}\big|S_i-1\big|^r\Bigg] \\
=\ & \mathbb{E}\big[N_o(t)+1\big]E\big[\big|S-1\big|^r\big], \qquad (36)
\end{aligned}$$

the final inequality following from Wald's identity. Combining (35) and (36) with the fact that the conditions of Corollary 8 are satisfied with $X_i = (S_i-1)I\big(N_o(t)+1\geq i\big), \mathcal{F}_i = \sigma(S_1,\ldots,S_i)$, and the fact that $\mathbb{E}\big[N_o(t)+1\big] \leq \mathbb{E}\Big[\big(N_o(t)+1\big)^{\frac{r}{2}}\Big]$ (since $r\geq 2$), we conclude that (33) is at most

$$2(20r)^r\mathbb{E}\big[|S-1|^r\big]\mathbb{E}\Big[\big(N_o(t)+1\big)^{\frac{r}{2}}\Big]. \qquad (37)$$

Combining (37) and Corollary 9, we conclude that (33) is at most

$$\begin{aligned}
&2(20r)^r\mathbb{E}\big[|S-1|^r\big]\exp(\theta)\bigg(\frac{50r}{1-\mathbb{E}[\exp(-\theta S)]}\bigg)^{\frac{r}{2}+2} t^{\frac{r}{2}} \qquad (38)\\
\leq\ & \exp(\theta)\bigg(\frac{2\times 10^3 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\bigg)^{r+1}\mathbb{E}\big[|S-1|^r\big]t^{\frac{r}{2}}. \qquad (39)
\end{aligned}$$

We next bound (31), by bounding

$$\mathbb{E}\Bigg[\bigg|\sum_{i=1}^{N_o(t)+1} S_i - t\bigg|^r\Bigg]. \qquad (40)$$

By definition, $\sum_{i=1}^{N_o(t)+1} S_i - t$ is the residual life of the renewal process N_o at time t, i.e. the remaining time until the next renewal (at time t), and it follows that w.p.1

$$\begin{aligned}
\bigg|\sum_{i=1}^{N_o(t)+1} S_i - t\bigg|^r &\leq S^r_{N_o(t)+1} \\
&\leq \sum_{i=1}^{N_o(t)+1} S_i^r.
\end{aligned}$$

Combining with Wald's identity, we conclude that (40) is at most $\mathbb{E}\big[N_o(t)+1\big]\mathbb{E}[S^r]$, which by Corollary 9 is at most

$$\exp(\theta)\big(\frac{100}{1-\mathbb{E}\big[\exp(-\theta S)\big]}\big)^3 t\mathbb{E}[S^r]. \qquad (41)$$

Using (39) to bound (33) (providing a bound for (30)), and (41) to bound (40) (providing a bound for (31)), and applying (28) and some straightforward algebra, we find that $\mathbb{E}\Big[\Big|N_o(t)-t\Big|^r\Big]$ is at most

$$\begin{aligned}
&3^{r-1}\exp(\theta)\big(\frac{2\times 10^3 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}\mathbb{E}\big[|S-1|^r\big]t^{\frac{r}{2}} \\
&\quad + \quad 3^{r-1}\exp(\theta)\big(\frac{100}{1-\mathbb{E}\big[\exp(-\theta S)\big]}\big)^3 t\mathbb{E}[S^r] \quad + \quad 3^{r-1} \\
\leq\ & 3^{r-1}\exp(\theta)\big(\frac{2\times 10^3 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}t^{\frac{r}{2}}\Big(\mathbb{E}\big[|S-1|^r\big]+\mathbb{E}[S^r]\Big)+3^{r-1} \\
\leq\ & 3^{r-1}\exp(\theta)\big(\frac{2\times 10^3 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}t^{\frac{r}{2}}\Big(2^{r-1}\big(\mathbb{E}[S^r]+1\big)+\mathbb{E}[S^r]\Big)+3^{r-1} \\
\leq\ & 3^{r-1}\exp(\theta)\big(\frac{2\times 10^3 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}t^{\frac{r}{2}}2^{r+1}\mathbb{E}[S^r]+3^{r-1} \\
\leq\ & \exp(\theta)\mathbb{E}[S^r]\big(\frac{10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}t^{\frac{r}{2}}.
\end{aligned}$$

Combining the above completes the proof.

We now extend Lemma 12 to the corresponding equilibrium renewal process. We note that given the results of Lemma 12, such an extension follows nearly identically to the proof of Lemma 8 of [30]. However, as we wish to make all quantities completely explicit, we include a self-contained proof in the appendix.

Corollary 10. *Suppose that $\mathbb{E}[S]=1$, and that $\mathbb{E}[S^r]<\infty$ for some $r\geq 2$. Then for all $t\geq 1$ and $\theta>0$,*

$$\mathbb{E}\Big[\Big|N_1(t)-t\Big|^r\Big]\leq \mathbb{E}[S^r]\exp(\theta)\big(\frac{10^7 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+2}t^{\frac{r}{2}}.$$

Before completing the proof of Lemma 11, we recall the celebrated Marcinkiewicz-Zygmund inequality, a close relative of the Rosenthal inequality. The precise result which we will use follows immediately from [75] Theorem 2, and we refer the interested reader to [29] for a further overview of related results. We note that for several results which we will state, it is not required that the r.v.s be identically distributed, although we only state the results for that setting.

Lemma 16 **([75] Theorem 2)**. *Suppose that for some $p\geq 2$, $\{X_i, i\geq 1\}$ is a collection of i.i.d. zero-mean r.v.s. s.t. $\mathbb{E}[|X_1|^p]<\infty$. Then for all $k\geq 1$,*

$$\mathbb{E}\big[\big|\sum_{i=1}^k X_i\big|^p\big]\leq (5p)^p\mathbb{E}[|X_1|^p]k^{\frac{p}{2}}.$$

For later use, here we also state a result similar to Lemma 16, but under different assumptions, e.g. requiring the r.v.s be non-negative but not necessarily centered, and only requiring finite first moment. The result follows immediately from [50] Theorem 2.5.

LEMMA 17 (**[50] Theorem 2.5**). *Suppose that for some $p \geq 1$, $\{X_i, i \geq 1\}$ is a collection of i.i.d. non-negative r.v.s. s.t. $\mathbb{E}[X_1^p] < \infty$. Then for all $k \geq 1$,*

$$\mathbb{E}\big[\big(\sum_{i=1}^{k} X_i\big)^p\big] \leq (2p)^p \max\big((k\mathbb{E}[X_1])^p, k\mathbb{E}[X_1^p]\big).$$

With Corollary 10 and Lemma 16 in hand, we now complete the proof of Lemma 11.

Proof of Lemma 11 Applying Lemma 16 with $X_i = N_i(t) - t$, we find that

$$\mathbb{E}\big[\big|\sum_{i=1}^{k} N_i(t) - kt\big|^r\big] \leq (5r)^r E\big[|N_1(t) - t|^r\big] k^{\frac{r}{2}}.$$

Combining with Corollary 10 and some straightforward algebra completes the proof.

5.2. Bounding the central moments of $\sum_{i=1}^{k} N_i(t)$ for $t \in [0,1]$.

In this subsection we bound the central moments of $\sum_{i=1}^{k} N_i(t)$ for $t \in [0,1]$. We will use an argument similar to that used in the proof of [30] Lemma 5. However, in contrast to the arguments of [30], here all quantities are made completely explicit.

LEMMA 18. *Suppose that $\mathbb{E}[S] = 1$. Then for all $k \geq 1, p \geq 2, t \in [0,1]$, and $\theta > 0$,*

$$\mathbb{E}\Big[\big|\sum_{i=1}^{k} N_i(t) - kt\big|^p\Big] \leq \exp(\theta)\big(\frac{10^5 p^4}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2} \max\big(kt, (kt)^{\frac{p}{2}}\big). \tag{42}$$

Our proof proceeds by first proving a somewhat weaker bound, and then leveraging this bound to prove the desired result.

5.2.1. A useful weaker bound.

We now establish the aforementioned weaker bound, which we will ultimately use to prove Lemma 18. Intuitively, this weaker bound follows by "interpreting" $\sum_{i=1}^{k} N_i(t)$ as a type of "modified binomial" distribution, where each renewal process has "a success probability" of having had at least one event. We note that this weaker bound, and its proof, are similar to that of Lemma 9 in [30] (although in [30] the corresponding results are non-explicit). In particular, we prove the following.

LEMMA 19. *Suppose that $\mathbb{E}[S] = 1$. Then for all $k \geq 1, p \geq 2, t \in [0,1]$, and $\theta > 0$,*

$$\mathbb{E}\big[\big|\sum_{i=1}^{k} N_i(t) - kt\big|^p\big] \leq \exp(\theta)\big(\frac{10^3 p^3}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2} \max\big(kt, (kt)^p\big). \tag{43}$$

Proof We note that here it is important to correctly capture the joint scaling of k and t, so e.g. a naive application of Lemma 16 will not suffice. Instead, we proceed as follows. It follows from (28) that the left-hand-side of (43) is at most

$$\mathbb{E}\big[\big(\sum_{i=1}^{k} N_i(t) + kt\big)^p\big] \leq 2^{p-1}\big(\mathbb{E}\big[\big(\sum_{i=1}^{k} N_i(t)\big)^p\big] + (kt)^p\big). \tag{44}$$

We now bound the term $\mathbb{E}\big[\big(\sum_{i=1}^{k} N_i(t)\big)^p\big]$ appearing in (44). Let us fix some $t \in [0,1]$, and let $\{B_i, i \geq 1\}$ denote a sequence of i.i.d. Bernoulli r.v. s.t $\mathbb{P}(B_i = 1) = p_t \triangleq \mathbb{P}(R(S) \leq t)$, and $\mathbb{P}(B_i = 0) = 1 - p_t$. Note that we may construct $\{N_i(t), i \geq 1\}$, $\{N_{o,i}(t), i \geq 1\}, \{B_i, i \geq 1\}$ on the same probability space s.t. w.p.1 $N_i(t) \leq B_i\big(1 + N_{o,i}(t)\big)$ for all $i \geq 1$, with $\{N_{o,i}(t), i \geq 1\}, \{B_i, i \geq 1\}$ mutually independent. Let $M_t \triangleq \sum_{i=1}^{k} B_i$. Then it follows from Lemma 17, Corollary 9, the fact that $t \leq 1$, and Jensen's inequality that

$$\begin{aligned}
\mathbb{E}\big[\big(\sum_{i=1}^{k} N_i(t)\big)^p\big] &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^{M_t}\big(1 + N_{o,i}(t)\big)\Big)^p\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\Big(\sum_{i=1}^{M_t}\big(1 + N_{o,i}(t)\big)\Big)^p | M_t\Big]\Big] \\
&\leq \mathbb{E}\Big[(2p)^p \max\Big(\big(M_t\mathbb{E}[1 + N_o(t)]\big)^p, M_t\mathbb{E}\big[\big(1 + N_o(t)\big)^p\big]\Big)\Big] \\
&\leq (2p)^p\mathbb{E}\big[\big(1 + N_o(t)\big)^p\big]\mathbb{E}\big[\max\big(M_t^p, M_t\big)\big] \\
&= (2p)^p\mathbb{E}\big[\big(1 + N_o(t)\big)^p\big]\mathbb{E}[M_t^p] \\
&\leq (2p)^p\mathbb{E}\big[\big(1 + N_o(1)\big)^p\big]\mathbb{E}[M_t^p] \\
&\leq \exp(\theta)\big(\frac{200p^2}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2}\mathbb{E}[M_t^p].
\end{aligned}$$

Further applying Lemma 17 to conclude that

$$\mathbb{E}[M_t^p] \quad = \quad \mathbb{E}\big[\big(\sum_{i=1}^{k} B_i\big)^p\big] \quad \leq \quad (2p)^p \max\big((kp_t)^p, kp_t\big),$$

we may combine the above and find that

$$\mathbb{E}\big[\big(\sum_{i=1}^{k} N_i(t)\big)^p\big] \leq \exp(\theta)\big(\frac{400p^3}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2} \max\big(kp_t, (kp_t)^p\big).$$

Since it follows from the definition of the equilibrium distribution and p_t that $p_t \leq t$, we conclude that the left-hand-side of (43) is at most

$$2^{p-1}\Bigg(\exp(\theta)\big(\frac{400p^3}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2} \max\big(kt, (kt)^p\big) + (kt)^p\Bigg).$$

Further combining with some straightforward algebra completes the proof.

5.2.2. Proof of Lemma 18. We now use Lemma 19 to complete the proof of the desired result Lemma 18, proceeding by a case analysis. We note that a similar, albeit non-explicit, analysis appeared in [30].

Proof of Lemma 18 Let us fix some $t \in [0,1]$. We proceed by a case analysis. First, suppose $t \leq \frac{1}{k}$. In this case $\max\big(kt, (kt)^p\big) = kt$, and the desired result follows from Lemma 19.

Next, suppose $t \in (\frac{1}{k}, \frac{2}{k}]$. In this case,

$$\max\big(kt, (kt)^p\big) \quad = \quad (kt)^p \quad \leq \quad 2^{\frac{p}{2}}(kt)^{\frac{p}{2}},$$

and the result then follows from Lemma 19.

Alternatively, suppose $t \in \big(\frac{2}{k}, 1\big]$. Let $n_1(t) \triangleq \lfloor kt \rfloor$. Noting that $t \geq \frac{2}{k}$ implies $n_1(t) > 0$, in this case we may define $n_2(t) \triangleq \lfloor \frac{k}{n_1(t)} \rfloor$. Then the left-hand-side of (42) equals

$$\mathbb{E}\big[\big|\sum_{m=1}^{n_1(t)}\sum_{l=1}^{n_2(t)}\big(N_{(m-1)n_2(t)+l}(t) - t\big) + \sum_{l=n_1(t)n_2(t)+1}^{k}\big(N_l(t) - t\big)\big|^p\big]$$

$$\leq 2^{p-1}\mathbb{E}\big[\big|\sum_{m=1}^{n_1(t)}\sum_{l=1}^{n_2(t)}\big(N_{(m-1)n_2(t)+l}(t) - t\big)\big|^p\big] \tag{45}$$

$$+2^{p-1}\mathbb{E}\big[\big|\sum_{l=n_1(t)n_2(t)+1}^{k}\big(N_l(t) - t\big)\big|^p\big] \quad \text{by (28).} \tag{46}$$

We now bound (45). It follows from Lemma 16 that (45) is at most

$$(10p)^p \times \big(n_1(t)\big)^{\frac{p}{2}} \times \mathbb{E}\big[\big|\sum_{l=1}^{n_2(t)}\big(N_l(t) - t\big)\big|^p\big]$$

$$\leq (10p)^p \times \big(n_1(t)\big)^{\frac{p}{2}} \times \exp(\theta)\big(\frac{10^3 p^3}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2} \times \max\Big(tn_2(t), \big(tn_2(t)\big)^p\Big), \tag{47}$$

the final inequality following from Lemma 19. We now bound the term $tn_2(t)$ appearing in (47). In particular,

$$tn_2(t) \quad = \quad t\lfloor\frac{k}{\lfloor kt \rfloor}\rfloor \quad \leq \quad \frac{kt}{kt-1}. \tag{48}$$

But since $t \geq \frac{2}{k}$ implies $kt \geq 2$, and $g(z) \triangleq \frac{z}{z-1}$ is a decreasing function of z on $(1, \infty)$, it follows from (48) that

$$tn_2(t) \leq 2.$$

Thus $\max\Big(tn_2(t), \big(tn_2(t)\big)^p\Big) \leq 2^p$. As in addition $n_1(t) \leq kt$, it then follows from (47) that (45) is at most

$$\exp(\theta)\big(\frac{2 \times 10^4 p^4}{1 - \mathbb{E}[\exp(-\theta S)]}\big)^{p+2}(kt)^{\frac{p}{2}}. \tag{49}$$

We now bound (46). Note that the sum $\sum_{l=n_1(t)n_2(t)+1}^{k}\big(N_l(t) - t\big)$ appearing in (46) is taken over $k - n_1(t)n_2(t)$ terms. Furthermore,

$$\begin{aligned} k - n_1(t)n_2(t) &= k - n_1(t)\lfloor\frac{k}{n_1(t)}\rfloor \\ &\leq k - n_1(t)\big(\frac{k}{n_1(t)} - 1\big) \\ &= n_1(t). \end{aligned}$$

As $n_1(t) \le kt$, it thus follows from Lemma 16 that (46) is at most

$$(10p)^p \times (kt)^{\frac{p}{2}} \times \mathbb{E}\big[|N_1(t) - t|^p\big]. \tag{50}$$

Further using Lemma 19 to bound $\mathbb{E}\big[|N_1(t) - t|^p\big]$ by $\exp(\theta)\big(\frac{10^3 p^3}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{p+2}$ (since $t \le 1$), we conclude that (46) is at most

$$\exp(\theta)\Big(\frac{10^4 p^4}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{p+2}(kt)^{\frac{p}{2}}. \tag{51}$$

Using (49) to bound (45) and (51) to bound (46) completes the proof.

5.3. Proof of main result Theorem 1.

In this subsection we complete the proof of our main result Theorem 1. We proceed by using Lemmas 11 and 18 to prove that the conditions of Theorem 4 are met, in conjunction with the stochastic comparison results Theorems 2 and 3. Before completing the proof, we provide one additional auxiliary result, bounding the central moments of $\mu_A \sum_{i=1}^k A_i$. The proof follows immediately from Lemma 16, the easily verified (using (28)) fact that for all $r \ge 2$,

$$\mathbb{E}\big[|\mu_A A - 1|^r\big] \le 2^{r-1}(\mathbb{E}[A^r]\mu_A^r + 1) \le 2^r \mathbb{E}[A^r]\mu_A^r,$$

and some straightforward algebra, and we omit the details.

Lemma 20. *Suppose that $0 < \mu_A < \infty$, and $\mathbb{E}[A^{r_3}] < \infty$ for some $r_3 \ge 2$. Then for all $k \ge 1$,*

$$\mathbb{E}\Big[|\mu_A \sum_{i=1}^{k} A_i - k|^{r_3}\Big] \le (10 r_3)^{r_3} \mathbb{E}[A^{r_3}]\mu_A^{r_3} k^{\frac{r_3}{2}}.$$

We now complete the proof of our main result Theorem 1.

Proof of Theorem 1 First, we note that it suffices to prove the result for the case $\mathbb{E}[S] = 1$. Indeed, if the result is proven in that special case, one can derive the general case by simply rescaling time (i.e. multiplying both the service and inter-arrival times by μ_S). This follows from the fact that such a rescaling does not change the distribution of $Q^n(\infty)$, and only impacts the proven bounds by replacing the term $\mathbb{E}[S^r]$ by $\mathbb{E}[(S\mu_S)^r]$, leaving all other quantities unchanged (as $\mathbb{E}[(A\mu_A)^r]$ and ρ are unchanged by such a rescaling).

Thus suppose $\mathbb{E}[S] = 1$. It follows from Lemmas 11, 18, and 20 that for each integer $n' \ge 1$ s.t. $n' > \mu_A$, the conditions of Theorem 4 are met with the following parameters:

$$r_1 = r_2 = r_3 = r \quad , \quad s_1 = s_3 = \frac{r}{2},$$

$$C_1 = \mathbb{E}[S^r]\exp(\theta)\Big(\frac{10^8 r^3}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{r+2},$$

$$C_2 = \exp(\theta)\Big(\frac{10^5 r^4}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{r+2},$$
$$C_3 = (10r)^r \mathbb{E}[A^r]\mu_A^r.$$

Thus, applying Theorem 4 and some straightforward algebra (e.g. the fact that $1+C_i \le 2C_i$ for $i=1,2,3$), we find that for all $z \ge 16$, $\mathbb{P}\Bigg(\sup_{t\ge 0}\bigg(A(t)-\sum_{i=1}^{n'} N_i(t)\bigg)\ge z\Bigg)$ is at most

$$\begin{aligned} 8 \times \bigg(\frac{10^6(3r)^5}{(\frac{r}{2}-1)^2(\frac{r}{2})^2(r-2)}\bigg)^{3r+1} &\times \bigg(\mathbb{E}[S^r]\exp(\theta)\Big(\frac{10^8 r^3}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{r+2}\bigg) \\ \times \bigg(\exp(\theta)\Big(\frac{10^5 r^4}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{r+2}\bigg) &\times \bigg((10r)^r\mathbb{E}[A^r]\mu_A^r\bigg) \\ \times \Bigg(n'^{\frac{r}{2}}(n'-\mu_{A,n'})^{-\frac{r}{2}} z^{-\frac{r}{2}} & \\ + \quad n'^{\frac{r}{2}}(n'-\mu_{A,n'})^{-\frac{r}{2}} z^{-\frac{r}{2}} & \\ + \quad (n'-\mu_{A,n'})^{-\frac{r}{2}}(n')^r \mu_{A,n'}^{-\frac{r}{2}} z^{-\frac{r}{2}} \Bigg), & \end{aligned}$$

which after some further straightforward algebra (and the fact that $n' > \mu_{A,n'}$) is itself bounded by

$$\mathbb{E}[S^r]\mathbb{E}[A^r]\mu_A^r\exp(2\theta)\Big(\frac{10^{23}r^8}{(r-2)^3\big(1-\mathbb{E}[\exp(-\theta S)]\big)}\Big)^{4r} \times \Big(\frac{n'}{\mu_{A,n'}}\Big)^{\frac{r}{2}} \times n'^{\frac{r}{2}} \times \Big(n'-\mu_{A,n'}\Big)^{-\frac{r}{2}} \times z^{-\frac{r}{2}}. \tag{52}$$

Next, observe that the definition of $\mu_{A,n'}$ ensures that

$$\frac{n'}{\mu_{A,n'}} \quad \le \quad 2 \quad , \quad \text{and} \quad n'-\mu_{A,n'} \quad \ge \quad \frac{1}{2}(n'-\mu_A), \tag{53}$$

the second inequality following from the fact that if $\mu_A < \frac{1}{2}n'$, then $n'-\mu_{A,n'} = \frac{1}{2}n' \ge \frac{1}{2}(n'-\mu_A)$; while if $\mu_A \ge \frac{1}{2}n'$, then $\mu_A = \mu_{A,n'}$, and hence $\frac{1}{2}(n'-\mu_A) = \frac{1}{2}(n'-\mu_{A,n'}) \le n'-\mu_{A,n'}$. Combining (52) and (53) with some straightforward algebra, we conclude that for all $n' \ge 1$ s.t. $\rho_{n'} < 1$, all $z \ge 16$, and all $\theta > 0$, $\mathbb{P}\Bigg(\sup_{t\ge 0}\bigg(A(t)-\sum_{i=1}^{n'} N_i(t)\bigg)\ge z\Bigg)$ is at most

$$\mathbb{E}[S^r]\mathbb{E}[A^r]\mu_A^r\exp(2\theta)\Big(\frac{10^{24}r^8}{(r-2)^3\big(1-\mathbb{E}[\exp(-\theta S)]\big)}\Big)^{4r} \times \Big(z(1-\rho_{n'})\Big)^{-\frac{r}{2}}. \tag{54}$$

Further noting that $z \in (0,16)$ implies (54) is at least one (by a straightforward calculation), we conclude that for all $n' \ge 1$ s.t. $\rho_{n'} < 1$, and all $z > 0$, $\mathbb{P}\Bigg(\sup_{t\ge 0}\bigg(A(t)-\sum_{i=1}^{n'} N_i(t)\bigg)\ge z\Bigg)$ is at most (54). We next show how to get rid of the term $1-\mathbb{E}[\exp(-\theta S)]$ by an appropriate choice of θ. Note that for all $\theta > 0$, w.p.1, $\exp(\theta S) \ge 1+\theta S$, and hence

$$\begin{aligned} \exp(-\theta S) &\le \frac{1}{1+\theta S} \\ &= 1-\theta S + \frac{\theta^2 S^2}{1+\theta S} \\ &\le 1-\theta S+\theta^2 S^2. \end{aligned}$$

It follows that for all $\theta > 0$,

$$1 - \mathbb{E}[\exp(-\theta S)] \geq \theta \mathbb{E}[S] - \theta^2 \mathbb{E}[S^2]. \tag{55}$$

Taking $\theta = \frac{\mathbb{E}[S]}{2\mathbb{E}[S^2]}$ and recalling that $\mathbb{E}[S] = 1$, we find that

$$\frac{1}{1 - \mathbb{E}[\exp(-\theta S)]} \leq 4\mathbb{E}[S^2] \leq 4\big(\mathbb{E}[S^r]\big)^{\frac{2}{r}}, \tag{56}$$

the final inequality following from Jensen's inequality and the fact that $r \geq 2$. Further noting that $\mathbb{E}[S] = 1$ implies $\exp(2\theta) \leq 3$, we may combine the above with (54) to conclude that for all $n' \geq 1$ s.t. $\rho_{n'} < 1$, and all $z \geq 0$, $\mathbb{P}\Bigg(\sup_{t\geq 0}\bigg(A(t) - \sum_{i=1}^{n'} N_i(t)\bigg) \geq z\Bigg)$ is at most

$$\big(\mathbb{E}[S^r]\big)^3 \mathbb{E}[A^r]\mu_A^r \Big(10^{26} r^8 (r-2)^{-3}\Big)^{4r} \times \Big(z(1-\rho_{n'})\Big)^{-\frac{r}{2}}. \tag{57}$$

Letting $n' = n$, plugging in $z = \frac{x}{1-\rho_n}$, and applying Theorem 2 completes the proof of the first part of the theorem.

To prove the second part, let us apply Theorem 3 by plugging in $n' = n - \lfloor \frac{1}{2}(n-\mu_A) \rfloor$ and $z = \lfloor \frac{1}{2}(n-\mu_A) \rfloor$ into (57). First, let us treat the case that $\rho_n \leq 1 - \frac{4}{n}$, which implies that

$$\frac{1}{4}(n-\mu_A) \leq \lfloor \frac{1}{2}(n-\mu_A) \rfloor \leq \frac{1}{2}(n-\mu_A),$$

from which we conclude (after some straightforward algebra) that $z(1-\rho_{n'})$ is at least

$$\begin{aligned} &\big(\frac{1}{4}(n-\mu_A)\big) \times \left(1 - \frac{\mu_A}{\left(n - \left(\frac{1}{2}(n-\mu_A)\right)\right)}\right) \\ &= \quad \frac{1}{4}\frac{(n-\mu_A)^2}{n+\mu_A} \quad \geq \quad \frac{1}{8}\frac{(n-\mu_A)^2}{n} \quad = \quad \frac{n}{8}(1-\rho_n)^2. \end{aligned}$$

Combining with some straightforward algebra, Theorem 3, and (57) proves that for all $n \geq 5$ s.t. $\rho_n \leq 1 - \frac{4}{n}$, the s.s.p.d. is at most

$$\big(\mathbb{E}[S^r]\big)^3 \mathbb{E}[A^r]\mu_A^r \Big(10^{27} r^8 (r-2)^{-3}\Big)^{4r} \times \Big(n(1-\rho_n)^2\Big)^{-\frac{r}{2}}. \tag{58}$$

Noting that 1: if $n \geq 5$ and $\rho_n > 1 - \frac{4}{n}$ then (58) is at least one, and 2: if $n \leq 4$ then (58) is at least one, completes the proof of the second part of the theorem.

6. Conclusion.

In this paper, we proved the first simple and explicit bounds for general $GI/GI/n$ queues which scale gracefully and universally as $\frac{1}{1-\rho}$, assuming only existence of finite $2+\epsilon$ moments. Our results provided the first multi-server analogue of the celebrated Kingman's bound, which has been an open problem for over fifty years. More generally we were able to prove such bounds for the steady-state queue-length and the steady-state probability of delay (s.s.p.d.), where the strength of our bounds (e.g. in the form of tail decay rate) was a function of how many moments of the inter-arrival and service distributions were assumed finite. In all cases, our bounds are a simple function of only a few normalized moments, and $\frac{1}{1-\rho}$, as in the original Kingman's bound. In certain settings we were actually able to provide bounds which scale better than $\frac{1}{1-\rho}$, capturing the intuition (from e.g. the Markovian case) that the relevant expectations should grow more slowly then $\frac{1}{1-\rho}$ when the s.s.p.d. goes to zero. Importantly, our simple and explicit bounds scale gracefully even when the number of servers grows large and the traffic intensity converges to unity simultaneously, as in the Halfin-Whitt scaling regime. By applying our results to queues under the Halfin-Whitt scaling, we also derived new explicit and uniform bounds for queues in that setting.

Our results leave many interesting directions for future research. First and foremost, there is the important question of reducing the prefactor and moment-dependency in our bounds. Indeed, we primarily view our results as a proof-of-concept that such bounds are theoretically possible, and believe that future work will be able to drastically reduce the demonstrated prefactor. As noted earlier, previous work of Daley (and others) has essentially conjectured that one should only need to assume finite second moments, and that the relevant prefactor should be $\frac{1}{2}$ (as in the original Kingman's bound for a single server). In our own proofs, we opted for simplicity over tightness in essentially all cases, and as such a reasonable first pass at reducing the prefactor may come from a similar but more careful analysis (also using e.g. optimized versions of various maximal inequalities and bounds from probability which appear in our proofs). It is also plausible that a completely different analysis could yield considerably tighter bounds, and for some interesting ideas along these lines we refer the reader to e.g. [27]. If this "bridge to practicality" were acheived, the corresponding results would essentially be as powerful as Kingman's bound, but in the multi-server setting, and potentially quite impactful both in theory and practice.

Second, it would be interesting to more directly connect our bounds to other known results in the literature. For example, in certain heavy-traffic regimes, e.g. both classical and Halfin-Whitt, it is known that the tail of the steady-state queue length has (asymptotically) an exponential decay [57, 30]. This would suggest that stronger explicit bounds, exhibiting such a decay (as opposed to our demonstrated polynomial decay), may be possible. The asymptotic results of [35] similarly suggest that stronger results may be possible for the s.s.p.d. Of course, we note that the connection between asymptotic tail behavior and the associated pre-limit systems can be quite subtle, as studied in e.g. [33]. A possible first step here may be to consider service distributions with finite exponential moments, and attempting to apply maximal inequalities and related bounds which hold in that setting [64].

A related set of questions centers around how our bounds relate to the results of [78, 79], which (in many settings) give necessary and sufficient conditions for the steady-state queue length to have finite rth moment. The results of [78, 79] actually show that having more servers leads to more moments being finite in a subtle way, and e.g. shows that in the Halfin-Whitt regime, the number of moments which are finite grows as the numbers of servers diverges. Although our bounds do not reflect this phenomena (as the number of moments which our bounds imply to be finite depends only on the underlying distributions and not on the number of servers), our results do speak explicitly to certain moments scaling gracefully with $\frac{1}{1-\rho}$, which does not follow from the results of [78, 79]. Understanding this disconnect, and the connection between finiteness of moments and the scaling of those moments is an interesting open question.

These questions also connect to the literature on heavy tails (e.g. the setting in which service times have infinite variance), for which essentially nothing is known in the Halfin-Whitt regime. Indeed, there may be quite subtle interactions between which moments of the service time distribution exist, which moments of the steady-state queue-length exist, and how those moments scale, where related questions have been previously studied in the single-server setting [22]. To what extent universal bounds similar to our own are possible in the heavy-tailed setting remains an interesting open question, although here additional challenges will likely arise as $\frac{1}{1-\rho}$ is known to be the incorrect scaling even in the single-server case [14].

One could also attempt to extend our analysis to a more general class of queueing systems, e.g. queues with abandonments, networks of queues, etc. We note that the results of [30] also give bounds for the transient setting, and could be naturally extended to consider time-varying arrival processes, another interesting setting in which one could aim to prove bounds analogous to our own. One could also attempt to derive simple, explicit, and universal bounds for other quantities of interest in the analysis of queues, e.g. the rate of convergence to stationarity, where we refer the reader to [31] for some relevant discussion in the Markovian setting.

On a final note, and taking a broader view of the literature on queueing theory, there is the meta-question of how to conceptualize the trade-off between simplicity/explicitness, and accuracy, in approximations for multi-server queues. This question is particularly interesting in the Halfin-Whitt regime, where the inherent complexity of the weak limits that arise brings this question front and center. The following are but a few interesting questions along these lines. What is the right notion of "complexity" in queueing approximations? How should one compare analytical bounds with results derived from simulation and numerical procedures? What is the formal algorithmic complexity of both numerical computation, and simulation, for the limiting processes which arise? And last, but by no means least, which types of approximations may be most useful in practice?

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7. Appendix.

7.1. Proof of Theorem 3.

We begin by citing the relevant result of [35].

LEMMA 21 (**[35] Theorem 4**). *Under the same assumptions as Theorem 2, for all* $x \geq 0$, $\mathbb{P}\big(Q^n(\infty) \geq x\big)$ *is at most*

$$\inf_{\substack{\delta \geq 0 \\ \eta \in [0,n]}} \mathbb{P}\Bigg(\max\Big(\sup_{0 \leq t \leq \delta} \big(A(t) - \sum_{i=1}^{\eta} N_i(t)\big) \ , \ \sup_{t \geq \delta} \big(A(t) - \sum_{i=1}^{n} N_i(t)\big) + \sum_{i=\eta+1}^{n} N_i(\delta) \Big) + \sum_{i=\eta+1}^{n} I(N_i(\delta) = 0) \ \geq \ x - \eta \Bigg).$$

With Lemma 21 in hand, we now complete the proof of Theorem 3.

Proof of Theorem 3 First, let us prove that $n\mu_S > \mu_A$ implies

$$\Big(n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor\Big)\mu_S > \mu_A. \tag{59}$$

Indeed,

$$\begin{aligned}\Big(n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor\Big)\mu_S &\geq \Big(n - \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\Big)\mu_S \\ &= \frac{1}{2}(n\mu_S + \mu_A) \quad > \quad \mu_A.\end{aligned}$$

Next, taking $x = n, \eta = n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor$, we conclude from Lemma 21 that $\mathbb{P}\big(Q^n(\infty) \geq n\big)$ is at most

$$\inf_{\delta \geq 0} \mathbb{P}\Bigg(\max\Big(\sup_{0 \leq t \leq \delta} \big(A(t) - \sum_{i=1}^{n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor} N_i(t)\big) \ , \ \sup_{t \geq \delta} \big(A(t) - \sum_{i=1}^{n} N_i(t)\big) + \sum_{i=n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor + 1}^{n} N_i(\delta) \Big) + \sum_{i=n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor + 1}^{n} I(N_i(\delta) = 0) \ \geq \ \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor \Bigg).$$

Applying monotonicity and a union bound, we further find that for all $\epsilon, T > 0$, $\mathbb{P}\big(Q^n(\infty) \geq n\big)$ is at most

$$\mathbb{P}\Bigg(\sup_{t \geq 0} \big(A(t) - \sum_{i=1}^{n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor} N_i(t)\big) \geq \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor - \epsilon \Bigg) \tag{60}$$

$$+ \ \mathbb{P}\Bigg(\sup_{t \geq T} \big(A(t) - \sum_{i=1}^{n} N_i(t)\big) + \sum_{i=n - \lfloor \frac{1}{2}(n - \frac{\mu_A}{\mu_S})\rfloor + 1}^{n} N_i(T) \geq 0 \Bigg) \tag{61}$$

$$+ \ \mathbb{P}\Bigg(\sum_{i=1}^{n} I(N_i(T) = 0) \geq \epsilon \Bigg). \tag{62}$$

We next bound (61), which by stationary increments and another union bound is at most

$$\mathbb{P}\Bigg(A(T) - \sum_{i=1}^{n-\lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor} N_i(T) \geq -T^{\frac{1}{2}}\Bigg) + \mathbb{P}\Bigg(\sup_{t\geq 0}\big(A(t) - \sum_{i=1}^{n} N_i(t)\big) \geq T^{\frac{1}{2}}\Bigg).$$

It follows from the well-known Strong law of large numbers for renewal processes, and (59), that

$$\lim_{T\to\infty}\mathbb{P}\Bigg(A(T) - \sum_{i=1}^{n-\lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor} N_i(T) \geq -T^{\frac{1}{2}}\Bigg) = 0 \quad , \quad \lim_{T\to\infty}\mathbb{P}\Bigg(\sup_{t\geq 0}\big(A(t) - \sum_{i=1}^{n} N_i(t)\big) \geq T^{\frac{1}{2}}\Bigg) = 0,$$

from which we conclude that

$$\lim_{T\to\infty}\mathbb{P}\Bigg(\sup_{t\geq T}\big(A(t) - \sum_{i=1}^{n} N_i(t)\big) + \sum_{i=n-\lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor+1}^{n} N_i(T) \geq 0\Bigg) = 0. \tag{63}$$

Furthermore, for all $\epsilon > 0$, it trivially holds that

$$\lim_{T\to\infty}\mathbb{P}\Bigg(\sum_{i=1}^{n} I(N_i(T) = 0) \geq \epsilon\Bigg) = 0. \tag{64}$$

Combining the above, we conclude that for all $\epsilon > 0$ (by taking the limit $T\to\infty$ above), $\mathbb{P}\big(Q^n(\infty) \geq n\big)$ is at most

$$\mathbb{P}\Bigg(\sup_{t\geq 0}\big(A(t) - \sum_{i=1}^{n-\lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor} N_i(t)\big) \geq \lfloor\frac{1}{2}(n-\frac{\mu_A}{\mu_S})\rfloor - \epsilon\Bigg). \tag{65}$$

Letting $\epsilon \downarrow 0$ and applying continuity completes the proof.

7.2. Proof of Lemma 2.

We begin by citing the relevant result of [59].

Lemma 22 (**[59] Theorem 2**). *Let $\{X_l, 1\leq l\leq L\}$ be a completely general sequence of r.v.s. Suppose there exist $\nu > 0$, $\gamma > 1$, and $C > 0$ such that for all $\lambda > 0$ and non-negative integers $1\leq i\leq j\leq L$, it holds that*

$$\mathbb{P}\big(|\sum_{k=i}^{j} X_k| \geq \lambda\big) \leq \big(C(j-i+1)\big)^{\gamma}\lambda^{-\nu}.$$

Then it must also hold that

$$\mathbb{P}\big(\max_{i\in[1,L]}|\sum_{k=1}^{i} X_k| \geq \lambda\big) \leq 2^{\gamma}\Big(1 + \big(2^{-\frac{1}{\nu+1}} - 2^{-\frac{\gamma}{\nu+1}}\big)^{-(\nu+1)}\Big)(CL)^{\gamma}\lambda^{-\nu}.$$

With Lemma 22 in hand, we now complete the proof of Lemma 2.

Proof of Lemma 2 As the conditions of Lemma 22 and Lemma 2 are identical, it suffices to prove that $\nu \geq \gamma$ (along with the other assumptions of Lemma 22) implies

$$2^{\gamma}\left(1+\left(2^{-\frac{1}{\nu+1}}-2^{-\frac{\gamma}{\nu+1}}\right)^{-(\nu+1)}\right) \leq \left(6\frac{\nu+1}{\gamma-1}\right)^{\nu+1}. \tag{66}$$

Note that

$$2^{-\frac{1}{\nu+1}}-2^{-\frac{\gamma}{\nu+1}} = 2^{-\frac{1}{\nu+1}}\left(1-2^{-\frac{\gamma-1}{\nu+1}}\right). \tag{67}$$

As our assumptions imply $0<\frac{\gamma-1}{\nu+1}<1$, and it is easily verified that $1-2^{-z}\geq \frac{z}{2}$ for all $z\in[0,1]$, we conclude that

$$\left(1-2^{-\frac{\gamma-1}{\nu+1}}\right)^{-(\nu+1)} \leq \left(2\frac{\nu+1}{\gamma-1}\right)^{\nu+1}. \tag{68}$$

Combining (67) and (68) with the fact that (by our assumptions) $(\frac{\nu+1}{\gamma-1})^{\nu+1}\geq 1$ and $2^{\gamma}\leq 2^{\nu}$, it follows that the left-hand-side of (66) is at most

$$\begin{aligned} & 2^{\gamma}\left(1+2\left(2\frac{\nu+1}{\gamma-1}\right)^{\nu+1}\right). \\ \leq\ & 3\times 2^{\gamma}\times\left(2\frac{\nu+1}{\gamma-1}\right)^{\nu+1} \\ \leq\ & 6\times 4^{\nu}\times\left(\frac{\nu+1}{\gamma-1}\right)^{\nu+1} \\ \leq\ & \left(6\frac{\nu+1}{\gamma-1}\right)^{\nu+1}, \end{aligned}$$

completing the proof.

7.3. Proof of Lemma 5.

Proof of Lemma 5 Note that for $\lambda>0$, $\mathbb{P}\left(\sup_{t\geq 0}\left(\phi(t)-\nu t\right)\geq\lambda\right)$ equals

$$\mathbb{P}\left(\left(\bigcup_{k=0}^{\infty}\{\phi(t)-\nu t\geq\lambda \text{ for some } t\in[2^k,2^{k+1}]\}\right)\bigcup\left\{\phi(t)-\nu t\geq\lambda \text{ for some } t\in[0,1]\right\}\right)$$

$$\leq \sum_{k=0}^{\infty}\mathbb{P}\left(\sup_{t\in[2^k,2^{k+1}]}\left(\phi(t)-\nu t\right)\geq\lambda\right) \tag{69}$$

$$+\ \mathbb{P}\left(\sup_{t\in[0,1]}\left(\phi(t)-\nu t\right)\geq\lambda\right). \tag{70}$$

We now bound (69), and proceed by bounding (for each $k\geq 0$)

$$\mathbb{P}\left(\sup_{t\in[2^k,2^{k+1}]}\left(\phi(t)-\nu t\right)\geq\lambda\right). \tag{71}$$

Since $t\in[2^k,2^{k+1}]$ implies $\nu t\geq \nu 2^k$, we conclude that (71) is at most

$$\mathbb{P}\left(\sup_{t\in[2^k,2^{k+1}]}\phi(t)\geq\lambda+\nu 2^k\right),$$

which by adding and subtracting $\phi(2^k)$, and applying stationary increments and a union bound, is at most

$$\begin{aligned}
&\mathbb{P}\bigg(\big(\sup_{t\in[2^k,2^{k+1}]}\phi(t)-\phi(2^k)\big)+\phi(2^k)\geq\lambda+\nu 2^k\bigg)\\
\leq\ &\mathbb{P}\bigg(\sup_{t\in[2^k,2^{k+1}]}\phi(t)-\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)+\mathbb{P}\bigg(\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)\\
=\ &\mathbb{P}\bigg(\sup_{t\in[0,2^k]}\phi(t)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)+\mathbb{P}\bigg(\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)\\
\leq\ &2\mathbb{P}\bigg(\sup_{t\in[0,2^k]}\phi(t)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg). \qquad (72)
\end{aligned}$$

We proceed to bound (72) by breaking the supremum into two parts, one part taken over integer points, one part taken over intervals of length one corresponding to the regions between these integer points. In particular, the assumptions of the lemma, combined with a union bound and stationary increments, ensure that

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{t\in[0,2^k]}\phi(t)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)\\
\leq\ &\mathbb{P}\bigg(\sup_{j\in\{0,\dots,2^k\}}\phi(j)+\sup_{\substack{j\in\{0,\dots,2^k-1\}\\ t\in[0,1]}}\big(\phi(j+t)-\phi(j)\big)\geq\frac{1}{2}(\lambda+\nu 2^k)\bigg)\\
\leq\ &\mathbb{P}\bigg(\sup_{j\in\{0,\dots,2^k\}}\phi(j)\geq\frac{1}{4}(\lambda+\nu 2^k)\bigg)+2^k\mathbb{P}\bigg(\sup_{t\in[0,1]}\phi(t)\geq\frac{1}{4}(\lambda+\nu 2^k)\bigg)\\
\leq\ &\frac{H_1 4^{r_1}2^{ks}}{(\lambda+\nu 2^k)^{r_1}}+\frac{H_2 4^{r_2}2^k}{(\lambda+\nu 2^k)^{r_2}}, \qquad (73)
\end{aligned}$$

where the final inequality is applicable since $\lambda\geq 4Z$ implies $\frac{1}{4}(\lambda+\nu 2^k)\geq Z$, in which case the inequality follows from our assumptions. Combining (72) and (73), we conclude that (69) is at most

$$2\sum_{k=0}^{\infty}\frac{H_1 4^{r_1}2^{ks}}{(\lambda+\nu 2^k)^{r_1}}+2\sum_{k=0}^{\infty}\frac{H_2 4^{r_2}2^k}{(\lambda+\nu 2^k)^{r_2}}. \qquad (74)$$

We now treat two cases. First, suppose $\lambda>\nu$. Then (74) is at most

$$\begin{aligned}
&2H_1 4^{r_1}\sum_{k=0}^{\lceil\log_2(\frac{\lambda}{\nu})\rceil-1}\frac{2^{ks}}{\lambda^{r_1}}\\
+\ &2H_2 4^{r_2}\sum_{k=0}^{\lceil\log_2(\frac{\lambda}{\nu})\rceil-1}\frac{2^{k}}{\lambda^{r_2}}\\
+\ &2H_1 4^{r_1}\sum_{k=\lceil\log_2(\frac{\lambda}{\nu})\rceil}^{\infty}\frac{2^{-(r_1-s)k}}{\nu^{r_1}}\\
+\ &2H_2 4^{r_2}\sum_{k=\lceil\log_2(\frac{\lambda}{\nu})\rceil}^{\infty}\frac{2^{-(r_2-1)k}}{\nu^{r_2}}
\end{aligned}$$

$$
\begin{aligned}
&= 2H_1 4^{r_1}\lambda^{-r_1}\frac{2^{\lceil \log_2(\frac{\lambda}{\nu})\rceil s}-1}{2^s-1}\\
&\quad + 2H_2 4^{r_2}\lambda^{-r_2}\big(2^{\lceil \log_2(\frac{\lambda}{\nu})\rceil}-1\big)\\
&\quad + 2H_1 4^{r_1}\nu^{-r_1}\frac{2^{-(r_1-s)\lceil \log_2(\frac{\lambda}{\nu})\rceil}}{1-2^{-(r_1-s)}}\\
&\quad + 2H_2 4^{r_2}\nu^{-r_2}\frac{2^{-(r_2-1)\lceil \log_2(\frac{\lambda}{\nu})\rceil}}{1-2^{-(r_2-1)}}\\
&\leq 4H_1 4^{r_1}\lambda^{-r_1}(\frac{\lambda}{\nu})^s\\
&\quad + 4H_2 4^{r_2}\lambda^{-r_2}\frac{\lambda}{\nu}\\
&\quad + 2H_1\big(1-2^{-(r_1-s)}\big)^{-1}4^{r_1}\nu^{-r_1}(\frac{\lambda}{\nu})^{-(r_1-s)}\\
&\quad + 2H_2\big(1-2^{-(r_2-1)}\big)^{-1}4^{r_2}\nu^{-r_2}(\frac{\lambda}{\nu})^{-(r_2-1)},
\end{aligned}
$$

with the first line of the final inequality following from the fact that $2^{\lceil \log_2(\frac{\lambda}{\nu})\rceil s}-1 \leq 2^s(\frac{\lambda}{\nu})^s$ and $2^s-1 \geq 2^{s-1}$. Combining with the fact that $r_2>2$ implies $\big(1-2^{-(r_2-1)}\big)^{-1}\leq 2$, we conclude that if $\lambda>\nu$, then (74) is at most

$$
\begin{aligned}
&6H_1\big(1-2^{-(r_1-s)}\big)^{-1}4^{r_1}\lambda^{-(r_1-s)}\nu^{-s}\\
&\quad + 8H_2 4^{r_2}\lambda^{-(r_2-1)}\nu^{-1}.
\end{aligned}
\tag{75}
$$

Combining with the fact that $\lambda>\nu>0$ and $r_2>2$ implies $\lambda^{-(r_2-1)}\nu^{-1}\leq(\lambda\nu)^{-\frac{r_2}{2}}$, we conclude that if $\lambda>\nu$, then (74) is at most

$$
\begin{aligned}
&6H_1\big(1-2^{-(r_1-s)}\big)^{-1}4^{r_1}\lambda^{-(r_1-s)}\nu^{-s}\\
&\quad + 8H_2 4^{r_2}(\lambda\nu)^{-\frac{r_2}{2}}.
\end{aligned}
\tag{76}
$$

Alternatively, suppose $\lambda\leq\nu$. Then (74) is at most

$$
\begin{aligned}
&2H_1 4^{r_1}\sum_{k=0}^{\infty}\frac{2^{-(r_1-s)k}}{\nu^{r_1}}\\
&\quad + 2H_2 4^{r_2}\sum_{k=0}^{\infty}\frac{2^{-(r_2-1)k}}{\nu^{r_2}}\\
&\leq 2H_1 4^{r_1}\nu^{-r_1}\big(1-2^{-(r_1-s)}\big)^{-1}\\
&\quad + 4H_2 4^{r_2}\nu^{-r_2}\\
&\leq 2H_1\big(1-2^{-(r_1-s)}\big)^{-1}4^{r_1}\lambda^{-(r_1-s)}\nu^{-s}\\
&\quad + 4H_2 4^{r_2}(\lambda\nu)^{-\frac{r_2}{2}},
\end{aligned}
\tag{77}
$$

the final inequality following from the fact that $\nu\geq\lambda, r_1>s, r_2>2$ implies $\nu^{-r_1}\leq\lambda^{-(r_1-s)}\nu^{-s}$, and $\nu^{-r_2}\leq(\lambda\nu)^{-\frac{r_2}{2}}$. Next, we claim that $\big(1-2^{-(r_1-s)}\big)^{-1}\leq 2(1+\frac{1}{r_1-s})$. Indeed, first, suppose $r_1-s<1$.

In this case, as it is easily verified that $1-2^{-z} \geq \frac{z}{2}$ for all $z \in (0,1)$, the result follows. Alternatively, if $r_1 - s \geq 1$, then $\big(1-2^{-(r_1-s)}\big)^{-1} \leq 2$, completing the proof. Combining with (76) and (77), and our assumptions, it follows that in all cases (74), and hence (69), is at most

$$(1+\frac{1}{r_1-s})4^{r_1+r_2+1}\bigg(H_1\nu^{-s}\lambda^{-(r_1-s)} + H_2(\lambda\nu)^{-\frac{r_2}{2}}\bigg). \tag{78}$$

We next bound (70). First, suppose $\lambda \geq \nu$. Then our assumptions (applied with $t_0 = 1$) imply that (70) is at most

$$H_2\lambda^{-r_2} \leq H_2(\lambda\nu)^{-\frac{r_2}{2}}. \tag{79}$$

Alternatively, suppose that $\lambda < \nu$. Then applying our assumptions with $t_0 = \frac{\lambda}{\nu}$, along with a union bound, we conclude that

$$\mathbb{P}\bigg(\sup_{t\in[0,1]}\big(\phi(t)-\nu t\big) \geq \lambda\bigg)$$

is at most

$$\mathbb{P}\bigg(\sup_{t\in[0,\frac{\lambda}{\nu}]}\big(\phi(t)-\nu t\big) \geq \lambda\bigg) \tag{80}$$

$$+\ \mathbb{P}\bigg(\sup_{t\in[\frac{\lambda}{\nu},1]}\big(\phi(t)-\nu t\big) \geq \lambda\bigg). \tag{81}$$

It follows from our assumptions that (80) is at most

$$\mathbb{P}\bigg(\sup_{t\in[0,\frac{\lambda}{\nu}]}\phi(t) \geq \lambda\bigg) \ \leq\ H_2(\frac{\lambda}{\nu})^{\frac{r_2}{2}}\lambda^{-r_2} \ =\ H_2(\lambda\nu)^{-\frac{r_2}{2}}. \tag{82}$$

We next bound (81), which by stationary increments, a union bound, and our assumptions is at most

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{t\in[\frac{\lambda}{\nu},1]}\big(\phi(\frac{\lambda}{\nu})+\phi(t)-\phi(\frac{\lambda}{\nu})-\nu(t-\frac{\lambda}{\nu})\big) \geq 2\lambda\bigg)\\
\leq\ & \mathbb{P}\big(\phi(\frac{\lambda}{\nu}) \geq \frac{1}{2}\lambda\big)\\
&+\ \mathbb{P}\bigg(\sup_{t\in[\frac{\lambda}{\nu},1]}\big(\phi(t)-\phi(\frac{\lambda}{\nu})-\nu(t-\frac{\lambda}{\nu})\big) \geq \frac{3}{2}\lambda\bigg)\\
=\ & \mathbb{P}\big(\phi(\frac{\lambda}{\nu}) \geq \frac{1}{2}\lambda\big)\\
&+\ \mathbb{P}\bigg(\sup_{s\in[0,1-\frac{\lambda}{\nu}]}\big(\phi(s+\frac{\lambda}{\nu})-\phi(\frac{\lambda}{\nu})-\nu s\big) \geq \frac{3}{2}\lambda\bigg)\\
\leq\ & \mathbb{P}\bigg(\sup_{t\in[0,\frac{\lambda}{\nu}]}\phi(t) \geq \frac{1}{2}\lambda\bigg)\\
&+\ \mathbb{P}\bigg(\sup_{s\in[0,1-\frac{\lambda}{\nu}]}\big(\phi(s+\frac{\lambda}{\nu})-\phi(\frac{\lambda}{\nu})-\nu s\big) \geq \frac{3}{2}\lambda\bigg)
\end{aligned}$$

$$\begin{aligned}
&\le H_2\left(\frac{\lambda}{\nu}\right)^{\frac{r_2}{2}}\left(\frac{\lambda}{2}\right)^{-r_2} \\
&\quad + \mathbb{P}\Big(\sup_{s\in[0,1-\frac{\lambda}{\nu}]}\big(\phi(s)-\nu s\big)\ge\frac{3}{2}\lambda\Big) \\
&\le 2^{r_2}H_2(\lambda\nu)^{-\frac{r_2}{2}} \qquad (83)\\
&\quad + \mathbb{P}\Big(\sup_{t\in[0,1]}\big(\phi(t)-\nu t\big)\ge\frac{3}{2}\lambda\Big). \qquad (84)
\end{aligned}$$

Let us define $f(z)\triangleq\mathbb{P}\Bigg(\sup_{t\in[0,1]}\big(\phi(t)-\nu t\big)\ge z\Bigg)$. Then using (82) to bound (80), and (83) - (84) to bound (81), we conclude that for all $z\in[2Z,\nu)$,

$$f(z)\le 2^{r_2+1}H_2(z\nu)^{-\frac{r_2}{2}}+f\left(\frac{3}{2}z\right). \qquad (85)$$

Let $j^*\triangleq\sup\{j\in Z^+\ :\ (\frac{3}{2})^j\lambda<\nu\}$. Then it follows from (85) that for all $j\in[0,j^*]$,

$$f\big((\frac{3}{2})^j\lambda\big)\le 2^{r_2+1}H_2(\lambda\nu)^{-\frac{r_2}{2}}\big((\frac{3}{2})^{\frac{r_2}{2}}\big)^{-j}+f\big((\frac{3}{2})^{j+1}\lambda\big). \qquad (86)$$

Combining (86) with a straightforward induction and our assumptions, and noting that f is a non-increasing function, we conclude that for all $\lambda\in[2Z,\nu)$,

$$\begin{aligned}
f(\lambda) &\le 2^{r_2+1}H_2(\lambda\nu)^{-\frac{r_2}{2}}\sum_{j=0}^{j^*}\big((\frac{3}{2})^{\frac{r_2}{2}}\big)^{-j}+f(\nu) \\
&\le 2^{r_2+1}H_2(\lambda\nu)^{-\frac{r_2}{2}}\sum_{j=0}^{\infty}(\frac{3}{2})^{-j}+H_2\nu^{-r_2} \\
&\le 2^{r_2+3}H_2(\lambda\nu)^{-\frac{r_2}{2}}+H_2\nu^{-r_2} \\
&\le 2^{r_2+4}H_2(\lambda\nu)^{-\frac{r_2}{2}}, \qquad (87)
\end{aligned}$$

the final inequality following from the fact that by assumption $\nu>\lambda$ and thus $\nu^{-r_2}\le(\lambda\nu)^{-\frac{r_2}{2}}$. Thus using (79) to bound (70) in the case $\lambda\ge\nu$, and (87) to bound (70) in the case $\lambda<\nu$, we conclude that in all cases (70) is at most

$$2^{r_2+4}H_2(\lambda\nu)^{-\frac{r_2}{2}}. \qquad (88)$$

Using (78) to bound (69), and (88) to bound (70), demonstrates that for all $\lambda\ge 4Z$, $\mathbb{P}\Bigg(\sup_{t\ge0}\big(\phi(t)-\nu t\big)\ge\lambda\Bigg)$ is at most

$$(1+\frac{1}{r_1-s})4^{r_1+r_2+2}\Bigg(H_1\nu^{-s}\lambda^{-(r_1-s)}+H_2(\lambda\nu)^{-\frac{r_2}{2}}\Bigg),$$

completing the proof.

7.4. Proof of Lemma 9.

We note that the proof of Lemma 9 follows quite similarly to the proof of Lemma 5, and as such whenever possible we will refer back to the proof of Lemma 5 for specific technical steps, etc.

Proof of Lemma 9 For $\lambda > 0$, $\mathbb{P}\Big(\sup_{k\geq 0}\big(\phi(k)-\nu k\big)\geq \lambda\Big)$ equals

$$\begin{aligned}&\mathbb{P}\Big(\bigcup_{k=0}^{\infty}\big\{\phi(i)-\nu i\geq\lambda \text{ for some } i\in[2^k,2^{k+1}]\big\}\Big)\\ \leq\ &\sum_{k=0}^{\infty}\mathbb{P}\Big(\sup_{i\in[2^k,2^{k+1}]}\big(\phi(i)-\nu i\big)\geq\lambda\Big). \qquad (89)\end{aligned}$$

We now bound (89), and proceed by bounding (for each $k\geq 0$)

$$\mathbb{P}\Big(\sup_{i\in[2^k,2^{k+1}]}\big(\phi(i)-\nu i\big)\geq\lambda\Big). \qquad (90)$$

Since $i\in[2^k,2^{k+1}]$ implies $\nu i\geq \nu 2^k$, we conclude that (90) is at most

$$\mathbb{P}\Big(\sup_{i\in[2^k,2^{k+1}]}\phi(i)\geq\lambda+\nu 2^k\Big),$$

which by adding and subtracting $\phi(2^k)$, and applying stationary increments and a union bound exactly as in the proof of Lemma 5, as well as the assumptions of the lemma, is at most

$$\begin{aligned}&\mathbb{P}\Big(\big(\sup_{i\in[2^k,2^{k+1}]}\phi(i)-\phi(2^k)\big)+\phi(2^k)\geq\lambda+\nu 2^k\Big)\\ \leq\ &\mathbb{P}\Big(\sup_{i\in[2^k,2^{k+1}]}\phi(i)-\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\Big)+\mathbb{P}\Big(\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\Big)\\ =\ &\mathbb{P}\Big(\sup_{i\in[1,2^k]}\phi(i)\geq\frac{1}{2}(\lambda+\nu 2^k)\Big)+\mathbb{P}\Big(\phi(2^k)\geq\frac{1}{2}(\lambda+\nu 2^k)\Big)\\ \leq\ &2\mathbb{P}\Big(\sup_{i\in[1,2^k]}\phi(i)\geq\frac{1}{2}(\lambda+\nu 2^k)\Big)\\ \leq\ &\frac{2H_3 4^{r_3}2^{ks_3}}{(\lambda+\nu 2^k)^{r_3}}.\end{aligned}$$

We conclude that (89) is at most

$$2H_3 4^{r_3}\sum_{k=0}^{\infty}\frac{2^{ks_3}}{(\lambda+\nu 2^k)^{r_3}}. \qquad (91)$$

As in the proof of Lemma 5, we now treat two cases, with each case largely mirroring the proof of Lemma 5. First, suppose $\lambda > \nu$. Then (91) is at most

$$2H_3 4^{r_3}\sum_{k=0}^{\lceil\log_2(\frac{\lambda}{\nu})\rceil-1}\frac{2^{ks_3}}{\lambda^{r_3}}+2H_3 4^{r_3}\sum_{k=\lceil\log_2(\frac{\lambda}{\nu})\rceil}^{\infty}\frac{2^{-(r_3-s_3)k}}{\nu^{r_3}}$$

$$
\begin{aligned}
&= 2H_3 4^{r_3}\left(\frac{2^{\lceil \log_2(\frac{\lambda}{\nu})\rceil s_3}-1}{2^{s_3}-1}\lambda^{-r_3}+\frac{2^{r_3}}{2^{r_3}-2^{s_3}}2^{-(r_3-s_3)\lceil \log_2(\frac{\lambda}{\nu})\rceil}\nu^{-r_3}\right)\\
&\le 8H_3 4^{r_3}\big(1-2^{-(r_3-s_3)}\big)^{-1}\nu^{-s_3}\lambda^{-(r_3-s_3)}\\
&\le 16H_3 4^{r_3}\big(1+\frac{1}{r_3-s_3}\big)\nu^{-s_3}\lambda^{-(r_3-s_3)}. \qquad (92)
\end{aligned}
$$

Alternatively, suppose $\lambda \le \nu$. Then (91) is at most

$$
\begin{aligned}
&2H_3 4^{r_3}\sum_{k=0}^{\infty}\frac{2^{-(r_3-s_3)k}}{\nu^{r_3}}\\
&= 2H_3 4^{r_3}\big(1-2^{-(r_3-s_3)}\big)^{-1}\nu^{-r_3}\\
&\le 4H_3 4^{r_3}\big(1+\frac{1}{r_3-s_3}\big)\nu^{-r_3}\\
&\le 4H_3 4^{r_3}\big(1+\frac{1}{r_3-s_3}\big)\nu^{-s_3}\lambda^{-(r_3-s_3)}. \qquad (93)
\end{aligned}
$$

Using (92) - (93) to bound (89) completes the proof.

7.5. Proof of Lemma 15.

Proof of Lemma 15 Note that for all $j \ge 1$ and $\theta > 0$,

$$
\begin{aligned}
\mathbb{P}\big(N_o(1)\ge j\big) &= \mathbb{P}\Big(\sum_{i=1}^{j} S_i \le 1\Big)\\
&= \mathbb{P}\left(\exp\Big(-\theta\sum_{i=1}^{j} S_i\Big)\ge \exp(-\theta)\right)\\
&\le \exp(\theta)\times \mathbb{E}^j[\exp(-\theta S)] \quad \text{by Markov's inequality.}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[N_o^p(1)] &= \sum_{j=0}^{\infty} j^p\,\mathbb{P}\big(N_o(1)=j\big)\\
&\le \sum_{j=0}^{\infty} j^p\,\mathbb{P}\big(N_o(1)\ge j\big)\\
&\le \exp(\theta)\times\sum_{j=0}^{\infty} j^p\big(\mathbb{E}[\exp(-\theta S)]\big)^j\\
&\le \exp(\theta)\times\left(1+\int_1^{\infty}(x+1)^p\big(\mathbb{E}[\exp(-\theta S)]\big)^x dx\right)\\
&\le \exp(\theta)\times 2^{\lceil p\rceil}\times\left(1+\int_1^{\infty}x^{\lceil p\rceil}\big(\mathbb{E}[\exp(-\theta S)]\big)^x dx\right) \quad \text{since} \quad \frac{x+1}{x}\le 2 \quad \text{for all} \quad x\ge 1\\
&\le \exp(\theta)\times 2^{\lceil p\rceil}\times\left(1+\int_0^{\infty}x^{\lceil p\rceil}\big(\mathbb{E}[\exp(-\theta S)]\big)^x dx\right)\\
&= \exp(\theta)\times 2^{\lceil p\rceil}\times\left(1+\lceil p\rceil!\log^{-\big(\lceil p\rceil+1\big)}\Big(\frac{1}{\mathbb{E}[\exp(-\theta S)]}\Big)\right)\\
&\le \exp(\theta)\times 2^{\lceil p\rceil}\times\left(1+\lceil p\rceil!\times\big(1-\mathbb{E}[\exp(-\theta S)]\big)^{-(\lceil p\rceil+1)}\right)\\
&\le \exp(\theta)\Big(\frac{24p}{1-\mathbb{E}[\exp(-\theta S)]}\Big)^{p+2},
\end{aligned}
$$

with the second-to-last inequality following from the fact that $\log(\frac{1}{x}) \geq 1-x$ for all $x \in (0,1)$, and the final inequality follows from some straightforward algebra. Combining the above completes the proof.

7.6. Proof of Corollary 10.

We note that the proof of Corollary 10 follows nearly identically to the proof of Lemma 8 of [30], although with all quantities made explicit (and using the results of Lemma 12). We include the entire proof for completeness.

Proof of Corollary 10 Let S^e denote the first renewal interval in $\mathcal{N}_1$, and f_{S^e} its density function, whose existence is guaranteed by the basic properties of the equilibrium distribution. Observe that we may construct $\mathcal{N}_1$ and $\mathcal{N}_o$ on the same probability space so that $\mathcal{N}_o$ is independent of S^e, and for all $t \geq 0$, w.p.1

$$N_1(t) - t = \Bigg(N_o\big((t-S^e)^+\big) - \big(t-S^e\big)^+\Bigg) I(S^e < t) + \Bigg(I(S^e \leq t) - \big(t - (t-S^e)^+\big)\Bigg).$$

Fixing some $t \geq 1$, it follows from (28) and the triangle inequality that $\mathbb{E}\big[|N_1(t)-t|^r\big]$ is at most

$$2^{r-1}\mathbb{E}\Bigg[\Big|N_o\big((t-S^e)^+\big) - \big(t-S^e\big)^+\Big|^r I(S^e < t)\Bigg] \tag{94}$$

$$+2^{r-1}\mathbb{E}\big[|I\big(S^e \leq t\big) - \big(t-(t-S^e)^+\big)|^r\big]. \tag{95}$$

We now bound the term $\mathbb{E}\Bigg[\Big|N_o\big((t-S^e)^+\big) - \big(t-S^e\big)^+\Big|^r I(S^e < t)\Bigg]$ appearing in (94), which equals

$$\int_0^{t-1} \mathbb{E}\big[|N_o\big(t-s\big) - \big(t-s\big)|^r\big] f_{S^e}(s)ds + \int_{t-1}^{t} \mathbb{E}\big[|N_o\big(t-s\big) - \big(t-s\big)|^r\big] f_{S^e}(s)ds. \tag{96}$$

Lemma 12 and Markov's inequality (after raising both sides to the rth power), combined with our assumptions on r and t, implies that the first summand of (96) is at most

$$\begin{aligned}
&\exp(\theta)\mathbb{E}[S^r]\big(\frac{10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1}\int_0^{t-1}(t-s)^{\frac{r}{2}} f_{S^e}(s)ds \\
\leq\ & \exp(\theta)\mathbb{E}[S^r]\big(\frac{10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1} t^{\frac{r}{2}} \int_0^{t-1} f_{S^e}(s)ds \\
\leq\ & \exp(\theta)\mathbb{E}[S^r]\big(\frac{10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+1} t^{\frac{r}{2}}.
\end{aligned}$$

Since $t-s \leq 1$ implies w.p.1 $|N_o(t-s)-(t-s)|^r \leq |N_o(1)+1|^r$, it follows from (28) and Lemma 15 that the second summand of (96) is at most

$$\begin{aligned}
&\mathbb{E}\big[|N_o(1)+1|^r\big] \times \mathbb{P}\big(S^e \in [t-1,t]\big) \\
\leq\ & 2^{r-1}\Bigg(\mathbb{E}\big[\big(N_o(1)\big)^r\big] + 1\Bigg) \\
\leq\ & 2^{r-1}\Bigg(\exp(\theta)\big(\frac{24r}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+2} + 1\Bigg) \\
\leq\ & \exp(\theta)\big(\frac{48r}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+2}.
\end{aligned}$$

Combining our bounds for (96) with some straightforward algebra, we find that (94) is at most

$$\mathbb{E}[S^r]\exp(\theta)\big(\frac{2\times 10^5 r^2}{1-\mathbb{E}[\exp(-\theta S)]}\big)^{r+2}t^{\frac{r}{2}}. \tag{97}$$

We now bound (95), which is at most

$$2^{2r-2}\bigg(1+\mathbb{E}\big[|\big(t-(t-S^e)^+\big)|^r\big]\bigg) \leq 2^{2r-2}\bigg(1+\big(\int_0^t s^r f_{S^e}(s)ds+\int_t^\infty t^r f_{S^e}(s)ds\big)\bigg). \tag{98}$$

It follows from the basic properties of the equilibrium distribution and Markov's inequality that for all $s \geq 0$,

$$f_{S^e}(s) = \mathbb{P}(S > s) \leq \mathbb{E}[S^r]s^{-r}.$$

Thus the term $\int_0^t s^r f_{S^e}(s)ds + \int_t^\infty t^r f_{S^e}(s)ds$ appearing in (98) is at most

$$\begin{aligned}\int_0^t s^r\big(\mathbb{E}[S^r]s^{-r}\big)ds + t^r\int_t^\infty \big(\mathbb{E}[S^r]s^{-r}\big)ds &= \mathbb{E}[S^r]\bigg(\int_0^t ds + t^r\int_t^\infty s^{-r}ds\bigg)\\ &= \mathbb{E}[S^r]\bigg(t + t^r(r-1)^{-1}t^{1-r}\bigg)\\ &\leq 2\mathbb{E}[S^r]t. \end{aligned}\tag{99}$$

Using (97) to bound (94), and (99) and (98) to bound (95), and combining with some straightforward algebra completes the proof.

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