

Randomized greedy algorithms for independent sets and matchings in regular graphs: Exact results and finite girth corrections.

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Abstract

We derive new results for the performance of a simple greedy algorithm for finding large independent sets and matchings in constant degree regular graphs. We show that for r -regular graphs with n nodes and girth at least g , the algorithm finds an independent set of expected cardinality $f(r)n - O\left(\frac{(r-1)^{\frac{g}{2}}}{\frac{g}{2}!}n\right)$, where $f(r)$ is a function which we explicitly compute. A similar result is established for matchings. Our results imply improved bounds for the size of the largest independent set in these graphs, and provide the first results of this type for matchings. As an implication we show that the greedy algorithm returns a nearly perfect matching when both the degree r and girth g are large. Furthermore, we show that the cardinality of independent sets and matchings produced by the greedy algorithm in *arbitrary* bounded degree graphs is concentrated around the mean. Finally, we analyze the performance of the greedy algorithm for the case of random i.i.d. weighted independent sets and matchings, and obtain a remarkably simple expression for the limiting expected values produced by the algorithm. In fact, all the other results are obtained as straightforward corollaries from the results for the weighted case.

1 Introduction

1.1 Regular graphs, independent sets, matchings and randomized greedy algorithms

An r -regular graph is a graph in which every node has degree exactly r . The girth g of a graph is the size of the smallest cycle. Let $G(g, r)$ denote the family of all r -regular graphs with girth at least g . For a graph G , we denote the set of nodes and edges by $V(G)$ and $E(G)$, respectively. A set of nodes I is defined to be an independent set if no two nodes of I are adjacent. For a graph G , let $\mathcal{I}(G)$ denote (any) maximum cardinality independent set (*MIS*) of G , and $|\mathcal{I}(G)|$ its cardinality. Throughout the paper we will drop the explicit reference to the underlying graph G when there is no ambiguity. For example we use \mathcal{I} instead of $\mathcal{I}(G)$ or V instead of $V(G)$.

Suppose the nodes of a graph are equipped with some non-negative weights $W_i, 1 \leq i \leq n \triangleq |V|$. The weight $W[I]$ of a given independent set I is the sum of the weights of the nodes in I . When the nodes of a graph are equipped with weights which are generated i.i.d. using a continuous distribution function $F(t) = \mathbb{P}(W_i \leq t)$ with non-negative support, we denote by \mathcal{I}_W the random unique with probability 1 (w.p.1) maximum weight independent set (*MWIS*) of G .

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A (partial) matching is a set of edges M in a graph G such that every node is incident to at most one edge in M . For a graph G , let \mathcal{M} denote (any) maximum cardinality matching (MM) of G . Suppose the edges of a graph are equipped with some non-negative weights $W_e, e \in E$. The weight $W[M]$ of a given matching M is the sum of the weights of the edges in M . When the edges of a graph G are equipped with weights generated i.i.d. using a continuous distribution function F with non-negative support, we denote by \mathcal{M}_W the random unique (w.p.1) maximum weight matching (MWM) of G .

In this paper we analyze the performance of a simple greedy algorithm, which we call *GREEDY*, for finding large independent sets and matchings. The description of *GREEDY* is as follows. For independent sets, *GREEDY* iteratively selects a node i uniformly at random (u.a.r) from all remaining nodes of the graph, adds i to the independent set, deletes all remaining nodes adjacent to i and repeats. Note that while the underlying graph is non-random, the independent set produced by *GREEDY* is random as it is based on randomized choices. For *MWIS*, *GREEDY* iteratively selects the node i with the greatest weight from all the remaining nodes, adds i to the independent set, deletes all the remaining nodes adjacent to i and repeats. Note that when acting on a fixed weighted graph, the action of *GREEDY* is non-random. In this setting, the randomness will come from the fact that the weighting itself is i.i.d. For matchings *GREEDY* operates similarly, except that it chooses edges instead of nodes, and deletes edges incident to the chosen edge.

Let $\mathcal{IG}(\mathcal{MG})$ denote the random independent set (matching) returned by *GREEDY* when run on an unweighted or (randomly) weighted graph G , depending on context. Denote by $W[\mathcal{IG}](W[\mathcal{MG}])$ the weight of $\mathcal{IG}(\mathcal{MG})$ (for the weighted case), and by $|\mathcal{IG}|(|\mathcal{MG}|)$ the respective cardinalities (in the unweighted case). Our goal is obtaining bounds on the expectation and variance of $|\mathcal{IG}|$, $|\mathcal{MG}|$, $W[\mathcal{IG}]$, $W[\mathcal{MG}]$, where the latter two will be considered for the case of i.i.d. continuous non-negative weight distributions. One of the motivations is to derive new lower bounds on largest independent set in constant degree regular graphs with large girth.

1.2 Summary of our results and prior work

Our main results are Theorems 1,2 which provide remarkably explicit upper and lower bounds on the expected weight of the independent set and matching produced by *GREEDY* in a regular graph of large fixed girth when the weights are generated i.i.d. from a continuous non-negative distribution. Since the gap between the upper and lower bound is of the order $\approx (r-1)^{g/2}/(g/2)!$, we also obtain the limiting expression for the weight of the independent set and matching produced by *GREEDY* in a regular graph when the girth diverges to infinity. These results are Corollaries 1,2.

As a corollary we obtain upper and lower bounds on $\mathbb{E}[|\mathcal{IG}|]$ and $\mathbb{E}[|\mathcal{MG}|]$, by considering a uniform distribution which is highly concentrated around 1. These results are stated as Theorems 4,5. Again the gap between the upper and lower bounds is of the order $\approx (r-1)^{g/2}/(g/2)!$ and we obtain a limiting expression when the girth diverges to infinity, as stated in Corollaries 3, 5. While Corollary 5 is a new result, Corollary 3 is not. This result was recently established by Lauer and Wormald [LW07] using a different approach called the ‘nibble’ method. Thus our Theorem 4 can be viewed as an explicit finite girth correction to the limiting result (Corollary 3) derived earlier in [LW07] and proved here using different methods.

Our results on the performance of the *GREEDY* algorithm, as well as the results of [LW07], are motivated by the problem of obtaining lower bounds on the size of the largest independent set in regular graphs, and specifically regular graphs with large girth. The history of this problem is long [HS82],[She83],[She91],[She95] with [LW07] and [FGS08] being the latest on the subject. In particular, the lower bounds obtained in [FGS08] are the best known for the case $r \geq 3$ and sufficiently large girth, and in this range they beat previous best bounds obtained by [LW07] (for $r \geq 7$) and

Shearer [She91] (for $r < 7$). Although these bounds are the best known as the girth diverges to infinity (for any fixed $r \geq 3$), no bounds are given in [FGS08] for fixed girth. Also, the bounds given in [LW07] for any fixed girth are very difficult to evaluate, as they are given implicitly as the solution to a large-scale optimization problem. Our bounds match those of [LW07] for any fixed r as the girth diverges to infinity, and give simple explicit bounds for any fixed girth as a finite girth correction of the order $\approx (r-1)^{g/2}/(g/2)!$. In addition, our bounds are superior for several instances discussed in [LW07] where bounds were derived numerically by lower-bounding the aforementioned large-scale optimization problem. The details of this comparison are presented in Section 6.

Our corresponding results for matchings are new, both the limiting version, Corollary 5, and the finite girth correction, Theorem 5. Interestingly, by considering the upper and lower bounds in Theorem 5 and taking a double limit $r, g \rightarrow \infty$, we find that the *GREEDY* algorithm produces a nearly perfect matching in the double limit $r, g \rightarrow \infty$. This partially answers an open problem posed by Frieze [Fri] regarding the construction of a simple, decentralized algorithm for finding a nearly perfect matching in constant degree regular graphs with large girth.

Our second set of results, Theorems 3 and 6, concerns the variance of the weight (cardinality) of the independent set and matching produced by *GREEDY* in *arbitrary* graphs with bounded degree. That is no additional assumptions on girth or regularity are adopted. We show that when the weights are i.i.d. and have finite second moment, and when the graph has bounded degree, the variance, appropriately normalized, is of the order $O(1/n)$ in both cases. We are also able to give explicit bounds in terms of the graph degree, the number of nodes, and the second moment of the weighting distribution. We also give similar results for the unweighted case. Thus the answers produced by *GREEDY* are highly concentrated around their means, and in this sense the *GREEDY* algorithm is very robust. We believe these are the first results on the variance of the *GREEDY* algorithm.

We note that since for any fixed r and d , a graph selected uniformly at random from the set of all r -regular graphs on n nodes with high probability has only a constant (independent of n) number of cycles of length $\leq d$, [LW07], such a graph will have the property that for large n most nodes have large constant-depth neighborhoods not containing any cycles. Thus our results concerning graphs with girth diverging to infinity, specifically Corollaries 1,2,3,5 can be extended to the setting of random regular graphs, since our analysis is both localized and asymptotic, only requiring that most nodes have regular trees appearing as constant size neighborhoods. However, we do not state and prove formally these results.

We now review some additional relevant literature. The *MIS*, *MWIS*, *MM* and *MWM* problems are obviously well-studied and central to the field of combinatorial optimization. The *MIS* problem is known to be NP-Complete, even for the case of cubic planar graphs [GJS76] and graphs of polynomially large girth [Mur92], and is known to be *MAX-SNP* complete even when restricted to graphs with degree at most 3 [BF94]. From both an approximation algorithm and existential standpoint, the *MIS* problem has been well-studied for bounded degree graphs [HR94], [HR97], [BF94]; graphs with large girth [MS85], [Mur92]; triangle-free graphs with a given degree sequence [AKS80], [AEKS81], [Gri83], [She83], [She91]; and large-girth graphs with a given degree sequence, including regular graphs with large girth [Bol80], [HS82], [She91], [Den94], [She95], [LW07]. We note that, as already mentioned, our Corollary 3 was derived earlier in [LW07] using different techniques.

Although the *MM* problem is solvable in polynomial time, much research has gone into finding specialized algorithms for restricted families of graphs. The most relevant graph families for which *MM* has been studied (often using *GREEDY* and related algorithms) are bounded-degree graphs, and bounded-degree graphs of girth at least 5 [DF91],[MP97]. However, there appears to be a gap in the literature for *MM* in regular graphs with large girth, barring a recent existential result that an r -regular graph with large girth g always contains a matching of size $\frac{n}{2} - O((r-1)^{-\frac{g}{2}}n)$ [FH07].

Namely, an asymptotically perfect matching exists in such graphs as the girth increases. It is of interest, however, to construct some *decentralized* and easy to implement algorithm for *MM* which leads to an asymptotically perfect matching, and our result Theorem 5 is a step towards this direction.

Our main method of proof uses the correlation decay technique, sometimes also called the local weak convergence (objective) method [Ald01], [AS03],[GNS08]. We establish that the choices made by the *GREEDY* algorithm are *asymptotically independent* for pairs of nodes (in the case of independent sets) and edges (in the case of matchings) which are far apart. That is, if two nodes i, j are at a large graph-theoretic distance, then $\mathbb{P}(i, j \in \mathcal{IG}) \approx \mathbb{P}(i \in \mathcal{IG})\mathbb{P}(j \in \mathcal{IG})$. A similar statement holds for matchings, and also for the weighted case with i.i.d. weights. This allows the reduction of the problem on a graph to the far simpler problem formulated on a regular tree, which can be solved in a very explicit way. Such an asymptotic independence was also observed in [LW07], but here we are able to characterize this decay in a more explicit manner. A similar phenomenon was also observed in [GNS08], which studied maximum weight independent sets and matchings for the case of i.i.d weights in r -regular graphs with girth diverging to infinity. There it was observed that for the case of i.i.d. exponentially distributed weights, such a decay of correlations occurs when $r = 3, 4$ and does not occur when $r \geq 5$, even as the girth diverges to infinity. Thus the techniques of [GNS08] were only able to analyze exponentially weighted independent sets in regular graphs of large girth when the degree was $r \leq 4$. In contrast we show that independent sets produced by the *GREEDY* *always* exhibits such a decay of correlations for any degree. This allows us to extend the analysis of [GNS08] to regular graphs of arbitrary constant degree. In Section 6 we will see that *GREEDY* is nearly optimal for the settings considered in [GNS08].

We now give an outline of the rest of the paper. In Section 2 we state our main results formally and show that our analysis for the case of i.i.d. weights encompasses the analysis for the unweighted case. In Section 3 we introduce the notion of an influence resistant subgraph, show that under an i.i.d. weighting most nodes (edges) will belong to such subgraphs, and show that these subgraphs determine the behavior of *GREEDY*. This enables us to prove certain locality properties of *GREEDY*, which we then apply to the setting of regular graphs of large constant girth. In Section 4 we introduce and study a bonus recursion that we will use to analyze the performance of *GREEDY* on infinite r -ary trees. Section 5 is devoted to proving results on the variance of *GREEDY*. In Section 6 we numerically evaluate our bounds and compare to earlier bounds in the literature. Finally, in Section 7 we provide directions for future work and summary remarks.

1.3 Notations and conventions

We close this section with some additional notations. Throughout the paper we consider simple undirected graphs $G = (V, E)$. Given a simple path P in a graph G , the length of P is the number of edges in P . Given two nodes $i, j \in V$, the distance $D(i, j)$ is the length of a shortest i to j path in G . Similarly, the distance $D(e_1, e_2)$ between two edges $e_1, e_2 \in E$ is the length of the shortest path in G that contains both e_1 and e_2 , minus one. Given a node $i \in V$, let the depth- d neighborhood $N_d(i)$ be the subgraph rooted at i induced by the set of nodes i' with $D(i, i') \leq d$. Given an edge e , let $N_d(e)$ denote the subgraph induced by the set of edges e' with $D(e, e') \leq d$. Specifically, for every node i and edge e , $N_0(i) = \{i\}$ and $N_0(e) = \{e\}$. For simplicity we write $N(\cdot)$ for $N_1(\cdot)$. $|N(i)|$ is the degree of the node i , and $\max_{i \in V} |N(i)|$ is defined to be the degree of the graph.

Given a rooted tree T , the depth of T is the maximum distance between the root r and any leaf, and the depth of a node i in T is $D(r, i)$. Given a node $i \in T$, the set of children of i is denoted by $C(i)$.

Suppose the nodes of an undirected graph G are equipped with weights W_i . We say that a path i_1, i_2, \dots, i_k is node increasing if $W_{i_1} < \dots < W_{i_k}$. Similarly, if the edges of G are weighted W_{ij} , we say that a path i_1, i_2, \dots, i_k is edge increasing if $W_{i_1 i_2} < \dots < W_{i_{k-1} i_k}$.

Denote by $T(r, d)$, $d \geq 1$ a depth- d tree where all non-leaf nodes have r children, and all leaves are distance d from the root. Denote by $T(r + 1, r, d)$, $d \geq 1$ the depth- d tree where the root has $r + 1$ children, all other non-leaf nodes have r children, and all leaves are distance d from the root. Note that if $G \in G(g, r)$ for some $g \geq 4$, then for every node $i \in V(G)$ and any $d \leq \lfloor \frac{g-2}{2} \rfloor$, $N_d(i)$ is (isomorphic to) $T(r, r - 1, d)$. By convention, $T(r, 0)$ and $T(r + 1, r, 0)$ both refer to a single node.

Throughout the paper we will only consider non-negative distribution functions, so the non-negativity qualification will be implicit. If X is a discrete r.v. taking values in \mathbb{Z}_+ , the corresponding probability generating function (p.g.f.) is denoted by $\phi_X(s) = \sum_{k=0}^{\infty} s^k \mathbb{P}(X = k)$. If two r.v. X and Y are equal in distribution, we write $X \stackrel{D}{=} Y$. When X is distributed according to distribution F , we will also write (with some abuse of notation) $X \stackrel{D}{=} F$. The m -fold convolution of a random variable X is denoted by $X^{(m)}$. Let W be a continuous r.v. and let X be a r.v. taking non-negative integer values. Denote by $W^{<X>}$ the r.v. $\max_{1 \leq i \leq X} W_i$ when $X > 0$ and 0 when $X = 0$. Here W_i are i.i.d. copies of W . Given two events \mathcal{A}, \mathcal{B} , let $\mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B}$ denote, respectively, the conjunction and disjunction events. Also \mathcal{A}^c denotes the complement of the event \mathcal{A} and $I(\mathcal{A})$ denotes the indicator function for the event \mathcal{A} .

2 Main results

2.1 Weighted case

The following is our main result for the performance of the *GREEDY* algorithm for finding largest weighted independent sets. Both in the context of independent sets and matchings we assume that the weights (of the nodes and edges) are generated i.i.d. from a non-negative continuous distribution F .

Theorem 1. *For every $g \geq 4$ and $r \geq 3$, and every continuous non-negative r.v. $W \stackrel{D}{=} F$ with density f and $\mathbb{E}[W] < \infty$,*

$$\begin{aligned} & \int_0^{\infty} x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{r}{r-2}} f(x) dx - \mathbb{E}[W] \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} \\ & \leq \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \leq \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \\ & \leq \int_0^{\infty} x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{r}{r-2}} f(x) dx + \mathbb{E}[W] \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \end{aligned}$$

As an immediate corollary, we obtain the following result.

Corollary 1. *For every $r \geq 3$ and every continuous non-negative r.v. $W \stackrel{D}{=} F$ with density f and $\mathbb{E}[W] < \infty$,*

$$\begin{aligned} \lim_{g \rightarrow \infty} \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] &= \lim_{g \rightarrow \infty} \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \\ &= \int_0^{\infty} x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{r}{r-2}} f(x) dx. \end{aligned} \quad (1)$$

We now present the results for matchings.

Theorem 2. For every $g \geq 4$ and $r \geq 3$, and every continuous non-negative r.v. $W \stackrel{D}{=} F$ with density f and $\mathbb{E}[W] < \infty$,

$$\begin{aligned} & \frac{r}{2} \int_0^\infty x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{2(r-1)}{r-2}} f(x) dx - \mathbb{E}[W] \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!} \\ & \leq \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{MG}]}{|V|} \right] \leq \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{MG}]}{|V|} \right] \\ & \leq \frac{r}{2} \int_0^\infty x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{2(r-1)}{r-2}} f(x) dx + \mathbb{E}[W] \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!}. \end{aligned}$$

An immediate implication is

Corollary 2. For every $r \geq 3$ and every continuous non-negative r.v. $W \stackrel{D}{=} F$ with density f and $\mathbb{E}[W] < \infty$,

$$\begin{aligned} \lim_{g \rightarrow \infty} \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{MG}]}{|V|} \right] &= \lim_{g \rightarrow \infty} \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{MG}]}{|V|} \right] \\ &= \frac{r}{2} \int_0^\infty x \left(r - 1 - (r - 2)F(x) \right)^{-\frac{2(r-1)}{r-2}} f(x) dx. \end{aligned} \quad (2)$$

We now state our main results on bounding the variance of $W[\mathcal{IG}]$ and $W[\mathcal{MG}]$.

Theorem 3. For every continuous non-negative r.v. $W \stackrel{D}{=} F$ with $\mathbb{E}[W^2] < \infty$, and for every graph G with degree $r \geq 3$,

$$\text{Var} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \leq \frac{9\mathbb{E}[W^2]r^2 e^{(r-1)^3}}{|V|}. \quad (3)$$

and

$$\text{Var} \left[\frac{W[\mathcal{MG}]}{|E|} \right] \leq \frac{33\mathbb{E}[W^2]r^2 e^{(r-1)^3}}{|E|}. \quad (4)$$

We stress that, unlike previous results, no assumption is made on the structure of the graph other than a bound on the maximum degree.

2.2 Unweighted case

As we will show in the following subsections, Theorems 1 and 2 lead to the following bounds on the cardinality of independent sets and matchings produced by *GREEDY* in regular unweighted graphs.

Theorem 4. For every $g \geq 4$ and $r \geq 3$,

$$\begin{aligned} \frac{1 - (r-1)^{-\frac{2}{r-2}}}{2} - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} &\leq \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \leq \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \\ &\leq \frac{1 - (r-1)^{-\frac{2}{r-2}}}{2} + \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \end{aligned} \quad (5)$$

The following immediate corollary is an analogue of Corollary 1 for the unweighted case.

Corollary 3. For every $r \geq 3$,

$$\begin{aligned} \lim_{g \rightarrow \infty} \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] &= \lim_{g \rightarrow \infty} \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \\ &= \frac{1 - (r-1)^{-\frac{2}{r-2}}}{2}. \end{aligned}$$

A second corollary is the following lower bound on the size of a maximum independent set in an r -regular graph with girth $\geq g$.

Corollary 4. For every $g \geq 4$ and $r \geq 3$,

$$\inf_{G \in G(g,r)} \frac{|\mathcal{I}|}{|V|} \geq \frac{1 - (r-1)^{-\frac{2}{r-2}}}{2} - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \quad (6)$$

Our results for matchings are as follows.

Theorem 5. For every $g \geq 4$ and $r \geq 3$,

$$\begin{aligned} \frac{1 - (r-1)^{-\frac{r}{r-2}}}{2} - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!} &\leq \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] \leq \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] \\ &\leq \frac{1 - (r-1)^{-\frac{r}{r-2}}}{2} + \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!}. \end{aligned} \quad (7)$$

Corollary 5. For every $r \geq 3$,

$$\begin{aligned} \lim_{g \rightarrow \infty} \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] &= \lim_{g \rightarrow \infty} \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] \\ &= \frac{1 - (r-1)^{-\frac{r}{r-2}}}{2}. \end{aligned}$$

As a result

$$\begin{aligned} \lim_{r \rightarrow \infty} \lim_{g \rightarrow \infty} \inf_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] &= \lim_{r \rightarrow \infty} \lim_{g \rightarrow \infty} \sup_{G \in G(g,r)} \mathbb{E} \left[\frac{|\mathcal{MG}|}{|V|} \right] \\ &= \frac{1}{2}. \end{aligned}$$

Namely, *GREEDY* finds a nearly perfect matching when both the degree and girth are large. A second corollary is the following lower bound on the size of a maximum matching in an r -regular graph with girth $\geq g$.

Corollary 6. For every $g \geq 4$ and $r \geq 3$,

$$\inf_{G \in G(g,r)} \frac{|\mathcal{M}|}{|V|} \geq \frac{1 - (r-1)^{-\frac{r}{r-2}}}{2} - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!}. \quad (8)$$

Bounds on the variance of $W[\mathcal{IG}]$, $W[\mathcal{MG}]$ will result in the following bounds for the variance of $|\mathcal{IG}|$, $|\mathcal{MG}|$.

Theorem 6. For every graph G with degree $r \geq 3$,

$$\text{Var}\left[\frac{|\mathcal{IG}|}{|V|}\right] \leq \frac{9r^2e^{(r-1)^3}}{|V|}. \quad (9)$$

and

$$\text{Var}\left[\frac{|\mathcal{MG}|}{|E|}\right] \leq \frac{33r^2e^{(r-1)^3}}{|E|}. \quad (10)$$

2.3 Converting the weighted case to the unweighted case

In this section, we prove that all of the results pertaining to *GREEDY*'s performance w.r.t. finding unweighted independent sets and matchings are implied by our analysis for the case of i.i.d. weights. This will allow us to focus only on the case of i.i.d. weights for the remainder of the paper.

Lemma 1. *Theorem 1 implies Theorem 4 and Theorem 2 implies Theorem 5.*

Proof. We first prove that Theorem 1 implies Theorem 4. Fix $\epsilon > 0$. Let F be a uniform distribution on $[1 - \epsilon, 1 + \epsilon]$. Applying Theorem 1 we have

$$\begin{aligned} & \int_{1-\epsilon}^{1+\epsilon} x((r-1) - (r-2)\frac{x-(1-\epsilon)}{2\epsilon})^{-\frac{r}{r-2}} \frac{1}{2\epsilon} dx - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} \\ & \leq \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \leq \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{W[\mathcal{IG}]}{|V|} \right] \\ & \leq \int_{1-\epsilon}^{1+\epsilon} x((r-1) - (r-2)\frac{x-(1-\epsilon)}{2\epsilon})^{-\frac{r}{r-2}} \frac{1}{2\epsilon} dx + \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \end{aligned}$$

Note that $(1-\epsilon)\mathbb{E}[|\mathcal{IG}|] \leq \mathbb{E}[W[\mathcal{IG}]] \leq (1+\epsilon)\mathbb{E}[|\mathcal{IG}|]$, and for $1-\epsilon \leq x \leq 1+\epsilon$ we have $((r-1) - (r-2)\frac{x-(1-\epsilon)}{2\epsilon})^{-\frac{r}{r-2}} \frac{1}{2\epsilon} \geq 0$. Thus

$$\begin{aligned} & (1-\epsilon) \int_{1-\epsilon}^{1+\epsilon} ((r-1) - (r-2)\frac{x-(1-\epsilon)}{2\epsilon})^{-\frac{r}{r-2}} \frac{1}{2\epsilon} dx - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} \\ & \leq (1+\epsilon) \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \leq (1+\epsilon) \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \\ & \leq \frac{(1+\epsilon)^2}{1-\epsilon} \int_{1-\epsilon}^{1+\epsilon} ((r-1) - (r-2)\frac{x-(1-\epsilon)}{2\epsilon})^{-\frac{r}{r-2}} \frac{1}{2\epsilon} dx + \frac{1+\epsilon}{1-\epsilon} \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \end{aligned}$$

Letting $u = \frac{x-(1-\epsilon)}{2\epsilon}$, we can apply integration by substitution to find that:

$$\begin{aligned} & (1-\epsilon) \int_0^1 ((r-1) - (r-2)u)^{-\frac{r}{r-2}} du - \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} \\ & \leq (1+\epsilon) \inf_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \leq (1+\epsilon) \sup_{G \in \mathcal{G}(g,r)} \mathbb{E} \left[\frac{|\mathcal{IG}|}{|V|} \right] \\ & \leq \frac{(1+\epsilon)^2}{1-\epsilon} \int_0^1 ((r-1) - (r-2)u)^{-\frac{r}{r-2}} du + \frac{1+\epsilon}{1-\epsilon} \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!}. \end{aligned}$$

Evaluating the integrals and letting $\epsilon \rightarrow 0$ then demonstrates the desired result. The proof that Theorem 2 implies Theorem 4 follows identically, using the bounds for matchings instead of those for independent sets. \square

Lemma 2. *Theorem 3 implies Theorem 6.*

Proof. We first prove that (3) implies (9). Let again F be a uniform distribution on $[1 - \epsilon, 1 + \epsilon]$. The following bounds are immediate

$$\begin{aligned} & (1 - \epsilon)^2 \mathbb{E}\left[\left(\frac{|\mathcal{IG}|}{|V|}\right)^2\right] - (1 + \epsilon)^2 \mathbb{E}^2\left[\frac{|\mathcal{IG}|}{|V|}\right] \\ & \leq \text{Var}\left[\frac{W[\mathcal{IG}]}{|V|}\right] \\ & \leq (1 + \epsilon)^2 \mathbb{E}\left[\left(\frac{|\mathcal{IG}|}{|V|}\right)^2\right] - (1 - \epsilon)^2 \mathbb{E}^2\left[\frac{|\mathcal{IG}|}{|V|}\right], \end{aligned}$$

which implies that

$$\left| \text{Var}\left[\frac{|\mathcal{IG}|}{|V|}\right] - \text{Var}\left[\frac{W[\mathcal{IG}]}{|V|}\right] \right| \leq (2\epsilon + \epsilon^2) \left(\mathbb{E}\left[\left(\frac{|\mathcal{IG}|}{|V|}\right)^2\right] + \mathbb{E}^2\left[\frac{|\mathcal{IG}|}{|V|}\right] \right).$$

Thus since the second moment of F is $1 + \frac{\epsilon^2}{3}$, by the triangle inequality and Theorem 3 we find that for any graph G of maximum degree r ,

$$\text{Var}\left[\frac{|\mathcal{IG}|}{|V|}\right] \leq \frac{9(1 + \frac{\epsilon^2}{3})r^2 e^{(r-1)^3}}{|V|} + (2\epsilon + \epsilon^2) \left(\mathbb{E}\left[\left(\frac{|\mathcal{IG}|}{|V|}\right)^2\right] + \mathbb{E}^2\left[\frac{|\mathcal{IG}|}{|V|}\right] \right).$$

Observing that $\left|\frac{|\mathcal{IG}|}{|V|}\right| \leq 1$, we see that (9) follows by letting $\epsilon \rightarrow 0$.

The proof of (10) from (4) is done similarly. □

3 Influence resistant subgraphs

In this section we introduce the notion of an influence resistant subgraph, and give a useful characterization of these subgraphs. We then bound the probability that a node (edge) of a bounded degree graph G is contained in (an appropriately) small influence resistant subgraph under an i.i.d. weighting from any continuous distribution function. Throughout this section we consider a graph whose nodes and edges are equipped with non-negative distinct (non-random unless otherwise stated) weights $W_i, i \in V$ and $W_e, e \in E$.

Definition 1. *A subgraph H of G is called an influence resistant subgraph (i.r.s.) if for every node (edge) $z \in H, W_z > \max_{y \in N(z) \setminus H} W_y$.*

Here $N(z) \setminus H$ means the set of nodes or edges (depending on the context) in $N(z)$ which do not belong to H . We now show that for any set of nodes (edges) Z there exists a unique minimal i.r.s. H containing Z , and give a simple characterization of this subgraph.

Lemma 3. *Given a set of nodes (edges) Z there exists a unique minimal i.r.s. H containing Z . Namely, for every other i.r.s. H' containing Z , H is a subgraph of H' . Moreover, H is characterized as the set of nodes (edges) z such that there exists a node (edge) increasing path z_1, \dots, z_k with $z_1 \in Z$ and $z_k = z$.*

We denote this unique minimal i.r.s. by $IR_G(Z)$, or $IR(Z)$ when the underlying graph G is unambiguous.

Proof. We first show that $IR_G(Z)$ is contained in every i.r.s. T containing Z . Suppose, for the purposes of contradiction, there exists an increasing path z_1, z_2, \dots, z_k such that $z_1 \in Z, z_k \notin T$. Let $l < k$ be the largest index such that $z_l \in T$. Then $z_{l+1} \in N(z_l) \setminus T$, but $W_{z_{l+1}} > W_{z_l}$, which is a contradiction to the fact that T is an i.r.s.

We now show that $IR(Z)$ is itself an i.r.s. containing Z . By definition $Z \subset IR(Z)$. Now let $z \in IR(Z)$ be arbitrary and let $z' \in N(z) \setminus IR(Z)$ be arbitrary as well. If $W_{z'} > W_z$, then since there exists an increasing path from Z to z , by appending z' to this path we obtain an increasing path from Z to z' and thus $z' \in IR(Z)$, which is a contradiction. Since the weights are distinct, we conclude $W_{z'} < W_z$, and the proof is complete. \square

We now show that the existence of a ‘small’ i.r.s. for $N(v)(N(e))$ is independent of $W_v(W_e)$ under an i.i.d. weighting.

Lemma 4. *Given an arbitrary node (edge) z , $IR(N(z)) \subset N_d(z)$ holds iff there does not exist a node (edge) increasing path between some node (edge) $z' \in N(z)$ and $z'' \in N_{d+1}(z) \setminus N_d(z)$ which is contained entirely in $G \setminus z$. As a result, if the node (edge) weights of G are generated i.i.d. from a continuous distribution F , then the event $IR(N(z)) \subset N_d(z)$ is (stochastically) independent from W_z .*

Proof. If there exists an increasing path z_1, \dots, z_k between $N(z)$ and $N_{d+1}(z) \setminus N_d(z)$, then the last element $z_k \in N_{d+1}(z) \setminus N_d(z)$ must belong to $IR(N(z))$ and thus $IR(N(z)) \subset N_d(z)$ cannot hold. Now suppose no increasing path exists between $N(z)$ and $N_{d+1}(z) \setminus N_d(z)$ inside $G \setminus z$. Then no increasing path can exist between $N(z)$ and $N_{d+1}(z) \setminus N_d(z)$ inside G either, since in any such path we can find a subpath which does not use z . This completes the proof of the first part of the lemma. The second line is an immediate implication since the random weights are distinct with probability one. \square

The usefulness of the i.r.s. comes from the following lemma, which informally states that the decisions taken by *GREEDY* inside an i.r.s. H are not affected by the complement of H in G .

Lemma 5. *Suppose H is an i.r.s. of G . Then $\mathcal{IG}(G) \cap V(H) = \mathcal{IG}(H)$ ($\mathcal{MG}(G) \cap E(H) = \mathcal{MG}(H)$), where the weights of H are induced from G .*

Proof. Let z_1, z_2, \dots, z_m be the nodes (edges) of H ordered in decreasing order by their weight. We show by induction in $k = 1, 2, \dots, m$ that $z_k \in \mathcal{IG}(G)$ ($z_k \in \mathcal{MG}(G)$) iff $z_k \in \mathcal{IG}(H)$ ($z_k \in \mathcal{MG}(H)$). For the base case $k = 1$ observe that z_1 is the heaviest element of H . Since H is an i.r.s. then also z_1 cannot have a heavier neighbor in $G \setminus H$. Thus *GREEDY* will select it both for G and H .

We now prove the induction step and assume the assertion holds for all $k' \leq k - 1 < m$. Suppose z_k was not accepted by *GREEDY* when it was operating on G . This means that *GREEDY* accepted some neighbor of z_k which was heavier than z_k and, as a result, deleted z_k . Since H is an i.r.s. this neighbor must be in H , namely it is $z_{k'}$ for some $k' < k$. By the inductive assumption *GREEDY* selected $z_{k'}$ when it was operating on H as well. Then all neighbors of $z_{k'}$ in H are deleted including z_k , and thus z_k cannot be accepted by *GREEDY* when operating on H . Similarly, suppose *GREEDY* did not select z_k when it was operating on H . Namely, *GREEDY* accepted some neighbor $z_{k'}$ of z_k with $k' < k$. By the inductive assumption the same holds for *GREEDY* operating on G : $z_{k'}$ was accepted and all neighbors, including z_k were deleted. This completes the proof of the induction step. \square

We now bound the probability that $IR(N(z))$ is contained in $N_d(z)$ when z is a node (edge) in a bounded degree graph G and the weights are random.

Lemma 6. *Let G be any graph of maximum degree $r \geq 3$, and suppose that the nodes and edges of G are equipped with i.i.d. weights from a continuous distribution F . Then for any node (edge) $i(e)$ and any $d \geq 0$,*

$$\begin{aligned}\mathbb{P}(IR(N(i)) \subset N_d(i)) &\geq 1 - \frac{r(r-1)^d}{(d+1)!}, \\ \mathbb{P}(IR(N(e)) \subset N_d(e)) &\geq 1 - \frac{2(r-1)^{d+1}}{(d+1)!},\end{aligned}$$

where the first (second) inequality is understood in the context of node (edge) weights.

Proof. Any path of (edge) length k equipped with i.i.d. node (edge) weights generated using a continuous distribution is a node (edge) increasing path with probability equal to $1/(k+1)!$ ($1/k!$). For every node $z \in G$ there exist at most $r(r-1)^d$ distinct length- d paths in $G \setminus z$ that originate on some node in $N(z) \setminus z$ and use exactly one node from $N(z)$. For every edge $z \in G$, there exist at most $2(r-1)^{d+1}$ distinct length- $(d+1)$ paths in $G \setminus z$ that originate on some edge in $N(z) \setminus z$ and use exactly one edge from $N(z)$. Observe that every node increasing path originating in $N(z) \setminus z$ and terminating in $N_{d+1}(z) \setminus N_d(z)$ must contain a length- d node increasing subpath originating in $N(z) \setminus z$ which uses exactly one node of $N(z)$. We then obtain the result by applying a simple union bound and Lemma 4. \square

We now state and prove the main result of this section.

Theorem 7. *Let $G \in G(g, r)$ for some $g \geq 4$, and $d \geq \lfloor \frac{g-2}{2} \rfloor$ be arbitrary. Let $T = T(r, r-1, d)$ have root 0. Suppose the nodes and edges of G and T are equipped with i.i.d. weights from a continuous distribution F . Then for every node $i \in V(G)$, edge $e \in E(G)$, and every child j of 0 in T*

$$\left| \mathbb{E}[W_i I(i \in \mathcal{IG}(G))] - \mathbb{E}[W_0 I(0 \in \mathcal{IG}(T))] \right| \leq \mathbb{E}[W] \frac{r(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor + 1)!} \quad (11)$$

and

$$\left| \mathbb{E}[W_e I(e \in \mathcal{MG}(G))] - \mathbb{E}[W_{0j} I((0, j) \in \mathcal{MG}(T))] \right| \leq \mathbb{E}[W] \frac{2(r-1)^{\lfloor \frac{g-2}{2} \rfloor}}{(\lfloor \frac{g-2}{2} \rfloor)!}, \quad (12)$$

where $W \stackrel{d}{=} F$. Also the limits

$$\lim_{d \rightarrow \infty} \mathbb{P}(0 \in \mathcal{IG}(T)), \quad \lim_{d \rightarrow \infty} \mathbb{P}((0, j) \in \mathcal{MG}(T)) \quad (13)$$

exist.

Remark : It is important to note that the bounds of this theorem hold for *any* value of $d \geq \lfloor \frac{g-2}{2} \rfloor$. It is this property which will ultimately lead to the existence of limits (13), as we will see shortly in the proof. Later on the existence of these limits will lead to a simple expression for the limiting value of $\mathbb{E}[W_i I(i \in \mathcal{IG}(T))]$ and $\mathbb{E}[W_e I(e \in \mathcal{MG}(T))]$.

Proof. Denote $IR(N(i))$ with respect to G by $H(i)$ and $IR(N(0))$ with respect to T by $H(0)$ for simplicity. Let $d_0 \triangleq \lfloor \frac{g-2}{2} \rfloor \leq d$. Then $N_{d_0}(i)$ is a $T(r, r-1, d_0)$ tree. We can construct a coupling in which $T = T(r, r-1, d)$ is the natural extension of this tree with additional node weights generated

independently from the node weights of G . In this setting the node i takes the role of the root 0 of T . We have

$$\begin{aligned} W_i I(i \in \mathcal{IG}(G)) &= W_i I(i \in \mathcal{IG}(G), H(i) \subset N_{d_0}(i)) + W_i I(i \in \mathcal{IG}(G), H(i) \not\subset N_{d_0}(i)) \\ &= W_0 I(0 \in \mathcal{IG}(T), H(0) \subset N_{d_0}(0)) + W_i I(i \in \mathcal{IG}(G), H(i) \not\subset N_{d_0}(i)), \end{aligned}$$

since by Lemma 5, $i \in \mathcal{IG}(G), H(i) \subset N_{d_0}(i)$ implies that

$$i \in \mathcal{IG}(H(i)) \subseteq \mathcal{IG}(N_{d_0}(i)) = \mathcal{IG}(T(r, r-1, d_0)). \quad (14)$$

This sum is upper bounded by

$$\leq W_0 I(0 \in \mathcal{IG}(T)) + W_i I(H(i) \not\subset N_{d_0}(i)).$$

It follows that

$$\begin{aligned} \mathbb{E}[W_i I(i \in \mathcal{IG}(G)) - W_0 I(0 \in \mathcal{IG}(T))] &\leq \mathbb{E}[W_i I(H(i) \not\subset N_{d_0}(i))] \\ &= \mathbb{E}[W] \mathbb{P}(H(i) \not\subset N_{d_0}(i)) \\ &\leq \mathbb{E}[W] \frac{r(r-1)^{d_0}}{(d_0+1)!}, \end{aligned}$$

where the equality follows from the second part of Lemma 4 and the last inequality follows from Lemma 6. We complete the proof of the bound (11) by establishing a similar bound with the roles of $W_i I(i \in \mathcal{IG}(G))$ and $W_0 I(0 \in \mathcal{IG}(T))$ reversed.

We now establish the last part of the theorem, namely the existence of limits (13). Consider any $d' > d$. Let $T' = T(r, r-1, d')$ be a natural extension of the tree T with the same root 0. Namely, the additional nodes of T' are weighted i.i.d. using F , independently from the weights of the nodes already in T . Let H' denote $IR(N(0))$ with respect to T' . We have

$$\begin{aligned} I(0 \in \mathcal{IG}(T)) &= I(0 \in \mathcal{IG}(T), H' \subset T) + I(0 \in \mathcal{IG}(T), H' \not\subset T) \\ &= I(0 \in \mathcal{IG}(T'), H' \subset T) + I(0 \in \mathcal{IG}(T), H' \not\subset T) \\ &\leq I(0 \in \mathcal{IG}(T')) + I(H' \not\subset T). \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{P}(0 \in \mathcal{IG}(T)) - \mathbb{P}(0 \in \mathcal{IG}(T')) &\leq \mathbb{P}(H' \not\subset T) \\ &\leq r(r-1)^d / (d+1)!, \end{aligned}$$

where the last inequality follows from Lemma 6. By reversing the roles of T and T' we obtain

$$\mathbb{P}(0 \in \mathcal{IG}(T')) - \mathbb{P}(0 \in \mathcal{IG}(T)) \leq r(r-1)^d / (d+1)!.$$

We conclude that the sequence $\mathbb{P}(0 \in \mathcal{IG}(T(r, r-1, d))), d \geq 1$ is Cauchy and therefore has a limit. This concludes the proof for the case of independent sets. The proof for the case of matchings is obtained similarly and is omitted. \square

4 Bonus, bonus recursion and proofs of the main results

4.1 Bonus and bonus recursion

In this subsection, we introduce the notion of a bonus for independent sets and matchings on trees. Consider a tree T with root 0, whose nodes (edges) are equipped with distinct positive weights $W_i, i \in T(W_{i,j}, (i, j) \in T)$.

Definition 2. For every node $i \in T$ let

$$S(i) = \begin{cases} W_i & \text{if } i \text{ is a leaf;} \\ W_i I(W_i > \max_{j \in C(i)} S(j)) & \text{otherwise;} \end{cases}$$

$$MS(i) = \begin{cases} 0 & \text{if } i \text{ is a leaf;} \\ \max_{j \in C(i)} (W_{ij} I(W_{ij} > MS(j))) & \text{otherwise;} \end{cases}$$

The quantities $S(i), MS(i)$ are called the bonus of i in the rooted tree T and will be used for the analysis of independent sets and matchings respectively. Let T_i be the subtree of T rooted at i . Note that the bonus of i depends only on the subtree T_i . To avoid ambiguity, for a subtree H of T rooted at i we let $MS_H(i)$ denote the bonus of i computed w.r.t. the subtree H . We now prove that $S(i)(MS(i))$ determines whether the root 0 belongs to $\mathcal{IG}(T)(\mathcal{MG}(T))$.

Proposition 1. Given a weighted rooted tree T with distinct positive weights on the nodes and edges, for every node i and edge (i, j) ,

1. [Independent sets]. $S(i) = W_i I(i \in \mathcal{IG}(T_i))$. Specifically, for the root 0 we obtain $S(0) = W_0 I(0 \in \mathcal{IG}(T))$.
2. [Matchings]. $MS(i) = \max_{j \in C(i)} W_{ij} I((i, j) \in \mathcal{MG}(T_i))$. Specifically, for the root 0 we obtain $MS(0) = \max_{j \in C(0)} W_{0j} I((0, j) \in \mathcal{MG}(T))$.
3. [Matchings]. For every $j \in C(i)$, $(i, j) \in \mathcal{MG}(T_i)$ iff $W_{ij} > \max(MS(j), MS_H(i))$, where H is the subgraph of T_i obtained by deleting $(i, j) \cup T_j$.

Proof. Let d be the depth of T . We first prove part 1. The proof proceeds by induction on the depth of a node, starting from nodes at depth d . Thus for the base case, suppose i belongs to level d of T , and, as a result, it is a leaf. Then $S(i) = W_i$. On the other hand, $T_i = \{i\}$ and $I(i \in \mathcal{IG}(T_i)) = 1$, and the claim follows.

For the induction part assume that the hypothesis is true for all nodes at depth $\geq k+1$ for $k \leq d-1$. Let i be some node at depth k . Observe that *GREEDY* selects node i for inclusion in $\mathcal{IG}(T_i)$ iff i is not adjacent to any nodes in T_i that are selected by *GREEDY* prior to node i being examined by *GREEDY*. The set of nodes in T_i examined by *GREEDY* before i are those nodes j such that $W_j > W_i$. Thus the event $i \in \mathcal{IG}(T_i)$ occurs iff for all $j \in C(i)$ s.t. $W_j > W_i$, we have $j \notin \mathcal{IG}(T_i)$. We claim that for each such j , $j \notin \mathcal{IG}(T_i)$ iff $j \notin \mathcal{IG}(T_j)$. Indeed, the event $j \notin \mathcal{IG}(T_i)$ is determined by a subgraph H of T_i induced by nodes with weights at least W_j . Therefore this subgraph does not include i if $W_j > W_i$. It follows that $H \cap T_j$ is disconnected from the rest of H and then the claim follows.

We conclude that $i \in \mathcal{IG}(T_i)$ iff for each $j \in C(i)$ either $W_i > W_j$, or $j \notin \mathcal{IG}(T_j)$. Combining, $i \in \mathcal{IG}(T_i)$ iff $W_i > \max_{j \in C(i)} W_j I(j \in \mathcal{IG}(T_j))$, but by the inductive hypothesis, $W_j I(j \in \mathcal{IG}(T_j)) = S(j)$. Therefore, $i \in \mathcal{IG}(T_i)$ iff $W_i > \max_{j \in C(i)} S(j)$ and the inductive assertion follows.

We now prove part 2. The proof is again by induction on the depth of a node. Base case: i is at lowest depth d and thus a leaf. In this case, $C(i) = \emptyset$, and thus $\max_{j \in C(i)} W_{ij} I(W_{ij} \in \mathcal{MG}(T_j)) = MS(i) = 0$.

For the induction step assume that the induction hypothesis is true for all nodes at depth $\geq k + 1, k \leq d - 1$. Let i be some node at depth k . If i is a leaf we use the same argument as for the base case. Thus assume i is not a leaf. Suppose $(i, j_1) \in \mathcal{MG}(T_i)$. We claim that then $W_{ij_1} > MS(j_1)$. Indeed, observe that *GREEDY* selects (i, j_1) for inclusion in $\mathcal{MG}(T_i)$ iff (i, j_1) is not itself adjacent to any edges in T_i that are selected by *GREEDY* prior to (i, j_1) being examined by *GREEDY*. Thus the event $(i, j_1) \in \mathcal{MG}(T_i)$ implies that for all $l \in C(j_1)$ s.t. $W_{j_1,l} > W_{ij_1}$, we have $(j_1, l) \notin \mathcal{MG}(T_i)$. Repeating the argument used for the case of independent sets, we claim that the event $(j_1, l) \notin \mathcal{MG}(T_i)$ occurs iff the event $(j_1, l) \notin \mathcal{MG}(T_{j_1})$ occurs. Therefore, the event $(i, j_1) \in \mathcal{MG}(T_i)$ implies that for each $l \in C(j_1)$ either $W_{ij_1} > W_{j_1,l}$ or $(j_1, l) \notin \mathcal{MG}(T_{j_1})$, namely the event $W_{ij_1} > \max_{l \in C(j_1)} W_{j_1,l} I((j_1, l) \in \mathcal{MG}(T_{j_1}))$ occurs, which by induction hypothesis is equivalent to the event $W_{ij_1} > MS(j_1)$, as claimed.

We now complete the proof of the induction step. First assume that $W_{ij} < MS(j)$ for all $j \in C(i)$. Then from the preceding claim we obtain that no edge (i, j) belongs to $\mathcal{MG}(T_i)$ and the claim is established. Otherwise, let $j_1 \in C(i)$ be such that W_{ij_1} is the largest weight among edges $W_{ij}, j \in C(i)$ satisfying $W_{ij} > MS(j)$. By the choice of j_1 and the preceding claim it follows that if $W_{ij'} > W_{ij_1}$, then $(i, j') \notin \mathcal{MG}(T_i)$. Thus it remains to show that $(i, j_1) \in \mathcal{MG}(T_i)$. *GREEDY* examines (i, j_1) after edges (i, j) with $W_{ij} > W_{ij_1}$, but before edges (i, j) with $W_{ij} < W_{ij_1}$. Since edges with $W_{ij} > W_{ij_1}$ were rejected, then whether (i, j_1) is accepted is determined completely by (i, j_1) plus the subtree $T(j_1)$. Repeating the argument above, we see that (i, j_1) is accepted iff $W_{ij_1} > \max_{l \in C(j_1)} W_{j_1,l} I((j_1, l) \in \mathcal{MG}(T_{j_1}))$, which, by the inductive hypothesis occurs iff $W_{ij_1} > MS(j_1)$, which is satisfied by the choice of j_1 .

To prove part 3, we repeat the arguments used to prove parts 1 and 2 to observe that *GREEDY* selects (i, j) iff for all neighbors l of j in $T(j)$ with $W_{j,l} > W_{ij}$, the edge (j, l) is rejected by *GREEDY* in $T(j)$, and for all neighbors $l \neq j$ of i in $T(i) \setminus ((i, j) \cup T(j))$, with $W_{il} > W_{ij}$, the edge (i, l) is rejected by *GREEDY* in $T(i) \setminus ((i, j) \cup T(j))$. \square

4.2 Distributional recursion for bonuses

We now introduce two sequences of recursively defined random variables $\{X_{d,r}\}, d \geq 0$, and $\{Y_{d,r}\}, d \geq 0$ for any given integer $r \geq 2$. These sequences will play a key role in understanding the probability distribution of the bonuses introduced in the previous subsection.

Given a positive integer k , let $B(k)$ denote a Bernoulli random variable with $\mathbb{P}(B(k) = 1) = 1/k$. Define

$$X_{d,r} \stackrel{D}{=} \begin{cases} 1 & d = 0; \\ (X_{d-1,r}^{(r)} + 1)B(X_{d-1,r}^{(r)} + 1) & d \geq 1; \end{cases} \quad (15)$$

$$Y_{d,r} \stackrel{D}{=} \begin{cases} 0 & d = 0; \\ \left((Y_{d-1,r} + 1)B(Y_{d-1,r} + 1) \right)^{(r)} & d \geq 1; \end{cases} \quad (16)$$

For an integer-valued r.v. $Z \geq 1$, the joint probability distribution of $Z, B(Z)$ is assumed to be $\mathbb{P}(Z = z, B = 1) = (1/z)\mathbb{P}(Z = z)$, i.e. $P(B = 1|Z = z) = 1/z$. Also, recall that for a r.v. $U, U^{(m)}$ is the sum of m i.i.d. copies of U .

It is immediate from these recursions that for all $d \geq 1$

$$\mathbb{E}[X_{d,r}] = 1, \quad \mathbb{E}[Y_{d,r}] = r. \quad (17)$$

In the following lemma we show that the distribution of the bonuses S and MS on regular trees have a very simple representation in terms of the constructed sequences $\{X_{d,r}\}, \{Y_{d,r}\}$.

Lemma 7. *Suppose the nodes and edges of a tree $T(r, d)$ with root 0 are equipped with i.i.d. weights generated according to a continuous distribution F . Then $S(0) \stackrel{D}{=} W^{<X_{d,r}>}$, and $MS(0) \stackrel{D}{=} W^{<Y_{d,r}>}$, where $W \stackrel{D}{=} F$ and $W^{<X>}$ is the maximum of X i.i.d. copies of W .*

Proof. We first prove the identity for $S(0)$. The proof proceeds by induction on d . For the base case, suppose $d = 0$. Then $S(0) \stackrel{D}{=} W$ and the conclusion trivially holds. For the induction step, assume the hypothesis is true for all $d' < d$. Let T_j denote the depth $d - 1$ subtree of $T(r, d)$ rooted at the j -th child of 0. By the inductive hypothesis we have that $S(j)$ is distributed as $W^{<X_{d-1,r}>}$. This implies that $\max_{j \in C(i)} S(j)$ is distributed as $(W^{<X_{d-1,r}>})^{<r>}$. Since W_0 is drawn independent of $\max_{j \in C(i)} S(j)$, we have that $S(0) \stackrel{D}{=} W_0 I(W_0 > (W^{<X_{d-1,r}>})^{<r>})$. The event underlying $I(\cdot)$ means that W_0 is the largest among $K + 1$ random variables distributed according to F , where $K \stackrel{D}{=} X_{d-1,r}^{(r)}$. The required identity then follows from the definition of $X_{d,r}$.

We now establish the identity for $MS(0)$ using induction in d . For the base case $d = 0$ we have $MS(0) = 0$ and the conclusion trivially holds. For the induction case, assume that the hypothesis is true for all $d' < d$. Let again T_j denote the depth $d - 1$ subtree of $T(r, d)$ rooted at the j -th child of 0. By the inductive hypothesis $MS(j)$ is distributed as $W^{<Y_{d-1,r}>}$. Since W_{0j} is drawn i.i.d., we have $MS(0) \stackrel{D}{=} \max_{j \in C(0)} W_{0j} I(W_{0j} > W^{<Y_{d-1,r}>})$, where $W_{0j} I(W_{0j} > W^{<Y_{d-1,r}>})$ is independent for each j . Note that for each j , $W_{0j} I(W_{0j} > W^{<Y_{d-1,r}>})$ is by definition distributed as the maximum of $(Y_{d-1,r} + 1)B(Y_{d-1,r} + 1)$ i.i.d. realizations of W . Thus $\max_{j \in C(0)} W_{0j} I(W_{0j} > W^{<Y_{d-1,r}>})$ is distributed as the maximum of r independent samples of the maximum of $(Y_{d-1,r} + 1)B(Y_{d-1,r} + 1)$ i.i.d. realizations of W , which by the basic properties of maxima is distributed as the maximum of $((Y_{d-1,r} + 1)B(Y_{d-1,r} + 1))^{(r)}$ i.i.d. realizations of W , from which the lemma follows. \square

Recall that ϕ_X denotes the probability generating function for a discrete r.v. X .

Lemma 8. *Suppose the nodes and edges of a tree $T = T(r, r - 1, d)$ with root 0 are equipped with i.i.d. weights generated from a continuous distribution F . Then*

$$\mathbb{E}[W_0 I(0 \in \mathcal{IG}(T))] = \mathbb{E}[W \phi_{X_{d-1,r-1}}^r(F(W))],$$

and for every $j \in N(0)$

$$\mathbb{E}[W_{0j} I((0, j) \in \mathcal{MG}(T))] = \mathbb{E}[W \phi_{Y_{d-1,r-1} + Y_{d,r-1}}(F(W))],$$

where W is distributed according to F , and random variables $W, X_{d-1,r-1}, Y_{d-1,r-1}, Y_{d,r-1}$ are independent.

Proof. We first prove the result for independent sets. By Proposition 1 and the definition of $S(0)$, $\mathbb{E}[W_0 I(0 \in \mathcal{IG}(T))] = E[W_0 I(W_0 > \max_{j \in C(0)} S(j))]$. By Lemma 7, for each $j \in C(0)$,

$S(j) \stackrel{D}{=} W^{<X_{d-1,r-1}>}$. It then follows that $\max_{j \in C(0)} S(j) \stackrel{D}{=} W^{<X_{d-1,r-1}^{(r)}>}$, which is independent from W_0 . Thus we have

$$\begin{aligned} \mathbb{E}[W_0 I(0 \in \mathcal{IG}(T))] &= \mathbb{E}[\mathbb{E}[W_0 I(0 \in \mathcal{IG}(T)) | W_0]] \\ &= \mathbb{E}[W_0 \mathbb{E}[I(W^{<X_{d-1,r-1}^{(r)}>} \leq W_0) | W_0]] \\ &= \mathbb{E}[W_0 \sum_{k=0}^{\infty} (F(W_0))^k \mathbb{P}(X_{d-1,r-1}^{(r)} = k)] \\ &= \mathbb{E}[W_0 \phi_{X_{d-1,r-1}^{(r)}}(F(W_0))] \\ &= \mathbb{E}[W_0 \phi_{X_{d-1,r-1}}^r(F(W_0))]. \end{aligned}$$

We now prove the result for matchings. From the third part of Proposition 1 we have

$$\mathbb{E}[W_{0j}I((0, j) \in \mathcal{IG}(T))] = \mathbb{E}[W_{0j}I(W_{0j} > \max(MS_H(0), MS_{T_j}(j)))],$$

where H is the subgraph of T obtained by deleting $(0, j)$ and T_j - the subtree of T rooted at j . Observe that H is an $r - 1$ regular tree with depth d , namely it is $T(r - 1, d)$, and T_j is an $r - 1$ regular tree with depth $d - 1$. Thus applying Lemma 7, $MS_H(0) \stackrel{D}{=} W^{<Y_{d,r-1}>}$ and $MS_{T_j}(j) \stackrel{D}{=} W^{<Y_{d-1,r-1}>}$. Repeating the line of argument used for independent sets, replacing $X_{d-1,r-1}^{(r)}$ with $Y_{d-1,r-1} + Y_{d,r-1}$, we obtain the result. \square

4.3 Limiting distribution of $X_{d,r}$ and $Y_{d,r}$

In this subsection we show that the sequences $\{X_{d,r}\}, d \geq 0$, and $\{Y_{d,r}\}, d \geq 0$ converge in distribution to some limiting random variables, by exploiting their recursive definitions. We then use this convergence along with Lemma 8 to express the quantities of interest in terms of the p.g.f. of these limiting random variables.

Lemma 9. *There exist r.v. $X_{\infty,r}, Y_{\infty,r}$ such that for all $k \geq 0$, $\lim_{d \rightarrow \infty} \mathbb{P}(X_{d,r} = k) = \mathbb{P}(X_{\infty,r} = k)$ and $\lim_{d \rightarrow \infty} \mathbb{P}(Y_{d,r} = k) = \mathbb{P}(Y_{\infty,r} = k)$.*

Proof. We begin by establishing the existence of the limit $\lim_{d \rightarrow \infty} \mathbb{P}(X_{d,r} = k)$ for $k = 0$. The case of $k \geq 1$ will be established by induction. Consider $T = T(r, d)$ with root 0 whose nodes are weighted i.i.d. with an arbitrary continuous distribution F . From Proposition 1, part 1, we have that $S(0) = 0$ iff $0 \notin \mathcal{IG}(T)$. Therefore by Lemma 7

$$\mathbb{P}(S(0) = 0) = \mathbb{P}(0 \notin \mathcal{IG}(T)) = \mathbb{P}(X_{d,r} = 0).$$

But the last quantity has a limit as $d \rightarrow \infty$ as asserted by the last part of Theorem 7.

Assume now that the limits exist for all $k' \leq k - 1$. We have

$$\mathbb{P}(X_{d,r} = k) = \frac{1}{k} \mathbb{P}(X_{d-1,r}^{(r)} = k - 1) = \frac{1}{k} \sum_{(k_1, k_2, \dots, k_r)} \prod_{1 \leq i \leq r} \mathbb{P}(X_{d-1,r} = k_i),$$

where the sum is over all partitions (k_1, k_2, \dots, k_r) with $k_i \geq 0, \sum_{1 \leq i \leq r} k_i = k - 1$. Since $k_i \leq k - 1$ for each i , by the inductive assumption the limits $\lim_{d \rightarrow \infty} \mathbb{P}(X_{d-1,r} = k_i)$ exist. The same assertion then follows for $\mathbb{P}(X_{d,r} = k)$ and the proof is complete.

Define $X_{\infty,r}$ by $\mathbb{P}(X_{\infty,r} = k) = \lim_{d \rightarrow \infty} \mathbb{P}(X_{d,r} = k)$. We need to show that $\sum_k \mathbb{P}(X_{\infty,r} = k) = 1$. Fix $\epsilon > 0$ and $K > 1/\epsilon$. Applying Markov's inequality to (17) we have $1 \geq \sum_{0 \leq k \leq K} \mathbb{P}(X_{d,r} = k) \geq 1 - 1/K > 1 - \epsilon$. Then the same applies to the limits as $d \rightarrow \infty$. The assertion then follows.

The proof for the matching case is similar. \square

The recursion properties (15) which are used to define $\{X_{d,r}\}, \{Y_{d,r}\}$ carry on to $X_{\infty,r}, Y_{\infty,r}$, which, as a result, satisfy recursive distributional equations.

Lemma 10. *The following equality in distribution takes place*

$$\begin{aligned} X_{\infty,r} &\stackrel{D}{=} (X_{\infty,r}^{(r)} + 1)B(X_{\infty,r}^{(r)} + 1), \\ Y_{\infty,r} &\stackrel{D}{=} \left((Y_{\infty,r} + 1)B(Y_{\infty,r} + 1) \right)^{(r)}. \end{aligned}$$

Proof. Applying Lemma 9, for each $k > 0$,

$$\begin{aligned}\mathbb{P}(X_{\infty,r} = k) &= \lim_{d \rightarrow \infty} \mathbb{P}(X_{d,r} = k) \\ &= \lim_{d \rightarrow \infty} \frac{1}{k} \sum_{(k_1, \dots, k_r)} \prod_{1 \leq l \leq r} \mathbb{P}(X_{d-1,r} = k_l) \\ &= \frac{1}{k} \sum_{(k_1, \dots, k_r)} \prod_{1 \leq l \leq r} \mathbb{P}(X_{\infty,r} = k_l),\end{aligned}$$

where the sums are over all partitions (k_1, \dots, k_r) , $k_l \geq 0$, $\sum_{1 \leq l \leq r} k_l = k - 1$. But the last expression is exactly the probability that $(X_{\infty,r}^{(r)} + 1)B(X_{\infty,r}^{(r)} + 1)$ takes value k . The assertion then follows. A similar argument shows the identity for $Y_{\infty,r}$. \square

4.4 Solving for the distribution of $X_{d,r}$ and $Y_{d,r}$

We now show that $\phi_{X_{\infty,r}}(s)$ and $\phi_{Y_{\infty,r}}(s)$ have a very simple explicit form. We first show that they satisfy simple differential equations.

Lemma 11. *For every $s \in [0, 1)$*

$$\begin{aligned}\frac{d}{ds} \phi_{X_{\infty,r}}(s) &= \phi_{X_{\infty,r}}^r(s), \\ \frac{d}{ds} \phi_{Y_{\infty,r}}^{\frac{1}{r}}(s) &= \phi_{Y_{\infty,r}}(s).\end{aligned}$$

Proof. We first prove the identity for $X_{\infty,r}$. Applying Lemma 10,

$$\phi_{X_{\infty,r}}(s) = \mathbb{P}(X_{\infty,r} = 0) + \sum_{k=0}^{\infty} \frac{1}{k+1} s^{k+1} \mathbb{P}(X_{\infty,r}^{(r)} = k).$$

Thus since the p.g.f. of any non-negative integer-valued r.v. is differentiable on $[0, 1)$, and can be differentiated term-by-term, we obtain

$$\begin{aligned}\frac{d}{ds} \phi_{X_{\infty,r}}(s) &= \frac{d}{ds} \sum_{k=0}^{\infty} \frac{1}{k+1} s^{k+1} \mathbb{P}(X_{\infty,r}^{(r)} = k) \\ &= \sum_{k=0}^{\infty} s^k \mathbb{P}(X_{\infty,r}^{(r)} = k) \\ &= \phi_{X_{\infty,r}}^r(s).\end{aligned}$$

As for for $Y_{\infty,r}$ we have from Lemma 10 that $\phi_{Y_{\infty,r}}^{\frac{1}{r}}(s)$ is equal to the p.g.f. of $(Y_{\infty,r} + 1)B(Y_{\infty,r} + 1)$. Therefore

$$\begin{aligned}\frac{d}{ds} \phi_{Y_{\infty,r}}^{\frac{1}{r}}(s) &= \sum_{k=0}^{\infty} \frac{d}{ds} \frac{1}{k+1} s^{k+1} \mathbb{P}(Y_{\infty,r} = k) \\ &= \phi_{Y_{\infty,r}}(s).\end{aligned}$$

\square

We now solve for the p.g.f. of $X_{\infty,r}, Y_{\infty,r}$.

Proposition 2. For every $s \in [0, 1]$,

$$\begin{aligned}\phi_{X_{\infty,r}}(s) &= (r - (r-1)s)^{-\frac{1}{r-1}} \\ \phi_{Y_{\infty,r}}(s) &= (r - (r-1)s)^{-\frac{r}{r-1}}.\end{aligned}$$

Proof. Applying the chain rule and Lemma 11

$$\begin{aligned}\frac{d}{ds}\phi_{\infty,r}^{-(r-1)}(s) &= -(r-1)(\phi_{X_{\infty,r}}(s))^{-r}\frac{d}{ds}\phi_{X_{\infty,r}}(s) \\ &= -(r-1)(\phi_{X_{\infty,r}}(s))^{-r}\phi_{X_{\infty,r}}^r(s) \\ &= -(r-1).\end{aligned}$$

We conclude that $\phi_{X_{\infty,r}}(s) = (c - (r-1)s)^{-\frac{1}{r-1}}$ for some constant c for all $s \in [0, 1]$. Since $\lim_{s \uparrow 1} \phi_{X_{\infty,r}}(s) = \phi_{X_{\infty,r}}(1) = 1$, we conclude that $c = r$ and therefore $\phi_{X_{\infty,r}}(s) = (r - (r-1)s)^{-\frac{1}{r-1}}$ and the required identity is established.

Similarly, we find

$$\begin{aligned}\frac{d}{ds}(\phi_{Y_{\infty,r}}(s))^{-\frac{r-1}{r}} &= \frac{d}{ds}\left(\phi_{Y_{\infty,r}}^{\frac{1}{r}}(s)\right)^{-(r-1)} \\ &= -(r-1)\left(\phi_{Y_{\infty,r}}^{\frac{1}{r}}(s)\right)^{-r}\frac{d}{ds}\phi_{Y_{\infty,r}}^{\frac{1}{r}}(s) \\ &= -(r-1)(\phi_{Y_{\infty,r}}(s))^{-1}\phi_{Y_{\infty,r}}(s) \\ &= -(r-1).\end{aligned}$$

Using this and $\phi_{Y_{\infty,r}}(1) = 1$ the required identity is established. \square

4.5 Proofs of Theorems 1 and 2

We now have all the necessary results to complete the proofs of our main theorems.

Proof of Theorem 1. Applying the last part of Lemma 9 and the Dominated Convergence Theorem (see [Dur96]) we have that if $W \stackrel{D}{=} F$, then

$$\lim_{d \rightarrow \infty} \mathbb{E}[W \phi_{X_{d-1,r-1}}^r(F(W))] = \mathbb{E}[W \phi_{X_{\infty,r-1}}^r(F(W))].$$

Here W serves as a dominating random variable. Applying Proposition 2 the right-hand side of this expression equals

$$\mathbb{E}[W(r-1 - (r-2)F(W))^{-\frac{r}{r-2}}] = \int_0^{\infty} x \left((r-1) - (r-2)F(x) \right)^{-\frac{r}{r-2}} f(x) dx.$$

Now observe that $\mathbb{E}[W[\mathcal{IG}]] = \sum_{i \in V(G)} \mathbb{E}[W_i I(i \in \mathcal{IG})]$. Applying part (11) of Theorem 7, Lemma 8, and letting $d \rightarrow \infty$ we obtain the result. \square

Proof of Theorem 2. Observe that $\mathbb{E}[W[\mathcal{MG}]] = \sum_{e \in E(G)} \mathbb{E}[W_e I(e \in \mathcal{MG})]$. The rest of the proof is similar to the case for independent sets. \square

5 The variance of *GREEDY*

In this section we prove our main results on the variance of *GREEDY*.

Proof of Theorem 3. Since $W[\mathcal{IG}] = \sum_{i \in V} W_i I(i \in \mathcal{IG})$, we have

$$\begin{aligned}
\text{Var}(W[\mathcal{IG}]) &= \sum_{i,j \in V} \left(\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG})] \mathbb{E}[W_j I(j \in \mathcal{IG})] \right) \\
&= \sum_{i \in V} \left(\mathbb{E}[W_i^2 I(i \in \mathcal{IG})] - (\mathbb{E}[W_i I(i \in \mathcal{IG})])^2 \right) \\
&\quad + \sum_{i \in V} \sum_{d \geq 0} \sum_{j \in N_{d+1}(i) \setminus N_d(i)} \left(\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG})] \mathbb{E}[W_j I(j \in \mathcal{IG})] \right) \\
&\leq n \mathbb{E}[W^2] \\
&\quad + \sum_{i \in V} \sum_{d \geq 0} \sum_{j \in N_{d+1}(i) \setminus N_d(i)} \left(\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG})] \mathbb{E}[W_j I(j \in \mathcal{IG})] \right). \tag{18}
\end{aligned}$$

Our proof approach is to show that the terms in parenthesis are sufficiently close to each other, provided that the distance between nodes i and j is sufficiently large.

Fix an arbitrary $i \in V$ and $j \in N_{d+1}(i) \setminus N_d(i)$ for $d \geq 2$. Recall the notion of the influence resistant subgraph from Section 3. Denote $IR(N(i))$ and $IR(N(j))$ by H_i and H_j for short. Let $l = \lfloor d/2 \rfloor - 1$. Consider the event $\mathcal{E} \triangleq (H_i \subset N_l(i) \wedge H_j \subset N_l(j))^c$.

We have

$$\begin{aligned}
\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] &= \mathbb{E}[W_i W_j I(i, j \in \mathcal{IG}, H_i \subset N_l(i), H_j \subset N_l(j))] \\
&\quad + \mathbb{E}[W_i W_j I(i, j \in \mathcal{IG}) I(\mathcal{E})]. \tag{19}
\end{aligned}$$

We first analyze the second summand.

$$\begin{aligned}
\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG}) I(\mathcal{E})] &\leq \mathbb{E}[W_i W_j I(\mathcal{E})] \\
&\leq \mathbb{E}[W_i W_j (1 - I(H_i \subset N_l(i)))] + \mathbb{E}[W_i W_j (1 - I(H_j \subset N_l(j)))] \\
&= \mathbb{E}[W_i (1 - I(H_i \subset N_l(i)))] \mathbb{E}[W_j] + \mathbb{E}[W_j (1 - I(H_j \subset N_l(j)))] \mathbb{E}[W_i]
\end{aligned}$$

where the equality holds since both W_i and the event $H_i \subset N_l(i)$ depend only on the weight configuration inside $N_{l+1}(i)$ which does not contain node j and vice versa. Next, applying the second part of Lemma 4 and Lemma 6 we have

$$\begin{aligned}
\mathbb{E}[W_i (1 - I(H_i \subset N_l(i)))] &= \mathbb{E}[W_i] (1 - \mathbb{P}(H_i \subset N_l(i))) \\
&\leq \mathbb{E}[W_i] r (r - 1)^l / (l + 1)!. \tag{20}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathbb{E}[W_i W_j I(i, j \in \mathcal{IG}) I(\mathcal{E})] &\leq 2 \mathbb{E}[W_i] \mathbb{E}[W_j] r (r - 1)^l / (l + 1)! \\
&= 2 \mathbb{E}[W]^2 r (r - 1)^l / (l + 1)! \tag{21}
\end{aligned}$$

We now analyze the first summand in (19). Let $\hat{H}_i = H_i \cap N_l(i)$, $\hat{H}_j = H_j \cap N_l(j)$. Namely, \hat{H}_i and \hat{H}_j are the subgraphs of $N_l(i)$ and $N_l(j)$ induced by nodes $V(H_i) \cap V(N_l(i))$ and $V(H_j) \cap V(N_l(j))$, respectively. Observe that the random variables $W_i I(i \in \mathcal{IG}(\hat{H}_i))$, $\hat{H}_i = H_i$ and $W_j I(j \in \mathcal{IG}(\hat{H}_j))$, $\hat{H}_j = H_j$ are

independent. Indeed, since $I(\hat{H}_i = H_i) = I(H_i \in N_l(i))$ and $I(\hat{H}_j = H_j) = I(H_j \in N_l(j))$, they are completely determined by the weights inside $N_{l+1}(i)$ and $N_{l+1}(j)$ (respectively) and those do not intersect. Therefore

$$\begin{aligned} & \mathbb{E}[W_i W_j I(i \in \mathcal{IG}(\hat{H}_i), \hat{H}_i = H_i, j \in \mathcal{IG}(\hat{H}_j), \hat{H}_j = H_j)] \\ &= \mathbb{E}[W_i I(i \in \mathcal{IG}(\hat{H}_i), \hat{H}_i = H_i)] \mathbb{E}[W_j I(j \in \mathcal{IG}(\hat{H}_j), \hat{H}_j = H_j)] \end{aligned} \quad (22)$$

On the other hand

$$\begin{aligned} I(i \in \mathcal{IG}(\hat{H}_i), \hat{H}_i = H_i) &= I(i \in \mathcal{IG}(\hat{H}_i), H_i \subset N_l(i)) \\ &= I(i \in \mathcal{IG}, H_i \subset N_l(i)) \end{aligned}$$

where the second equality follow from Lemma 5. Similarly we obtain

$$I(j \in \mathcal{IG}(\hat{H}_j), \hat{H}_j = H_j) = I(j \in \mathcal{IG}, H_j \subset N_l(j)).$$

Thus, we can rewrite (22) as

$$\begin{aligned} & \mathbb{E}[W_i W_j I(i, j \in \mathcal{IG}, H_i \subset N_l(i), H_j \subset N_l(j))] \\ &= \mathbb{E}[W_i I(i \in \mathcal{IG}, H_i \subset N_l(i))] \mathbb{E}[W_j I(j \in \mathcal{IG}, H_j \subset N_l(j))]. \end{aligned}$$

We recognize the left-hand side of this equation as the first summand in (19). Returning to (19) we obtain

$$\begin{aligned} & \left| \mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG}, H_i \subset N_l(i))] \mathbb{E}[W_j I(j \in \mathcal{IG}, H_j \subset N_l(j))] \right| \\ & \leq 2\mathbb{E}[W]^2 r(r-1)^l / (l+1)!. \end{aligned} \quad (23)$$

Also we have

$$\mathbb{E}[W_i I(i \in \mathcal{IG})] = \mathbb{E}[W_i I(i \in \mathcal{IG}, H_i \subset N_l(i))] + \mathbb{E}[W_i I(i \in \mathcal{IG}(H_i))(1 - I(H_i \subset N_l(i)))],$$

and

$$\mathbb{E}[W_i I(i \in \mathcal{IG}(H_i))(1 - I(H_i \subset N_l(i)))] \leq \mathbb{E}[W_i (1 - I(H_i \subset N_l(i)))] \leq \mathbb{E}[W] r(r-1)^l / (l+1)!,$$

where the second inequality is (20). It follows

$$\left| \mathbb{E}[W_i I(i \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG}, H_i \subset N_l(i))] \right| \leq \min(\mathbb{E}[W] r(r-1)^l / (l+1)!, \mathbb{E}[W_i I(i \in \mathcal{IG})]).$$

A similar inequality holds for j . Putting these two bounds together we obtain

$$\begin{aligned} & \left| \mathbb{E}[W_i I(i \in \mathcal{IG})] \mathbb{E}[W_j I(j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG}, H_i \subset N_l(i))] \mathbb{E}[W_j I(j \in \mathcal{IG}, H_j \subset N_l(j))] \right| \\ & \leq 2\mathbb{E}[W]^2 r(r-1)^l / (l+1)!, \end{aligned}$$

where trivial bounds $\mathbb{E}[W_i I(i \in \mathcal{IG})] \leq \mathbb{E}[W_i]$, $\mathbb{E}[W_j I(j \in \mathcal{IG})] \leq \mathbb{E}[W_j]$ are used. Combining with bound (23) we obtain

$$\begin{aligned} & \left| \mathbb{E}[W_i W_j I(i, j \in \mathcal{IG})] - \mathbb{E}[W_i I(i \in \mathcal{IG})] \mathbb{E}[W_j I(j \in \mathcal{IG})] \right| \\ & \leq 4\mathbb{E}[W]^2 r(r-1)^l / (l+1)! \\ & = 4 \frac{r}{r-1} \mathbb{E}[W^2] (r-1)^{l+1} / (l+1)!. \end{aligned}$$

We now use this estimate in (18). Observe that $|N_{d+1}(i) \setminus N_d(i)| \leq r(r-1)^d$. Recall that $l+1 = \lfloor d/2 \rfloor$. Then for each i , considering the cases of odd and even d separately and observing that the estimate is also trivially an upper bound for the $d = 0, 1$ cases, the double sum $\sum_{d \geq 0} \sum_{j \in N_{d+1}(i) \setminus N_d(i)}$ in (18) is upper bounded by

$$\begin{aligned} 4 \frac{r}{r-1} \mathbb{E}[W^2] \sum_{d \geq 0} r(r-1)^d \frac{(r-1)^{\lfloor \frac{d}{2} \rfloor}}{(\lfloor \frac{d}{2} \rfloor)!} &= 4 \frac{r^2}{r-1} \mathbb{E}[W^2] \sum_{k \geq 0} \frac{(r-1)^{3k}}{k!} + 4 \frac{r^2}{r-1} \mathbb{E}[W^2] \sum_{k \geq 0} \frac{(r-1)^{3k+1}}{k!} \\ &< 8 \frac{r^2}{r-1} \mathbb{E}[W^2] \sum_{k \geq 0} \frac{(r-1)^{3k+1}}{k!} \\ &< 8 \mathbb{E}[W^2] r^2 \exp((r-1)^3). \end{aligned}$$

Our final upper bound on $\text{Var}(W[\mathcal{IG}])$ becomes

$$n \mathbb{E}[W^2] + 8n \mathbb{E}[W^2] r^2 \exp((r-1)^3) < 9n \mathbb{E}[W^2] r^2 \exp((r-1)^3)$$

This completes the proof. The proof for matchings follows similarly, and is omitted. \square

6 Numerical results

In this section we numerically evaluate the performance of *GREEDY* in several settings, and compare our results to the prior work. We first compare our bound, marked NEW in the table below, on the cardinality (normalized by the number of nodes) of a *MIS* in an r -regular graph of girth at least g (Corollary 4) to the previous bounds in [She91] and [LW07]. The bounds of [She91] are coming from their Theorem 3 (when $g < 127$) and their Theorem 4 (when $g \geq 127$), with $w_i = 1$ for all i (their formulas involve a notion of weighted girth). The bounds of [LW07] are coming from their Table 2. Omitted values are those for which no corresponding results are given or the given bounds are trivial. Certain values of the form $2k+3$ are emphasized to be compatible with the Table 2 given in [LW07]. All values are rounded up to the nearest thousandth. As we see our new bounds are the strongest for

Table 1: Comparison of bounds for the cardinality of *MIS* in r -regular large-girth graphs

$\frac{r}{g}$	5			7			10		
	NEW	[She91]	[LW07]	NEW	[She91]	[LW07]	NEW	[She91]	[LW07]
50	.302	.288	-	.256	.239	-	.160	.194	-
100	.302	.294	-	.256	.243	-	.211	.197	-
203	.302	.304	.262	.256	.250	-	.211	.201	.169
403	.302	.306	.277	.256	.251	-	.211	.202	.184
2003	.302	.308	.294	.256	.252	-	.211	.203	.202

many calculated values of g and $r \geq 7$. Recall that for $r \geq 7$, our bounds are asymptotically (as $g \rightarrow \infty$) equivalent to those of [LW07], and superior to those of [She91]. Note that our bounds converge to their limit much faster than the bounds of [She91] and [LW07].

We now give our bounds for the cardinality of a *MM* (also normalized by the number of nodes) in an r -regular graph of girth at least g (Corollary 6). These are the first results for *MM* in this setting.

Table 2: Bounds for the cardinality of *MM* in r -regular large-girth graphs

$\frac{r}{g}$	3	4	5	6	7	10	13
25	.437	.427	-	-	-	-	-
40	.438	.444	.450	.454	.424	-	-
50	.438	.444	.450	.455	.459	-	-
75	.438	.444	.450	.455	.459	.468	.449
100	.438	.444	.450	.455	.459	.468	.473

Note that as r increases, the asymptotic (in r) size of a *MM* approaches that of a perfect matching ($\frac{r}{2}$), as expected from Corollary 5.

We now give our results for *MWIS* and *MWM* with i.i.d Exp(1) (exponentially distributed with parameter 1) weights, and compare to the results given in [GNS08]. The *GREEDY* columns show the expected asymptotic weight (normalized by the number of nodes) of the weighted independent set and matching returned by *GREEDY* as given in Theorems 1 and 2, while the [GNS08] columns reflect the expected asymptotic weight of a true *MWIS* and *MWM* as computed in [GNS08]. We only give results for r -regular graph with limiting girth, as no results for fixed girth are given in [GNS08].

Table 3: Exact *MWIS* and *MWM* vs. *GREEDY* for r -regular large-girth graphs with i.i.d Exp(1) weights

r	<i>MWIS</i>		<i>MWM</i>	
	[GNS08]	<i>GREEDY</i>	[GNS08]	<i>GREEDY</i>
3	.6077	.5966	.7980	.7841
4	.5631 ¹	.5493	.9022	.8826
5	-	.5119	.9886	.9643
10	-	.3967	1.282	1.242

In all cases, *GREEDY* is nearly optimal.

7 Conclusion

We have provided new results for the performance of a simple randomized greedy algorithm, *GREEDY*, for finding large independent sets and matchings in regular graphs with large finite girth. This provided new constructive and existential results in several settings. In addition we established concentration results for the values produced by *GREEDY*. One of the interesting insights from this work is that *GREEDY* exhibits a correlation decay property, which aids greatly in our analysis.

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¹reported incorrectly in [GNS08].

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