Beating the curse of dimensionality in options pricing and optimal stopping

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The fundamental problems of pricing high-dimensional path-dependent options, and more generally optimal stopping, are central to applied probability, financial engineering, stochastic control, and operations research, and have generated a considerable amount of interest and research from both academics and practitioners. Modern approaches, often relying on approximate dynamic programming, simulation, and/or the martingale duality theory for optimal stopping, typically have limited rigorous performance guarantees, which may scale poorly and/or require previous knowledge of good basis functions. A key difficulty with many approaches is that to yield stronger theoretical performance guarantees, they would necessitate the computation of deeply nested conditional expectations, where the depth of nesting scales with the time horizon T. In practice this is often overcome through approximations which sidestep these deeply nested conditional expectations, but which typically do not come with strong theoretical guarantees.

We overcome this fundamental obstacle by providing an algorithm which can trade-off between the guaranteed quality of approximation and the level of nesting (conditional expectations) required in a principled manner, without requiring a set of good basis functions. We develop a novel pure-dual approach, inspired by a connection to network flows. This leads to a representation for the optimal value as an infinite sum for which: 1. each term is the expectation of a certain natural and elegant recursively defined infimum; 2. the first k terms only require k levels of nesting; and 3. truncating at the first k terms yields an error of $\frac{1}{k}$. This enables us to devise a simple randomized algorithm whose runtime is effectively independent of the dimension, beyond the need to simulate sample paths of the underlying process. Indeed, our algorithm is completely data-driven in that it only needs the ability to simulate the original process (possibly conditioned on partial histories), and requires no prior knowledge of the underlying distribution. Our method allows one to elegantly trade-off between accuracy and runtime through a parameter ϵ controlling the associated performance guarantee, with computational and sample complexity both polynomial in T (and effectively independent of the dimension) for any fixed ϵ, in contrast to past methods typically requiring a complexity scaling exponentially in these parameters.

Key words: optimal stopping, options pricing, high-dimensional, non-Markovian, martingale duality, nested conditional expectations, simulation, network flows, prophet inequalities, polynomial-time approximation scheme, stochastic control, Robbins' problem, American option, Bermudan option

1. Introduction

1.1. Overview of problem and literature

The fundamental problem of pricing a stock option is central to applied probability and financial engineering, and has a rich history (Poitras (2009)). Here our focus will be on Bermudan options, which are a special case of discrete time optimal stopping problems, themselves central to stochastic control and operations research. We make no attempt to survey either of these areas of research, and instead point the interested reader to Jarrow and Rudd (1983), Belomestny and Schoenmakers (2018), Myneni (1992), Ahn et al. (2011), Haugh and Kogan (2007), Ferguson (1989), Duffie (2010), Carmona and Touzi (2008), Lempa (2014), Robbins, Sigmund, and Chow (1971), Aydogan et al. (2018), Cont and Tankov (2003), Bank and Follmer (2003), and Lucier (2017) for additional background. Recall that in the general setting of pricing a Bermudan option, there is some underlying (possibly non-Markovian and high-dimensional) stochastic process $\mathbf{Y} = \{Y_t, t \in [1,T]\}$, and sequence of general (possibly time-dependent) payout functions $\{g_t, t \in [1,T]\}$. For $t \in [1,T]$, and a vector $\mathbf{X} = (X_1, \ldots, X_T)$, let $X_{[t]} \triangleq (X_1, \ldots, X_t)$. Then $g_t(Y_{[t]})$ denotes the payout from executing the stock option at time $t \in [1,T]$, and the problem of pricing a Bermudan option is that of computing $\sup_{\tau \in \mathcal{T}} E\big[g_\tau(Y_{[\tau]})\big]$, with $\mathcal{T}$ the set of all integer-valued stopping times in $[1,T]$ adapted to the natural filtration $\mathcal{F} = \{\mathcal{F}_t, t \in [1,T]\}$ generated by $\mathbf{Y}$ (McKean (1965)). In this work (and without loss of generality, i.e. w.l.o.g) it will be convenient to instead consider the associated minimization $\inf_{\tau \in \mathcal{T}} E\big[g_\tau(Y_{[\tau]})\big]$ with the additional assumption that g_t is non-negative, and we assume this throughout (later we will comment in greater depth on the relevant transformations). Occasionally it will also be convenient to assume that $\{g_t, t \in [1,T]\}$ has been normalized, so that $g_t \in [0,1]$ for all $t \in [1,T]$, instead of making repeated reference to some upper bound on the support and/or appropriate truncations - and we will make clear whenever such an assumption is in force.

It is well-known that for most problems of interest in financial applications, such optimal stopping problems have no simple analytical solution, and one must resort to numerical / computational methods. Here we focus exclusively on two of the most popular modern approaches, approximate dynamic programming (ADP) and duality. We note that although there is also a vast literature on alternative (e.g. PDE, quadrature) methods, those approaches generally suffer from the same complexity-related issues in the presence of high-dimensionality, and we refer the reader to Achdou et al. (2005), Peskir et al. (2006), Pasucci (2011) , and Haug (2007) for further details.

In typical applications, $\mathbf{Y}$ will be a high-dimensional vector of stock prices and economic indicators, which evolves in a non-Markovian fashion (Rogers (2002)). Unfortunately, this combination of high-dimensionality and path-dependence ensures that any naive attempt at DP (the natural

way to formulate the problem) will take time growing exponentially in the dimension, time horizon, and/or both. From the late 1990's to the mid-2000's (and continuing today), one of the main approaches taken was ADP. Here one uses sampling, and perhaps a judicious choice of basis functions, to approximate the DP equations and yield tractable algorithms. Seminal papers in this area include Tsitsiklis and Van Roy (2001), and especially Longstaff and Schwartz (2001). There was much subsequent work, including work on e.g. policy iteration, neural networks, and deep learning, also in the multiple-stopping setting (Kolodko et al. (2006), Lai et al. (2004), Kohler et al. (2010), Becker et al. (2018), Bender et al. (2006a,b), Kohler (2010), Bender et al. (2008), Beveridge (2013)). Often these approaches were enhanced using simulation-based methodologies (Broadie and Glasserman (2004), Broadie et al. (2000), Broadie and Detemple (2004), Kan et al. (2009), Hong and Juneja (2009), Belomestny et al. (2013), Bolia et al. (2004), Liu and Hong (2009), Kashtanov (2017), Boyle et al. (2003), Ibanez et al. (2004), Meinshausen et al. (2004), Egloff et al. (2007), Kargin (2005), Bolia et al. (2005), Lemieux et al. (2005), Staum (2002), Chen and Hong (2007), Belomestny et al. (2015), Agarwal et al. (2016)), where we note that simulation had also been a popular tool in options pricing even before the popularity of ADP (Broadie and Glasserman (1997)). Although these methods often provide good bounds in practice, rigorous performance guarantees in the non-Markovian and path-dependent setting are limited, and typically either: 1. require one to be given a good set of basis functions and/or initial approximations; 2. have bounds which can degrade rapidly in the dimension and other parameters; 3. have bounds in terms of quantities which are difficult to control and interpret; or 4. have essentially no rigorous bounds. Such analyses appear in e.g. Avramidis et al. (2002), Clement et al. (2002), Bouchard et al. (2012), Glasserman and Yu (2004b), Bezerra et al. (2017), Belomestny et al. (2018), Stentoft (2004), Belomestny (2011,b), Agarwal and Juneja (2013), Glasserman and Yu (2004), Egloff et al. (2005), Zanger (2013), Kim (2017), and Del Moral et al. (2011).

Building on the seminal work of Davis and Karatzas (1994), significant progress was made simultaneously by Haugh and Kogan (2004) and Rogers (2002) in their formulation of a dual methodology for options pricing. In particular, for $x \in \mathcal{R}$, let $\mathcal{M}^x$ denote the set of mean-x, T-dimensional martingales with respect to (w.r.t) filtration $\mathcal{F}$. Then the above works prove that $\inf_{\tau\in\mathcal{T}} E\big[g_\tau(Y_{[\tau]})\big] = \sup_{M\in\mathcal{M}^0} E\big[\min_{t\in[1,T]}\big(g_t(Y_{[t]}) - M_t\big)\big]$, which provides an alternate way to solve the problem. Other dual representations were subsequently discovered, also in the multiple-stopping setting (Jamshidian (2007), Joshi (2015), Lerche et al. (2010), Schoenmakers (2012), Chandramouli et al. (2012), Lerche et al. (2007)); and the methodology was extended to more general control problems (Brown (2010), Brown and Haugh (2017), Rogers (2007), Bender (2018)). Hybrid primal-dual, ADP (including methods approximating the optimal dual martingale by appropriate basis functions), and related iterative approaches (based on e.g. consumption processes) have

since led to substantial algorithmic progress (Andersen and Broadie (2004), Chen and Glasserman (2007), Belomestny and Milstein (2006), Belomestny et al. (2013), Ibanez et al. (2017), Desai et al. (2012), Christensen (2014), Belomestny (2013, 2017), Lelong (2018), Rogers (2015), Chandramouli et al. (2018), Fuji et al. (2011), Mair et al. (2013), Belomestny et al. (2009), Hepperger (2013), Broadie and Cao (2008), Zhu et al. (2015)). Unfortunately, the performance guarantees in all of the aforementioned works suffer from the same shortcomings mentioned previously. In spite of this lack of theoretical justification, we do note that many of these algorithms, also those discussed earlier in the context of ADP, simulation, and policy iteration, seem to perform quite well across a range of numerical examples.

More recently, Rogers (2010) proposed a so-called pure-dual approach, in which he gave a new and explicit representation for the optimal martingale using backwards induction, without having to circularly refer back to the cost-to-go-functions (as the optimal martingale was typically defined in past literature through the Doob-Meyer decomposition of processes associated with the so-called Snell envelope, see e.g. Davis and Karatzas (1994)). Unfortunately, as Rogers notes, this construction requires a depth-T nesting of conditional expectations, which in the high-dimensional and path-dependent setting cannot be efficiently simulated. Indeed, it is well-known that even the original primal problem has a similarly impractical "explicit solution" in terms of such fully nested conditional expectations. This approach was subsequently extended in several works (Bender et al. (2015), Schoenmakers et al. (2013), Belomestny et al. (2013), Belomestny (2017)). We note that an important earlier work of Chen and Glasserman (2007) in some sense anticipates this notion of a pure-dual solution, as the authors define an iterative procedure based on so-called supersolutions which leads to an expansion for the optimal dual martingale (similar ideas also appear in Jamshidian (2007)). Furthermore, we note that several of the algorithms referenced above, including most of the primal-dual methods as well as Kolodko et al. (2006), Belomestny and Milstein (2006), Beveridge (2013), and Chen and Glasserman (2007), are of an iterative nature (yielding bounds over a series of rounds) and seem to yield strong bounds after only a few iterations, at least on the numerical examples studied in many of the referenced papers. As these methods can be efficiently implemented for a small number of iterations without using deeply nested conditional expectations, this suggests that in practice many of these iterative methods can often yield good bounds efficiently. **However, to our knowledge none of these methods have strong theoretical guarantees ensuring good performance after only a small number of iterations, and prior to this work it remained a fundamental open challenge to devise such an iterative method which allowed one to explicitly trade-off between the number of iterations / depth of nested conditional expectations required and the accuracy achieved.**

1.2. Summary of state-of-the-art

In summary, the current state-of-the-art for pricing high-dimensional path-dependent Bermudan options involves either: 1. ADP and primal-dual methods where performance and runtime guarantees are limited and may rapidly degrade in the problem parameters; 2. pure-dual methods which either cannot be implemented efficiently or have no performance guarantees; 3. iterative methods which seem to yield good bounds on numerical examples after only few iterations (for which they can be implemented efficiently) yet which have no theoretical guarantees along these lines; or 4. alternative approaches such as PDE methods which suffer from many of the same problems. The presence of deeply nested conditional expectations is a fundamental bottleneck to many current approaches with rigorous theoretical guarantees, especially among those that do not require a good set of basis functions as input.

2. Main Results

2.1. Additional notations

We now state our main results, and begin by formalizing some additional notations. Recall that $\mathbf{Y} = \{Y_t, t \in [1,T]\}$ is a general (possibly high-dimensional and non-Markovian) discrete-time stochastic process. For concreteness, we fix the dimension of each Y_t to be some fixed value D (which may be very large, even relative to T and any other parameters). For $t \in [1,T]$, let $\aleph^t$ denote the set of all D by t matrices (i.e. D rows, t columns) with entries in $\mathcal{R}$, so $Y_{[t]} \in \aleph^t$. Recall that $\mathcal{F}_t$ is the σ-field generated by $Y_{[t]}$; $\mathcal{F}$ the corresponding filtration; and $\mathcal{T}$ the set of integer-valued stopping times τ adapted to $\mathcal{F}$ s.t. $\tau \in [1,T]$ w.p.1. Let $Z_t \triangleq g_t(Y_{[t]})$, where we write $Z_t(Y_{[t]})$ if we wish to make the dependence explicit, and assume that Z_t is non-negative and integrable for all t. More generally, for a stochastic process $\mathbf{X}$ adapted to $\mathcal{F}, t \in [1,T]$, and $\gamma \in \aleph^t$, we let $X_t(\gamma)$ denote the value of X conditional on the event $\{Y_{[t]} = \gamma\}$, where if this event is measure-zero $X_t(\gamma)$ should be interpreted in the sense of so-called regular conditional probabilities (Faden (1985)). In general we will assume that $\mathbf{Y}$ and $\{g_t, t \in [1,T]\}$ are sufficiently non-pathological to ensure that all relevant conditionings and conditional distributions, also of the various derived random variables (r.v.s) of interest, are well-defined (again at least in the sense of regular conditional probabilities), and leave a more formal investigation of such technical matters for future research. All logarithms should be read as base e. For an event A, I(A) will denote the corresponding indicator function. Let $\text{OPT} \triangleq \inf_{\tau \in \mathcal{T}} E[Z_\tau]$.

2.2. Novel pure-dual solution

We begin by stating our new pure-dual representation. Here we only state the implications regarding the optimal value, and leave a formal discussion of the associated dual martingales to Section 3. We begin by defining an inductive sequence of r.v.s on a common probability space. For $t \in [1,T]$,

let $Z_t^1 \triangleq Z_t$. For $k \geq 1$ and $t \in [1,T]$, let $Z_t^{k+1} \triangleq Z_t^k - E\big[\min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\big]$. Let $\mathbf{Z}$ and $\mathbf{Z}^k$ denote the respective stochastic processes. It follows from the basic properties of conditional expectations and non-negativity of $\mathbf{Z}^1$ that $\mathbf{Z}^k$ is non-negative for all $k \geq 1$. Then our pure-dual representation for OPT is as follows. For $k \geq 1$, let $H_k \triangleq E[\min_{t\in[1,T]} Z_t^k]$.

THEOREM 1. *$OPT = \sum_{k=1}^{\infty} H_k$.*

We note that in many ways the statement of Theorem 1 is quite surprising, as it asserts that the value of a general path-dependent optimal stopping problem has a representation which looks very much like a closed-form solution. To make this point clear, let us more explicitly give the first few terms.

$$H_1 = E\big[\min_{t\in[1,T]} g_t(Y_{[t]})\big] \quad ; \quad H_2 = E\Bigg[\min_{t\in[1,T]}\Bigg(g_t(Y_{[t]}) - E\big[\min_{i\in[1,T]} g_i(Y_{[i]})|\mathcal{F}_t\big]\Bigg)\Bigg];$$

$$H_3 = E\Bigg[\min_{t\in[1,T]}\Bigg(g_t(Y_{[t]}) - E\big[\min_{i\in[1,T]} g_i(Y_{[i]})|\mathcal{F}_t\big] - E\Bigg[\min_{i\in[1,T]}\Bigg(g_i(Y_{[i]}) - E\big[\min_{j\in[1,T]} g_j(Y_{[j]})|\mathcal{F}_i\big]\Bigg)|\mathcal{F}_t\Bigg]\Bigg)\Bigg].$$

Note that the first term, H_1, corresponds to the obvious lower bound. Later terms are the expectations of elegant and explicit infima, each of which can be computed by simulation, where the kth term involves only k levels of nested conditional expectations.

We now state a corresponding result for the optimal stopping boundary. Let τ_* denote the stopping time which stops the first time that $Z_t - E\big[\sum_{k=1}^{\infty} \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\big] = 0$, and stops at time T if no such time exists in $[1,T]$.

COROLLARY 1. *W.p.1 $\exists\ t \in [1,T]$ s.t. $Z_t - E\big[\sum_{k=1}^{\infty} \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\big] = 0$, and τ_* is an optimal solution to the stopping problem $\inf_{\tau\in\mathcal{T}} E[Z_\tau]$. Namely, $E[Z_{\tau_*}] = OPT$.*

2.3. Approximation guarantees and rate of convergence

The power of Theorem 1 is that it allows for rigorous approximation guarantees when the infinite sum is truncated. Let $E_k \triangleq \sum_{i=1}^{k} H_i$.

THEOREM 2. *Suppose w.p.1 $Z_t \in [0,1]$ for all $t \in [1,T]$. Then for all $k \geq 1$, $0 \leq OPT - E_k \leq \frac{1}{k+1}$.*

Thus truncating our expansion after k terms yields an absolute error at most $\frac{1}{k+1}$. Note that this is in stark contrast to the known results for other pure-dual methods (and essentially all other methods for this problem under general path-dependence), in that our approach allows us to explicitly trade-off between the approximation error and the level of nesting (of conditional expectations) required.

One might hope that if one's original stopping problem was somehow "easy", then one could show a faster rate of convergence. By recursively applying a celebrated prophet inequality of Hill and Kertz (1983), our next result demonstrates this is indeed the case. Let us define $h_1 : [0,1] \to [0,1]$ to be the function s.t. $h_1(x) \triangleq (1-x)\log(\frac{1}{1-x})$. For $k \geq 2$ and $x \in [0,1]$, let $h_k(x) \triangleq h_1\big(h_{k-1}(x)\big)$, i.e. the function h_1 composed with itself k times. Then we prove the following.

THEOREM 3. *Suppose w.p.1 $Z_t \in [0,1]$ for all $t \in [1,T]$. Then for all $k \geq 1$, $OPT - E_k \leq h_k(OPT)$. In addition, for each fixed $x \in [0,1]$, $\{h_k(x), k \geq 1\}$ is a monotone decreasing sequence converging to 0; and $\lim_{x \downarrow 0} h_1(x) = \lim_{x \uparrow 1} h_1(x) = 0$.*

Theorem 3 implies that if OPT is close to 0 or 1, then after only 1 round our approach will already have very small error.

Although a normalization argument immediately extends Theorems 2 and 3 to the case in which w.p.1 $Z_t \in [0,U]$ for some general upper bound U, such an approach may be undesirable as the error is then relative to a (possibly large) upper bound, and furthermore the process may be unbounded. We now present a general bound which requires no such normalization assumption, at the expense of a slower rate of convergence. Whether such a slowdown is fundamental, or simply an artifact of our analysis, remains an interesting open question.

THEOREM 4. *Under no assumptions on $\mathbf{Z}$ beyond non-negativity, for all $k \geq 1$,*

$$OPT - E_k \leq 2 \times \left(\frac{E[(Z_T)^2]}{OPT^2} \right)^{\frac{1}{3}} \times k^{-\frac{1}{3}} \times OPT.$$

We note that so long as $\left(E[(Z_T)^2]\right)^{\frac{1}{2}}$ is not too many times larger than OPT, Theorem 4 shows that our method converges rapidly in relative error as well. In Section 4, we describe a modification of our approach which also converges rapidly in relative error even when such an assumption does not hold, as is the case in the i.i.d. setting and the celebrated Robbins' problem, a fundamental open problem in the theory of path-dependent optimal stopping (and only still-unsolved original variant of the so-called secretary problem) popularized by the statistician and probabilist Herbert Robbins, who famously stated (before his death) that he "should like to see this problem solved before I died" at the 1990 International Conference on Search and Selection in Real Time (Bruss (2005)).

We now show that in general the linear convergence of Theorem 2 cannot be improved, through the following lower-bound result.

THEOREM 5. *For any $n \geq 2$, there exists an optimal stopping problem with $T = 2$, $P(Z_t \in [0,1]) = 1$ for $t \in [1,2]$, yet $OPT - E_k \geq \frac{1}{4n}$ for all $k \leq n$.*

Of course, Theorem 5 is worst-case, and for many problems our method may converge more quickly than dictated by Theorem 2. In Section 4, we describe the convergence properties of some additional examples, and leave developing a deeper understanding of the instance-specific rate of convergence of our approach as an interesting direction for future research.

2.4. Algorithmic results

We now describe the main algorithmic implications of Theorems 1 and 2, and take the natural approach of simulating H_i for an appropriate range of i.

2.4.1. Formal computational and sampling model for algorithm analysis.

Access to samples and data-driven algorithms : In our analysis we will be interested in understanding exactly what kind of "access to randomness" is needed, as we will want to certify that our methods are "data-driven". Such a feature is highly desirable in an options pricing setting, as one is unlikely to have access to e.g. joint density functions (Broadie et al. (2000)). To this end, we will at times carefully specify that a given algorithm is only able to access randomness by accessing a certain "base simulator" which only has the ability to generate a single sample path (possibly conditioned on a partial history) of the underlying process $\mathbf{Y}$. We now formally define a subroutine (randomized algorithm) $\mathcal{B}$, which we will informally refer to as the "base simulator", and which will provide the only means for our algorithms to access information about $\mathbf{Y}$. For $t \in [1,T]$ and $\gamma \in \aleph^t$, let $\mathbf{Y}(\gamma)$ denote a random matrix distributed as $\mathbf{Y}$, conditioned on the event $\{Y_{[t]} = \gamma\}$. We suppose there exists a randomized algorithm $\mathcal{B}$ with the following properties. $\mathcal{B}$ takes as input $t \in [1,T]$ and $\gamma \in \aleph^t$, and outputs an independent sample of $\mathbf{Y}(\gamma)$, independent of any previous samples $\mathcal{B}$ has generated. Also, we let $\mathcal{B}(0,\emptyset)$ return an independent sample of $\mathbf{Y}$ (unconditioned). We suppose that $\mathcal{B}$ takes C units of computational time to terminate and generate any one such sample. Here C may of course depend on T, D, and other parameters, although we assume C does not depend on the particular input t, γ to $\mathcal{B}$. We note that for many processes even generating simulated sample paths may be quite challenging, and there are many interesting questions surrounding how to combine our framework with settings in which generating individual sample paths is very costly and/or one can only generate approximate sample-paths. Such questions will generally be beyond the scope of this paper, and left as directions for future research, where we refer the interested reader to e.g. Glasserman (2013), Dieker (2004) , and Blanchet et al. (2017) for additional related background.

Computational model and runtime analysis : Next, we must formalize a computational model for how to analyze the run-time of algorithms that use $\mathcal{B}$. As our goal here is more to understand the algorithmic performance from a practical simulation perspective as opposed to a formal complexity theory perspective, we suppose that addition, subtraction, multiplication, division, maximum, and minimum of any two numbers can be done in one unit of time, irregardless of the values of those numbers. We will ignore all computational costs associated with reading, writing, and storing numbers in memory, as well as inputting numbers to functions. We also suppose that for any $t \in [1,T]$ and $\gamma \in \aleph^t$, we may compute $g_t(\gamma)$ in G units of computational time, where again G may depend on T, D, and other parameters, although we again assume G does not depend on the particular choice of t, γ. We further assume $C, G \geq 1$.

2.4.2. Main algorithmic results. We now state our main algorithmic results, which allow one to trade-off between the accuracy desired (in terms of a parameter ϵ) and the runtime and samples required in the general high-dimensional path-dependent setting, analogous to the concept of a polynomial-time-approximation-scheme (PTAS) in the theory of approximation algorithms (Shmoys and Williamson (2011)). **The key insight of our results is that even in the presence of full path-dependence and high-dimensionality, for any given error parameter ϵ one can obtain an ϵ-approximation in time polynomial in T, and depending on the dimension (and state-space more generally) only through the cost of simulating individual sample paths, where only a polynomial number of such simulations are needed. Furthermore, our methods are completely data-driven, and require no knowledge of any distributions beyond the ability to generate samples. To our knowledge, these results are the first of their kind, and act as a proof-of-concept that such a result is possible even in the path-dependent and high-dimensional setting.** We also note that our analysis and bounds are worst-case, and in almost all cases we have opted for simplicity of analysis over tightness of bounds. Furthermore, the fact that all relevant terms must actually be estimated by simulations, whose errors propagate through our expansion, implies that our achieved runtime (although polynomial in T for any fixed ϵ) will be considerably slower than the rate implied by Theorem 2. We leave providing a tighter analysis and devising faster algorithms as interesting directions for future research, especially when the underlying problem exhibits additional structure. Although for simplicity we state and prove all our simulation results under the assumption that $P(Z_t \in [0,1]) = 1$ for all t, slight modifications of our proofs can also treat the case in which one has a general upper bound U on the support (and even more general settings), in which case all computational and sample complexities scale by additional polynomial terms. We note that our algorithms are constructed recursively, building a method for efficiently simulating $\mathbf{Z}^{k+1}$ out of one for efficiently simulating $\mathbf{Z}^k$, and that such nested schemes (albeit of a different nature) have been considered previously in the literature (Kolodko et al. (2006)).

For $k \geq 1$ and $\epsilon, \delta \in (0,1)$, let us define

$$f_k(\epsilon,\delta) \triangleq 10^{2(k-1)^2} \times \epsilon^{-2(k-1)} \times (T+2)^{k-1} \times \big(1 + \log(\frac{1}{\delta}) + \log(\frac{1}{\epsilon}) + \log(T)\big)^{k-1}.$$

THEOREM 6. *Suppose w.p.1 $Z_t \in [0,1]$ for all $t \in [1,T]$. Then for all $k \geq 1$, there exists a randomized algorithm $\hat{\mathcal{B}}^k$ which takes as input any $\epsilon, \delta \in (0,1)$, and achieves the following. In total computational time at most $(C+G+1)f_{k+1}(\epsilon,\delta)$, and with only access to randomness at most $f_{k+1}(\epsilon,\delta)$ calls to the base simulator $\mathcal{B}$, returns a random number X satisfying $P\big(|X - H_k| \leq \epsilon\big) \geq 1-\delta$.*

Combining Theorem 6 with Theorem 2 and a simple union bound, we are led to the following.

Corollary 2. *Suppose w.p.1 $Z_t \in [0,1]$ for all $t \in [1,T]$. Then there exists a randomized algorithm $\mathcal{A}$ which takes as input any $\epsilon, \delta \in (0,1)$, and achieves the following. In total computational time at most $(C+G+1)\exp(100\epsilon^{-2})T^{3\epsilon^{-1}}\big(1+\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}}$, and with only access to randomness at most $\exp(100\epsilon^{-2})T^{3\epsilon^{-1}}\big(1+\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}}$ calls to the base simulator $\mathcal{B}$, returns a random number X satisfying $P\big(|X-OPT| \leq \epsilon\big) \geq 1-\delta$.*

Of course, for the explicit bounds of Theorem 6 and Corollary 2, many known methods will (for many parameter regimes) exhibit much faster runtimes. However, we emphasize that essentially all of the aforementioned ADP methods have runtimes which scale exponentially in the dimension (also typically requiring a Markovian assumption), and all of the aforementioned iterative or pure-dual methods have runtimes which scale exponentially in the time horizon, if one requires theoretical guarantees of good performance. Indeed, the existence of an algorithm which can yield a solution with strong theoretical approximation guarantees in time polynomial in both the dimension and time horizon was not previously known to exist. We leave as a pressing direction for future research devising tighter bounds and more practical algorithms, based on the insights from this work, and also understanding which methods may be best for which parameter settings and instance-specific features. We also note that although such polynomial bounds were not known for any existing methods, it is also an interesting question whether (especially in light of our own work) such bounds can be proven, perhaps for suitable modifications of those methods.

We next state our algorithmic results w.r.t. implementing efficient stopping strategies with analogous performance guarantees. There is a subtlety here, as one might think that since our previous results yield approximate value-function evaluations, it should be immediate from known black-box reductions (Singh and Yee (1994)) that we also get a good approximate stopping strategy. However, the problem with such an approach is it would typically require one to approximate the value function to an additive error going to 0 as T grows large (Singh and Yee (1994), Chen and Glasserman (2007), Van Roy (2009)). In our framework, this would not work, as it would require deeply nested conditional expectations. Fortunately, en route to our main results we will prove strong pathwise convergence results, which will allow us to overcome this problem, yielding the following results. We note that our results are stated in terms of the existence of an efficiently implementable randomized stopping time, and we refer the reader to Chalasani et al. (2001) and Levin and Peres (2017) for associated standard definitions regarding the formal definition of a randomized stopping time as an appropriate mixture of $\mathcal{F}$-adapted stopping times. We also note that several past works have explicitly studied the connection between value-function-approximation and approximately optimal stopping times in the context of optimal stopping and options pricing, but their results are incomparable to our own (Van Roy (2009), Belomestny (2011)).

Corollary 3. *Suppose w.p.1 $Z_t \in [0,1]$ for all $t \in [1,T]$. Then for all $\epsilon \in (0,1)$, there exists a randomized stopping time τ_ϵ s.t. $E[Z_{\tau_\epsilon}] - OPT \leq \epsilon$, and with the following properties. At each time step, the decision of whether to stop (if one has not yet stopped) can be implemented in total computational time at most $(C+G+1)f_{\lceil \frac{4}{\epsilon} \rceil}(\frac{\epsilon}{4}, \frac{\epsilon}{4T})$, and with only access to randomness at most $f_{\lceil \frac{4}{\epsilon} \rceil}(\frac{\epsilon}{4}, \frac{\epsilon}{4T})$ calls to the base simulator $\mathcal{B}$.*

Although we defer the details of the stopping time τ_ϵ and all relevant algorithms to Section 5, we note that intuitively one can (roughly) take τ_ϵ to be the stopping time which stops the first time that (a simulated approximation of) $Z_t^{\lceil \frac{4}{\epsilon} \rceil}$ is less than $\frac{1}{2}\epsilon$.

We conclude this section by briefly circling back to our statement in Section 1 regarding the fact that the problem $\sup_{\tau \in \mathcal{T}} E[Z_\tau]$, the setting of primary interest in the context of options pricing, can w.l.o.g. be transformed into a problem of the form $\inf_{\tau \in \mathcal{T}} E[Z'_\tau]$ for appropriate non-negative Z'. Of course, if there exists a finite known upper bound U on $\mathbf{Z}$, one can set $Z'_t = U - Z_t$, in which case $\sup_{\tau \in \mathcal{T}} E[Z_\tau] = U - \inf_{\tau \in \mathcal{T}} E[Z'_\tau]$. Alternatively, if such an upper bound is unavailable or computationally undesirable, one can set $Z'_t = E[\max_{i \in [1,T]} Z_i | \mathcal{F}_t] - Z_t$, in which case (by optional stopping) $\sup_{\tau \in \mathcal{T}} E[Z_\tau] = E[\max_{t \in [1,T]} Z_t] - \inf_{\tau \in \mathcal{T}} E[Z'_\tau]$. Under both transformations, one can then apply our approach to the associated minimization problem, where we note that all relevant run-time analyses remain unchanged in their essential parts with only minor modification, e.g. under the second transformation all recursions must be carried out to one greater depth as even Z'_t must be computed by estimating an appropriate conditional expectation.

2.5. Outline of rest of paper

The remainder of the paper proceeds as follows. We derive our pure-dual martingale representation, compare to related approaches from the literature, and give an interpretation in terms of network flows in Section 3. In Section 4 we prove several bounds on the rate of convergence of our methodology, and provide several illustrative examples. We derive our main algorithmic results in Section 5, proving explicit bounds on the required computational and sample complexity needed to achieve a given performance guarantee. We provide concluding remarks and some interesting directions for future research in Section 6. We also provide a technical appendix in Section 7, which contains several proofs from throughout the paper.

3. Proof of Theorem 1, dual martingales, and network flows.

In this section we formalize our pure-dual approach and prove Theorem 1, put our results in the broader context of other related work, and give a formulation in terms of network flows.

3.1. Simple intuition

We begin by giving the simple intuition behind our approach. We wish to compute $\text{OPT} = \inf_{\tau\in\mathcal{T}} E[Z^1_\tau]$. It follows from a simple sample-path argument that $\inf_{\tau\in\mathcal{T}} E[Z^1_\tau] \geq E[\min_{t\in[1,T]} Z^1_t]$. We now observe that since $\big\{E[\min_{i\in[1,T]} Z^1_i|\mathcal{F}_t], t\in[1,T]\big\}$ is a martingale w.r.t $\mathcal{F}$, the optional stopping theorem implies that

$$\inf_{\tau\in\mathcal{T}} E[Z^1_\tau] = E[\min_{t\in[1,T]} Z^1_t] + \inf_{\tau\in\mathcal{T}} E\big[Z^1_\tau - E[\min_{i\in[1,T]} Z^1_i|\mathcal{F}_\tau]\big].$$

Since by definition $Z^2_t = Z^1_t - E[\min_{i\in[1,T]} Z^1_i|\mathcal{F}_t]$, this is equivalent to

$$\text{OPT} = \inf_{\tau\in\mathcal{T}} E[Z^1_\tau] = E[\min_{t\in[1,T]} Z^1_t] + \inf_{\tau\in\mathcal{T}} E[Z^2_\tau].$$

Now, we simply observe that we may recursively repeat this process on the problem $\inf_{\tau\in\mathcal{T}} E[Z^2_\tau]$, and then all subsequent problems, to conclude the following.

LEMMA 1. *For all $k\geq 1$, $OPT = \sum_{i=1}^k E[\min_{t\in[1,T]} Z^i_t] + \inf_{\tau\in\mathcal{T}} E[Z^{k+1}_\tau]$. In addition, w.p.1 Z^k_t is non-negative for all $k\geq 1$ and $t\in[1,T]$; and for each $t\in[1,T]$, w.p.1 $\{Z^k_t, k\geq 1\}$ is a monotone decreasing sequence of random variables.*

We would be done (at least with the proof of Theorem 1) if we could show that $\lim_{k\to\infty}\inf_{\tau\in\mathcal{T}} E[Z^{k+1}_\tau] = 0$.

3.2. Proof of Theorem 1

We now prove that $\lim_{k\to\infty}\inf_{\tau\in\mathcal{T}} E[Z^{k+1}_\tau] = 0$, completing the proof of Theorem 1.

Proof of Theorem 1: It follows from monotone convergence that $\{\mathbf{Z}^k, k\geq 1\}$ converges a.s., and thus $\{Z^{k+1}_T - Z^k_T, k\geq 1\}$ converges a.s. to 0. Since by definition, for all $k\geq 1$, w.p.1 $Z^{k+1}_T = Z^k_T - E\big[\min_{i\in[1,T]} Z^k_i|\mathcal{F}_T\big]$, and by measurability w.p.1 $E\big[\min_{i\in[1,T]} Z^k_i|\mathcal{F}_T\big] = \min_{i\in[1,T]} Z^k_i$, we conclude that

$$\{\min_{i\in[1,T]} Z^k_i, k\geq 1\} \quad \text{converges a.s. to 0.} \tag{1}$$

Thus, for any $j\geq 1$, there exists K_j s.t. $k\geq K_j$ implies $P\big(\min_{i\in[1,T]} Z^k_i \geq \frac{1}{j}\big) < \frac{1}{j^2}$. It follows that there exists a strictly increasing sequence of integers $\{K'_j, j\geq 1\}$ s.t. $P\big(\min_{i\in[1,T]} Z^{K'_j}_i \geq \frac{1}{j}\big) < \frac{1}{j^2}$.

For stopping problem $\inf_{\tau\in\mathcal{T}} E[Z^{K'_j}_\tau]$, consider the stopping time τ'_j which stops the first time $Z^{K'_j}_t \leq \frac{1}{j}$, and stops at time T if no such time has yet occurred by the end of the horizon. Let I'_j be the indicator for the event $\big\{\min_{i\in[1,T]} Z^{K'_j}_i > \frac{1}{j}\big\}$. Note that w.p.1, $Z^{K'_j}_{\tau'_j} \leq \overline{Z}_j \triangleq \frac{1}{j} + I'_j Z^{K'_j}_T$. Recall that under our definitions, $\{\mathbf{Z}^{K'_j}, j\geq 1\}$ are all constructed on the same probability space, and thus $\{Z^{K'_j}_T, j\geq 1\}$ is monotone decreasing. It follows from Borel-Cantelli that $\{I'_j, j\geq 1\}$ equals 0 after some a.s. finite time, and thus $\{\overline{Z}_j, j\geq 1\}$ converges a.s. to 0. But by monotonicity, integrability, and non-negativity, we may apply dominated convergence to conclude that $\lim_{j\to\infty} E[\overline{Z}_j] = 0$,

which implies that $\lim_{j\to\infty} E\big[Z_{\tau_j'}^{K_j'}\big]=0$, and thus $\lim_{j\to\infty}\inf_{\tau\in\mathcal{T}} E[Z_\tau^{K_j'}]=0$. Thus the sequence $\{\inf_{\tau\in\mathcal{T}} E[Z_\tau^j], j\geq 1\}$, which is monotone decreasing by non-negativity and Lemma 1, has a subsequence which converges to 0, and thus must itself converge to 0. Letting $k\to\infty$ in Lemma 1 then completes the proof. *Q.E.D.*

3.3. The optimal dual martingale and Proof of Corollary 1

We now explicitly describe the optimal martingale derived from our approach. First, we provide some additional background on martingale duality.

3.3.1. Background on martingale duality. There are many different, essentially equivalent statements of the relevant duality, and we take as our starting point a formulation essentially identical to what is referred to in several papers as a dual satisfying 0-sure-optimality (Schoenmakers et al. (2013), Belomestny et al. (2013), Belomestny (2017)).

LEMMA 2 **(0-sure-optimal dual for optimal stopping)**.

$$OPT = \sup\left\{x\in\mathcal{R}: \exists\ M\ \in\mathcal{M}^x\ s.t.\ P\left(\min_{t\in[1,T]}\big(Z_t^1-M_t\big)=0\right)=1\right\}.$$

We note that prior works studying 0-sure-optimal martingale duality actually imply that any martingale M adapted to $\mathcal{F}$ s.t. $P\big(\min_{t\in[1,T]}\big(Z_t^1-M_t\big)=0\big)=1$ must have mean equal to OPT, but for our purposes an optimization-oriented formulation as asserted in Lemma 2 will be convenient.

3.3.2. Our optimal dual martingale. Let $S\triangleq\sum_{k=1}^\infty \min_{i\in[1,T]} Z_i^k$, where we note that S is non-negative with finite mean by Theorem 1 and monotone convergence. Let **MAR** denote the Doob martingale s.t. $\mathrm{MAR}_t = E\big[S|\mathcal{F}_t\big]$, $t\in[1,T]$. We now prove that **MAR** is the optimal dual martingale arising from our pure-dual approach. Recall from Lemma 1 (and monotone convergence) that $\{\mathbf{Z}^k, k\geq 1\}$ converges a.s. to a limiting T-dimensional random vector $\mathbf{Z}^\infty=\{Z_t^\infty, t\in[1,T]\}$.

LEMMA 3. *$E[MAR_1]=OPT$, and $P\left(\min_{t\in[1,T]}\big(Z_t^1-MAR_t\big)=0\right)=1$.*

Proof: For all $t\in[1,T]$, $Z_t^1-\mathrm{MAR}_t$ equals

$$Z_t^1-E\big[\sum_{k=1}^\infty \min_{i\in[1,T]} Z_i^k|\mathcal{F}_t\big] \;=\; Z_t^1-E\big[\min_{i\in[1,T]} Z_i^1|\mathcal{F}_t\big]-E\big[\sum_{k=2}^\infty \min_{i\in[1,T]} Z_i^k|\mathcal{F}_t\big] \;=\; Z_t^2-E\big[\sum_{k=2}^\infty \min_{i\in[1,T]} Z_i^k|\mathcal{F}_t\big].$$

By applying the above inductively, we find that for all $j\geq 1$ and $t\in[1,T]$, w.p.1

$$Z_t^1-\mathrm{MAR}_t = Z_t^j - E\big[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k|\mathcal{F}_t\big]. \tag{2}$$

By taking limits in (2), we now prove that w.p.1

$$Z_t^1-\mathrm{MAR}_t = Z_t^\infty \text{ for all } t\in[1,T]. \tag{3}$$

Indeed, we already know that $\{Z_t^j, j \geq 1\}$ converges a.s. to Z_t^∞ for all $t \in [1,T]$. Next, we claim that $\left\{E\left[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\right], j \geq 1\right\}$ converges a.s. to 0 for all $t \in [1,T]$. Since $\left\{\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k, j \geq 1\right\}$ is monotone and non-negative, and conditional expectation preserves almost sure dominance, it follows that $\left\{E\left[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\right], j \geq 1\right\}$ is monotone and non-negative. Hence by monotone convergence, $\left\{E\left[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\right], j \geq 1\right\}$ converges almost surely to a limiting non-negative r.v. Q^t, and $E[Q^t] = \lim_{j\to\infty} E\left[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k\right]$. Combining with Theorem 1, we conclude that $E[Q^t] = 0$, and thus by non-negativity $Q^t = 0$ w.p.1, completing the proof that $\left\{E\left[\sum_{k=j}^\infty \min_{i\in[1,T]} Z_i^k | \mathcal{F}_t\right], j \geq 1\right\}$ converges a.s. to 0 for all $t \in [1,T]$.

Taking limits (in j) in the right-hand-side of (2) then completes the proof of (3). Next, note that the basic properties of a.s. convergence, continuity of the min function, and a.s. convergence of $\{\mathbf{Z}^k, k \geq 1\}$ to $\mathbf{Z}^\infty$, imply that $\{\min_{i\in[1,T]} Z_i^k, k \geq 1\}$ converges a.s. to $\min_{i\in[1,T]} Z_i^\infty$. Combining with (1), we conclude that $\min_{i\in[1,T]} Z_i^\infty = 0$ w.p.1. Combining the above, it follows that $P\left(\min_{t\in[1,T]}\left(Z_t^1 - \mathrm{MAR}_t\right) = 0\right) = 1$. As Theorem 1 and the definition of **MAR** imply that $E[\mathrm{MAR}_1] = \mathrm{OPT}$, this completes the proof. *Q.E.D.*

3.3.3. Proof of Corollary 1.

Proof of Corollary 1 : The proof follows immediately from Lemmas 3 and 2, combined with optional stopping, and we omit the details. *Q.E.D.*

3.4. Optimization formulations for duality, non-negativity, and network flows

In this section we describe a novel connection between optimal stopping and max-flow problems. For an overview of max-flow problems and network optimization more generally, we refer the reader to Ahuja et al. (2014) and Christiano et al. (2011). Linear programming (LP) formulations for optimal stopping have been studied by several authors, and we refer the reader to the very relevant work Chen and Glasserman (2007), which connects LP duality to martingale duality, as well as other works such as Buchbinder et al. (2010). However, it seems that previous authors never made the leap from such LPs to even more structured max-flow problems. We do note that some relevant considerations of non-negativity were previously studied for the multiplicative form of the martingale dual in Jamshidian (2007), although no connection to max-flow was made.

3.4.1. Additional notations. For simplicity, we suppose (in this section only) that there is a finite (possibly very large) set $\mathcal{S}$ of D-dimensional vectors s.t. for all $t \in [1,T]$, and all D by t matrices γ one can form by drawing all columns from $\mathcal{S}$ (in an arbitrary manner, possibly with repetition), it holds that $P(Y_{[t]} = \gamma) > 0$, while $P(Y_{[t]} = \gamma') = 0$ for any $\gamma' \in \aleph^t$ which cannot be formed in this way. Letting $\overline{\aleph}^t$ denote the set of all D by t matrices γ one can form by drawing all columns from $\mathcal{S}$, this is equivalent to assuming $P(Y_{[t]} \in \overline{\aleph}^t) = 1$, and $P(Y_{[t]} = \gamma) > 0$ for all $\gamma \in \overline{\aleph}^t$. Note that our assumption further implies that for all $t \in [1, T-1], v \in \mathcal{S}$, and $\gamma \in \overline{\aleph}^t$,

$P(Y_{t+1}=v|Y_{[t]}=\gamma)>0$. Of course, these probabilities may be very different for different t,γ,s. We further assume that all $v\in\mathcal{S}$ have rational entries, and that $P(Y_{[t]}=\gamma)$ is rational for all $t\in[1,T]$ and $\gamma\in\overline{\aleph}^t$, to preclude certain pathologies when talking about flows. These conditions, although not strictly necessary, will simplify notations considerably. Furthermore, it follows from standard approximation arguments that these assumptions are essentially w.l.o.g. For any martingale M adapted to $\mathcal{F}$, any $t\in[1,T]$, and any $\gamma\in\overline{\aleph}^t$, let $M_t(\gamma)$ denote the value of the martingale at time t when $Y_{[t]}=\gamma$. For $t\in[1,T-1]$, $\gamma\in\overline{\aleph}^t$, and $v\in\mathcal{S}$, let $\gamma|v$ denote the element of $\overline{\aleph}^{t+1}$ s.t. $\gamma|v_{[t]}=\gamma,\gamma|v_{t+1}=v$, i.e. the matrix derived by appending v to the right of γ.

3.4.2. Optimization formulations and non-negativity. We begin by observing that the dual characterization for OPT given in Lemma 2 can be formulated as an optimization problem as follows. This follows from the basic definitions associated with martingales, and is generally known (Chen and Glasserman (2007)).

LEMMA 4. *OPT is equivalent to the value of the following optimization problem OPT1, with variables* $\{M_t(\gamma),t\in[1,T],\gamma\in\overline{\aleph}^t\}$.

$$\begin{aligned}
\max\quad & \textstyle\sum_{v\in\mathcal{S}} M_1(v)P(Y_1=v)\\
& M_t(\gamma)=\textstyle\sum_{v\in\mathcal{S}} M_{t+1}(\gamma|v)P\big(Y_{t+1}=v|Y_{[t]}=\gamma\big)\quad \textit{for all } t\in[1,T-1],\gamma\in\overline{\aleph}^t;\\
& M_t(\gamma)\le Z_t(\gamma)\quad \textit{for all } t\in[1,T],\gamma\in\overline{\aleph}^t;\\
& \textit{For all } \gamma\in\overline{\aleph}^T\textit{, there exists } t\in[1,T] \textit{ s.t. } M_t(\gamma_{[t]})=Z_t(\gamma_{[t]}).
\end{aligned}$$

However, what we believe prior works failed to do was fully utilize the fact that OPT1 has an optimal solution which is non-negative, where we note that most dual martingale solutions proposed previously in the literature would not necessarily be non-negative. Indeed, it follows from Lemma 3 that OPT1 has a non-negative optimal solution. Imposing such non-negativity, combining with a straightforward contradiction argument which then allows us to drop the final constraint (which is always binding in the optimal solution to any max-flow problem), and performing the transformation $F_t(\gamma)=M_t(\gamma)P(Y_{[t]}=\gamma)$, we arrive at the following reformulation.

LEMMA 5. *OPT is equivalent to the value of the following linear program LP2, with variables* $\{F_t(\gamma),t\in[1,T],\gamma\in\overline{\aleph}^t\}$.

$$\begin{aligned}
\max\quad & \textstyle\sum_{v\in\mathcal{S}} F_1(v)\\
& F_t(\gamma)=\textstyle\sum_{v\in\mathcal{S}} F_{t+1}(\gamma|v)\quad \textit{for all } t\in[1,T-1],\gamma\in\overline{\aleph}^t;\\
& 0\le F_t(\gamma)\le Z_t(\gamma)P(Y_{[t]}=\gamma)\quad \textit{for all } t\in[1,T],\gamma\in\overline{\aleph}^t.
\end{aligned}$$

3.4.3. Connection to max-flow min-cut. We now observe that LP2 is a standard max s-t flow problem. Indeed, consider a flow network N with source node s and sink node t, constructed as follows. For all $i\in[1,T]$ and $\gamma\in\overline{\aleph}^i$, there is a node n_γ, in addition to the source node s and sink

node t. For all $\gamma \in \overline{\aleph}^1$, there is an undirected edge (s, n_γ), i.e. between node s and node n_γ, with capacity $Z_1(\gamma)P(Y_{[1]} = \gamma)$. For all $\gamma \in \overline{\aleph}^T$, there is an undirected edge (n_γ, t) with capacity ∞. For all $i \in [1, T-1], \gamma \in \overline{\aleph}^i, v \in \mathcal{S}$, there is an undirected edge $(n_\gamma, n_{\gamma|v})$ with capacity $Z_{i+1}(\gamma|v)P(Y_{[i+1]} = \gamma|v)$. Then standard arguments from max-flow theory (the details of which we omit) yield the following.

LEMMA 6. *OPT is equivalent to the value of a maximum s-t flow in network N. Using the relation $F_t(\gamma) = M_t(\gamma)P(Y_{[t]} = \gamma)$, one may recover a 0-sure-optimal dual martingale from an optimal flow in network N, and visa-versa.*

Interestingly, max-flow min-cut then reveals a very natural interpretation of optimal stopping in terms of finding the minimum cut in N, where we note that there is indeed a straightforward bijection between minimal cuts in N and stopping times. Although a closely related observation was made in Chen and Glasserman (2007), it seems this explicit connection between optimal stopping and max-flow / min-cut is novel. We leave further exploring this connection, e.g. applying other algorithms from the well-established theory of max-flows to optimal stopping and options pricing, as an interesting direction for future research. In addition, we note that optimal stopping and options pricing thus provide another simple yet elegant testament to the power of network flows as a modeling tool and optimization framework.

N's special structure endows the underlying max-flow problem with some special features, which may be helpful in devising other flow-inspired algorithms for optimal stopping. For example, let us recall that in an s-t flow problem, a blocking flow is a feasible flow such that every s-t path has at least one saturated edge, i.e. edge whose flow equals the capacity of that edge. It is well-known that in a general max s-t flow problem, a blocking flow need not be optimal, although any optimal flow must be a blocking flow. However, in a tree network such as N, it is actually true that every blocking flow is optimal. The proof follows from a straightforward contradiction argument and is generally well-known, and we omit the details. This fact provides a different way to interpret several properties of 0-sure-optimal dual martingales which appear in previous works, which all boil down to the fact that if a certain process has a zero along every sample path then a certain martingale must be dual-optimal.

As max-flow is already a polynomial-time solvable problem, and the max-flow problem on N is even easier due to its tree structure, one can of course ask why path-dependent optimal stopping should be considered a hard problem. The answer is that even very fast max-flow algorithms typically focus on a runtime polynomial in the number of nodes and edges, which will grow exponentially in T for path-dependent optimal stopping. Our pure-dual representation and Theorem 1 can be interpreted as a very fast randomized algorithm for approximating such flow problems.

3.5. Sure-optimal dual and algorithmic speedup

In this section we comment on a stronger sense in which certain dual martingales may be optimal, note that our approach does not have this property, and conjecture that this may play an important role in allowing our method to yield approximate solutions so quickly. For $1 \le t_1 \le t_2 \le T$, let $\mathcal{T}^{t_1,t_2}$ denote the set of all integer-valued stopping times τ, adapted to $\mathcal{F}$, s.t. w.p.1 $t_1 \le \tau \le t_2$. Let us say that $M \in \mathcal{M}^0$ has the sure-optimal property (w.r.t. $\mathbf{Z}$) if: 1. $P\left(\min_{t\in[1,T]}\left(Z_t - M_t\right) = \mathrm{OPT}\right) = 1$; and 2. for all $i \in [1,T]$, $P\left(\min_{t\in[i,T]}\left(Z_t - M_t + M_i\right) = \inf_{\tau\in\mathcal{T}^{i,T}} E[Z_\tau|\mathcal{F}_i]\right) = 1$. We note that here we refer only to 0-mean martingales, as it simplifies the relevant discussions (and is w.l.o.g by appropriate translations and known results). As argued in Schoenmakers et al. (2013), every optimal stopping problem has such a sure-optimal dual solution, and this property may be helpful in algorithm design as it yields an optimality characterization in terms of certain suprema having zero variance. Schoenmakers et al. (2013) also notes that there are some dual approaches which yield 0-sure-optimal martingales, but not sure-optimal martingales, although most approaches indeed yield sure-optimal dual solutions. Interestingly, our approach does not necessarily yield a sure-optimal martingale, as we now show through a simple example, where we defer all associated proofs to the technical appendix.

Indeed, consider the setting in which the dimension $D = 1$, the horizon $T = 3$; $Y_1 = 0$ w.p.1; $Y_2 = 1$ w.p.1; Y_3 equals $\frac{1}{2}$ w.p. $\frac{1}{2}$, and equals 1 w.p. $\frac{1}{2}$; and the stopping problem is simply $\inf_{\tau\in\mathcal{T}} E[Y_\tau]$, i.e. $g_t(Y_{[t]}) = Y_t$ for all $t \in [1,3]$.

Lemma 7. *In this setting, the unique (0-mean) surely-optimal dual martingale M satisfies $M_1(0) = M_2(0,1) = 0, M_3(0,1,1) = \frac{1}{4}, M_3(0,1,\frac{1}{2}) = -\frac{1}{4}$. It follows that $P(M_1 = 0) = P(M_2 = 0) = 1$, while $P(M_3 = \frac{1}{4}) = P(M_3 = -\frac{1}{4}) = \frac{1}{2}$. However, the optimal dual martingale **MAR** derived from our approach satisfies $P(MAR_t = 0) = 1$ for all t.*

We suspect that the lack of sure-optimality may be fundamental in allowing our method to yield quick approximations. Intuitively, any method which yields a sure-optimal dual martingale must (in some sense) be solving all of the subproblems defined on all the subtrees of N. Our method of approximation is fast precisely because it does not have to do this. We leave formalizing this line of reasoning, and understanding associated lower bounds and the connection of our approach to other dual formulations in general, as an interesting direction for future research.

4. Rate of convergence and Proofs of Theorems 2 - 5

In this section we prove Theorems 2 - 5, our main results regarding rate of convergence. Along the way, we prove a much stronger path-wise convergence result, which will later enable us to construct provably good approximate policies, beyond the standard framework of approximate

value functions. We also prove that a slight modification of our approach converges rapidly in relative error, even when $\big(E[(Z_T)^2]\big)^{\frac{1}{2}}$ is much larger than OPT, and apply these results to several problems of interest in the optimal stopping literature - the i.i.d. setting and Robbins' problem. Finally, we provide several additional examples illustrating that our approach may converge much more quickly for certain instances.

4.1. Proof of Theorem 2

In this section we prove our main rate of convergence result Theorem 2. First, we prove a much stronger path-wise convergence result, essentially that after k iterations of our expansion, the minimum of every sample path is at most $\frac{1}{k+1}$. This is a much stronger statement than that given in Theorem 2, which only regards expectations. We begin by proving the following strong convergence result, whose proof is surprisingly simple.

LEMMA 8. *Suppose w.p.1 $Z_t \in [0,U]$ for all $t \in [1,T]$. Then for all $k \geq 1$, w.p.1 $\min_{t\in[1,T]} Z_t^k \leq \frac{U}{k}$.*

Proof : Note that by definitions, measurability, and Lemma 1, for all $k \geq 1$, $Z_T^{k+1} = Z_T - \sum_{i=1}^k \min_{t\in[1,T]} Z_t^i \geq 0$ w.p.1. By the monotonicity ensured by Lemma 1, it follows that w.p.1, $k \times \min_{t\in[1,T]} Z_t^k \leq Z_T \leq U$, and the desired result follows. *Q.E.D.*

Proof of Theorem 2 : By Lemma 1, $\text{OPT} = \sum_{i=1}^k E[\min_{t\in[1,T]} Z_t^i] + \inf_{\tau\in\mathcal{T}} E[Z_\tau^{k+1}]$. Letting τ_{k+1} denote the stopping time which stops the first time $Z_t^{k+1} \leq \frac{1}{k+1}$ (and stops at time T if no such time exists), Lemma 8 implies that $E[Z_{\tau_{k+1}}^{k+1}] \leq \frac{1}{k+1}$. It follows that $\inf_{\tau\in\mathcal{T}} E[Z_\tau^{k+1}] \leq \frac{1}{k+1}$, and combining the above completes the proof. *Q.E.D.*

4.2. Alternate bounds with prophet inequalities and Proof of Theorem 3

Some of the most celebrated results in optimal stopping involve the so-called prophet inequalities, relating $\inf_{\tau\in\mathcal{T}} E[Z_\tau]$ to $E[\min_{t\in[1,T]} Z_t]$ under various assumptions on $\mathbf{Z}$, including boundedness, independence, etc. We make no attempt to survey the vast literature on such inequalities here, instead referring the reader to the classical survey of Hill and Kertz (1992), and the more recent survey (with a more economics-oriented perspective) of Lucier (2017). We note that many of these results in the literature hold only under an independence assumption, which will not be well-suited to our purposes, as even if $\{Z_t, t \in [1,T]\}$ is i.i.d., $\{Z_t^2, t \in [1,T]\}$ will in general not be so. However, there are some notable exceptions, including the following result of Hill and Kertz (1983). Although originally stated in terms of maximization, we here state the corresponding version for minimization, which is easily derived from the original result of Hill and Kertz (1983), and we omit the details. Recall that for $x \in [0,1]$, $h_1(x) = (1-x)\log(\frac{1}{1-x})$; and for $k \geq 2$ and $x \in [0,1]$, $h_k(x) = h_1\big(h_{k-1}(x)\big)$.

LEMMA 9 **(Prophet inequality for bounded sequences (Hill and Kertz (1983)))**. *Suppose $P\big(Z_t \in [0,1]\big) = 1$ for all $t \in [1,T]$. Then $OPT - E[\min_{t\in[1,T]} Z_t] \leq h_1(OPT)$.*

4.2.1. Proof of Theorem 3.

Proof of Theorem 3 : We begin by stating some basic properties of h_1 which are easily verified by straightforward calculus arguments, and we omit the details. First, $h_1(x) \in [0,1]$ for all $x \in [0,1]$. Second, $h_1(x) \leq x$ for all $x \in [0,1]$, from which it follows by a straightforward induction argument that for each fixed $x \in [0,1]$, $\{h_k(x), k \geq 1\}$ is monotone decreasing. Third, h_1 is strictly increasing on $[0, 1-\exp(-1)]$. Fourth, $h_1(x) \leq \exp(-1)$ for all $x \in [0,1]$.

Next, we prove that for all $k \geq 1$, $\mathrm{OPT} - E_k \leq h_k(\mathrm{OPT})$, and proceed by induction. Since Lemma 1 implies that $\mathrm{OPT} = E_k + \inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}]$ for all $k \geq 1$, the desired result is equivalent to proving that, for all $k \geq 1$, $\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}] \leq h_k\big(\mathrm{OPT}\big)$. The base case $k = 1$ follows immediately from Lemma 1 and Lemma 9. Now, let us proceed by induction. Suppose the desired statement is true for some $k \geq 1$. Then from definitions, optional stopping, and Lemma 9, $\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+2}]$ equals $\inf_{\tau \in \mathcal{T}} E\big[Z_\tau^{k+1} - E[\min_{t \in [1,T]} Z_t^{k+1} | \mathcal{F}_\tau]\big]$, itself equal to

$$\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}] - E[\min_{t \in [1,T]} Z_t^{k+1}] \quad \leq \quad h_1\Big(\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}]\Big).$$

By our induction hypothesis, $\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}] \leq h_k\big(\mathrm{OPT}\big)$. As we have already shown that $h_k\big(\mathrm{OPT}\big) \leq h_1\big(\mathrm{OPT}\big) \leq \exp(-1) < 1 - \exp(-1)$, and that h_1 is increasing on $[0, 1-\exp(-1)]$, it follows that

$$h_1\Big(\inf_{\tau \in \mathcal{T}} E[Z_\tau^{k+1}]\Big) \quad \leq \quad h_1\Big(h_k\big(\mathrm{OPT}\big)\Big) \quad = \quad h_{k+1}(\mathrm{OPT}).$$

Combining the above completes the induction, and further combining with some straightforward algebra completes the proof of the theorem. *Q.E.D.*

We note that unfortunately, it can be shown that $h_k(x)$ does not in general (i.e. if one takes the worst-case over all $x \in [0,1]$) yield a better bound than Theorem 2 as $k \to \infty$. However, it may yield strong bounds for particular values of OPT. For example, if OPT is close to 1, then Theorem 3 implies that our approach has small error after even a single iteration. More generally, Lemma 9 can also be used to prove alternative bounds on both $\mathrm{OPT} - E_k$ and how much progress is made during the kth iteration of our expansion, e.g. by using the fact (which follows from a straightforward Taylor expansion) that $h_1(x) \sim x - \frac{x^2}{2}$ as $x \downarrow 0$, although we do not pursue such an analysis here.

4.3. Alternate bounds when Z is unbounded and Proof of Theorem 4

We now complete the proof of Theorem 4, which provides a bound on the rate of convergence which is completely general, i.e. neither requiring normalization, nor rescaling of the absolute error by any upper bound, nor even the existence of any upper bound.

Proof of Theorem 4 : First, we claim that for all $k \geq 1$,

$$E[\min_{t\in[1,T]} Z_t^k] \leq \frac{1}{k} \times \text{OPT}. \tag{4}$$

Indeed, this follows directly from monotonicity and Lemma 1, which also implies (by some straightforward algebra) that to prove the overall theorem, it suffices to prove that

$$\inf_{\tau\in\mathcal{T}} E[Z_\tau^{k+1}] \leq 2 \times \left(\text{OPT} \times E[(Z_T)^2]\right)^{\frac{1}{3}} \times \frac{1}{k^{\frac{1}{3}}}. \tag{5}$$

To proceed, let us consider the performance of the following threshold policy. Let $x_k \triangleq \left(E[(Z_T)^2] \times \text{OPT} \times k^{-1}\right)^{\frac{1}{3}}$. Consider the policy which stops the first time that $Z_t^{k+1} \leq x_k$, and simply stops at time T if no such t exists in $[1,T]$. Denoting this stopping time by τ_k, note that w.p.1 $Z_{\tau_k}^{k+1} \leq x_k + I\left(\min_{t\in[1,T]} Z_t^{k+1} > x_k\right) Z_T^{k+1}$, which by monotonicity (implying $Z_T^{k+1} \leq Z_T$ w.p.1) is at most $x_k + I\left(\min_{t\in[1,T]} Z_t^{k+1} > x_k\right) Z_T$. Taking expectations and applying Cauchy-Schwartz, we find that $E[Z_{\tau_k}^{k+1}]$ is at most $x_k + \left(E[(Z_T)^2]\right)^{\frac{1}{2}} \times \left(P\left(\min_{t\in[1,T]} Z_t^{k+1} > x_k\right)\right)^{\frac{1}{2}}$. Further applying Markov's inequality and (4), we conclude that $E[Z_{\tau_k}^{k+1}]$ is at most $x_k + \left(\frac{\text{OPT}\times k^{-1}\times E[(Z_T)^2]}{x_k}\right)^{\frac{1}{2}} = 2x_k$, completing the proof. *Q.E.D.*

4.4. Alternate bounds for the i.i.d. setting and Robbins' problem

In some cases, it may be desirable to have guarantees not just on the absolute error, but on the relative error. Namely, one might hope that after a few rounds, the error is at most a small fraction of OPT itself. Theorem 4 yields such a result so long as $\left(E[(Z_T)^2]\right)^{\frac{1}{2}}$ is not too many times larger than OPT. However, many particular theoretical stopping problems studied intensely in the literature, e.g. the setting in which $\mathbf{Z}$ is i.i.d. from a uniform distribution on [0,1] (i.e. U[0,1]), or the celebrated Robbins' problem (Bruss (2005)), have the feature that when T is large, $\left(E[(Z_T)^2]\right)^{\frac{1}{2}}$ is many times larger than OPT. In such a setting, the bounds of Theorems 2 - 4 may require k to be very large (scaling with T) before achieving small relative error.

We now show that a small modification of our approach can overcome this. We note that although we suspect that our approach, unmodified, also achieves good performance w.r.t. relative error for such problems (not captured by our proven bounds), we leave this as an interesting open question. Let $\mathbf{Y}^1 = \{Y_i^1, i \geq 1\}$ denote an i.i.d. sequence of U[0,1] r.v.s. For $T \geq 1$ and $t \in [1,T]$, let $g_t^U(Y_{[t]}^1) \triangleq Y_t^1$; and $g_t^R(Y_{[t]}^1) \triangleq \sum_{i=1}^t I(Y_i^1 \leq Y_t^1) + (T-t)Y_t^1$. Then it is easily verified that the problem of optimal stopping for an i.i.d. U[0,1] sequence of r.v.s is equivalent to the problem $\text{OPT}_U(T) \triangleq \inf_{\tau\in\mathcal{T}} E\left[g_\tau^U(Y_{[\tau]}^1)\right]$. We note that as this problem is classical and its solution well-understood (Gilbert and Mosteller (1966), Kennedy and Kertz (1991)), e.g. it is known that $\lim_{T\to\infty} T \times \text{OPT}_U(T) = 2$, in this case our results only illustrate the adaptability of our framework.

It may be similarly verified (Bruss (2005)) that the celebrated Robbins' problem is equivalent to the problem $\mathrm{OPT}_R(T) \triangleq \inf_{\tau\in\mathcal{T}} E\big[g_\tau^R(Y_{[\tau]}^1)\big]$. However, for this problem much less is known, and furthering our understanding of the value of $\mathrm{OPT}_R(T)$ and the nature of an (approximately) optimal policy remain open problems of considerable recent interest (Bruss (2005), Dendievel et al. (2016), Gnedin and Iksanov (2011), Meier et al. (2017)). For example, although it is known that $\lim_{T\to\infty}\mathrm{OPT}_R(T)$ exists, the exact limiting value remains an open problem. One of the aspects that makes Robbins' problem so difficult is that it exhibits full history-dependence, even in the limit as $T\to\infty$ (Bruss (2005)). Of course, for our approach such a dependence is not a problem.

We proceed as follows. Observe that since Z_T may be much larger than OPT, it does not suffice to simply stop at time T if one has gotten unlucky and $\min_{t\in[1,T]} Z_t^k$ was larger than expected. Instead, we use the fact that for both the i.i.d. stopping of U[0,1] r.v.s and Robbins' problem, for any fixed $\eta\in(0,1)$, if T is large (for the fixed η), then it is very likely that there exists $t\in[(1-\eta)T,T]$ such that Z_t is not too many times larger than OPT (here "too many" will be some function of η), and furthermore that some stopping time (over this interval) can nearly achieve this in expectation. We thus proceed by applying a modified expansion in which we only take the minimum over the first $(1-\eta)T$ time periods, leaving us enough time to self-correct if we get unlucky. The result is then proven by taking a double-limit in which η is set to an appropriate function of the desired accuracy ϵ, and several additional bounds are applied. We note that our results for the i.i.d. setting and Robbins' problem follow from a more general bound we prove in Lemma 20 of the technical appendix, which may be useful for other such stopping problems.

More formally, let us define a modified expansion as follows. For $\eta\in(0,1)$ and $t\in[1,T]$, let $Z_{\eta,t}^1 \triangleq Z_t^1$. For $k\geq 1, \eta\in(0,1), t\in[1,T]$, let $Z_{\eta,t}^{k+1} \triangleq Z_{\eta,t}^k - E\big[\min_{i\in[1,\lceil(1-\eta)T\rceil]} Z_{\eta,i}^k|\mathcal{F}_t\big]$. Then the corresponding convergence result is as follows. Let $H_k(\eta)\triangleq E[\min_{1\leq t\leq\lceil(1-\eta)T\rceil} Z_{\eta,t}^k]$, and $E_k(\eta)\triangleq \sum_{i=1}^k H_i(\eta)$.

THEOREM 7. *There exists an absolute strictly positive finite constant C_0, independent of anything else, s.t. for all $T\geq 1$ and $k\geq 1$, $\big|OPT_U(T)-E_k\big((k+1)^{-\frac{1}{5}}\big)\big|\leq C_0\times(k+1)^{-\frac{1}{5}}\times OPT_U(T)$; and $\big|OPT_R(T)-E_k\big((k+1)^{-\frac{1}{5}}\big)\big|\leq C_0\times(k+1)^{-\frac{1}{5}}\times OPT_R(T)$.*

We defer the proof of Theorem 7 to the technical appendix. The important aspect of Theorem 7 is that the number of terms needed in the expansion to obtain a good relative error is independent of the time horizon T. We believe a similar approach can be applied to other optimal stopping problems from the literature, and that stronger convergence rates can be demonstrated, which we leave as interesting directions for future research.

4.5. Additional illustrative examples and proof of lower bound Theorem 5

In this section, we provide some examples which illustrate that the rate of convergence may be much faster than that proven in Theorem 2. Our examples also illustrate that even for toy problems our expansion leads to non-trivial dynamics, and we leave the question of developing a deeper understanding of these dynamics as an interesting direction for future research.

4.5.1. Example of convergence in one iteration. Consider the setting in which the horizon length T is general, $P(Z_t \in \{0,1\}) = 1$ for all $t \in [1,T]$, and otherwise the joint distribution of $\mathbf{Z}$ is general. Namely, w.p.1 Z_t is either 0 or 1 for all $t \in [1,T]$.

LEMMA 10. *In this setting, $E_1 = OPT$.*

Proof : There are many ways to see this. For example, it follows from a straightforward contradiction that an optimal strategy is stop at the first time t s.t $.Z_t = 0$, and stop at time T otherwise. As such, $\text{OPT} = P\big(\min_{t\in[1,T]} Z_t = 1\big) = E_1$. *Q.E.D.*

We note that convergence also occurs in one iteration whenever $T = 1$, or each Z_t has zero variance.

4.5.2. Example of fast and slow exponential convergence and Proof of Theorem 5. For general $n \geq 1$, consider the setting in which $T = 2, D = 1, Z_t = Y_t$ for $t \in [1,2]$, and $P(Y_1 = \frac{1}{n}) = 1$, $P(Y_2 = 1) = \frac{1}{n}, P(Y_2 = 0) = 1 - \frac{1}{n}$.

LEMMA 11. *In this setting, for all $k \geq 1$, $OPT - E_k = \frac{1}{n} \times (1 - \frac{1}{n})^k$.*

Proof : First, we note that by a straightforward induction, $Var[Z_1^k] = 0$ for all $k \geq 1$, i.e. Z_1^k is always w.p.1 a constant (possibly depending on k). As $\mathbf{Z}$ is a martingale, it follows from optional stopping that $\text{OPT} = \frac{1}{n}$. It then follows from definitions and the basic preservation properties of the martingale property that $\mathbf{Z}^k$ is a martingale for all $k \geq 1$, and thus by optional stopping $\inf_{\tau\in\mathcal{T}} E\big[Z_\tau^{k+1}\big] = E\big[Z_1^{k+1}\big]$ for all $k \geq 1$. Thus by Lemma 1, to prove the desired result it suffices to prove that $E\big[Z_1^k\big] = \frac{1}{n} \times (1 - \frac{1}{n})^{k-1}$ for all $k \geq 1$. As Z_1^k is always some constant, it thus suffices to prove by induction that $P\big(Z_1^k = \frac{1}{n} \times (1 - \frac{1}{n})^{k-1}\big) = 1$ for all $k \geq 1$. The base case $k = 1$ is trivial. Now, suppose the induction holds for some $k \geq 1$. Using the martingale property and the inductive hypothesis, it follows that $Z_1^k = \frac{1}{n} \times (1 - \frac{1}{n})^{k-1}$, and Z_2^k equals $(1 - \frac{1}{n})^{k-1}$ w.p. $\frac{1}{n}$, and 0 w.p. $1 - \frac{1}{n}$. It follows that Z_1^{k+1} equals

$$Z_1^k - E\big[\min_{t\in[1,2]} Z_t^k\big] \quad = \quad \frac{1}{n}(1 - \frac{1}{n})^{k-1} - (\frac{1}{n})^2(1 - \frac{1}{n})^{k-1} \quad = \quad \frac{1}{n}(1 - \frac{1}{n})^k,$$

completing the proof. *Q.E.D.*

Note that if n is small, e.g. 2, then Lemma 11 indicates rapid exponential convergence. However, if n is large, Lemma 11 indicates an exponential rate of convergence but with very small rate. We now use this insight to complete the proof of Theorem 5.

Proof of Theorem 5 : Noting that for any $n \geq 2$, $(1-\frac{1}{n})^k \geq \frac{1}{4}$ for all $k \leq n$ completes the proof. *Q.E.D.*

4.5.3. A variety of 2-period examples. We now provide a variety of simple 2-period examples. In all cases, the r.v. in the first period is a constant, and the r.v. in the second period is either an exponential distribution or a uniform distribution. Interestingly, we find that even in this simple setting a variety of non-trivial behaviors are possible. For two r.v.s X_1, X_2, let $X_1 =_d X_2$ denote equivalence in distribution. Let X be an Expo(1) r.v., i.e. an exponentially distributed r.v. with mean 1; and for $p \in [0,1]$, let $B(p)$ denote a bernoulli r.v. with success probability p, i.e. $P\big(B(p)=1\big) = p = 1 - P\big(B(p)=0\big)$. Further suppose $B(p)$ is independent of X. For $x > 0$, let $U(x)$ denote a r.v. distributed uniformly on $[0,x]$, with $\{U(x), x > 0\}, \{B(p), p \in [0,1]\}$ mutually independent. We defer all proofs to the technical appendix.

Exponential distribution : balanced means. Consider the setting in which $T = 2, D = 1, Z_t = Y_t$ for $t \in [1,2]$, $P(Y_1 = 1) = 1$, and $Y_2 =_d X$.

LEMMA 12. *In this setting,* $\lim_{k\to\infty} k \times (OPT - E_k) = 1$.

Note that although this setting falls beyond the scope of Theorem 2, as the r.v.s are unbounded, the rate of convergence is still $\Theta(\frac{1}{k})$.

Exponential distribution : unbalanced means. Consider the setting in which $T = 2, D = 1, Z_t = Y_t$ for $t \in [1,2]$, $P(Y_1 = \frac{1}{2}) = 1$, and $Y_2 =_d X$.

LEMMA 13. *In this setting,* $\lim_{k\to\infty} k^{-1} \times \log(OPT - E_k) = -\log(2)$.

Interestingly, the rate of convergence is very different in the case of balanced and unbalanced means, as Lemma 13 indicates that for any $\epsilon > 0$ and all sufficiently large k, $\mathrm{OPT} - E_k < \exp\big(-(\log(2) - \epsilon)k\big)$.

Uniform distribution : balanced means. Consider the setting in which $T = 2, D = 1, Z_t = Y_t$ for $t \in [1,2]$, $P(Y_1 = 1) = 1$, and $Y_2 =_d U(2)$.

LEMMA 14. *In this setting,* $\lim_{k\to\infty} k^2 \times (OPT - E_k) = 4$.

Interestingly, the rate of convergence in this setting is $\Theta(\frac{1}{k^2})$, which is faster than the worst-case $O(\frac{1}{k})$ convergence proven in Theorem 2, as well as the $\Theta(\frac{1}{k})$ convergence when Z_2^1 is exponentially distributed (with balanced means).

5. Simulation analysis, and proof of Theorem 6 and Corollaries 2 - 3

In this section, we complete the proof of our algorithm analysis, Theorem 6 and Corollary 2. We then combine with our convergence results to develop efficient stopping strategies with similar performance guarantees, completing the proof of Corollary 3.

5.1. Efficient randomized algorithm for computing Z_t^k

Recall that for each $k \geq 1$ and $t \in [1,T]$, Z_t^k can be thought of as a deterministic function from $\aleph^t$ to $\mathcal{R}^+$ (under our aforementioned caveat that all r.v.s are sufficiently non-pathological to ensure that all relevant conditionings are appropriately well-defined). We now formalize the fact that by exploiting the recursive definitions, we can build a "good" algorithm for approximating $Z_t^{k+1}(\gamma)$ given "good" algorithms for approximating $Z_t^k(\gamma')$ for all relevant γ'. We note that as our algorithms only ever attempt to compute $Z_t^k(\gamma)$ for γ actually output by the simulator, it will be implicit that our methods never attempt to compute ill-defined quantities. For $\epsilon, \delta \in (0,1)$, let $N(\epsilon,\delta) \triangleq \lceil \frac{1}{2\epsilon^2} \log(\frac{2}{\delta}) \rceil$.

5.1.1. Formal definition of algorithms. We now recursively define the relevant sequence of algorithms, with algorithm $\mathcal{B}^k(t,\gamma,\epsilon,\delta)$ returning an (additive) ϵ-approximation to $Z_t^k(\gamma)$ w.p. at least $1-\delta$ (we will soon make this completely precise). Recall that for $t \in [1,T]$ and $\gamma \in \aleph^t$, $\mathbf{Y}(\gamma)$ denotes a random matrix distributed as $\mathbf{Y}$, conditioned on the event $\{Y_{[t]} = \gamma\}$. We also recall that the base simulator $\mathcal{B}$ is a randomized algorithm that takes as input $t \in [1,T]$ and $\gamma \in \aleph^t$, and outputs an independent sample of $\mathbf{Y}(\gamma)$. Furthermore, recall that for $t \in [1,T]$ and $\gamma \in \aleph^t$, $g_t(\gamma)$ is equivalent to the value of Z_t conditioned on the event $\{Y_{[t]} = \gamma\}$. For a D by T matrix M, and $t \in [1,T]$, let $M_{[t]}$ denote the submatrix consisting of the first t columns of M.

As a notational convenience, it will be helpful to first define an algorithm $\mathcal{B}^1$, which also takes inputs t,γ,ϵ,δ, although we note that formally this is redundant as the algorithm will simply return $g_t(\gamma)$ which will be exactly the correct value.

Algorithm $\mathcal{B}^1(t,\gamma,\epsilon,\delta)$:

Return $g_t(\gamma)$

For $k \geq 1$, we define $\mathcal{B}^{k+1}$ inductively as follows.

Algorithm $\mathcal{B}^{k+1}(t, \gamma, \epsilon, \delta)$:

Create a length-$N(\frac{\epsilon}{4}, \frac{\delta}{4})$ vector $\mathbf{A}^0$

For i = 1 to $N(\frac{\epsilon}{4}, \frac{\delta}{4})$

Generate an ind. call to $\mathcal{B}(t, \gamma)$ and store in D by T matrix $\mathbf{A}^1$

Create a length-T vector $\mathbf{A}^2$

For j = 1 to T

Generate an ind. call to $\mathcal{B}^k\big(j, \mathbf{A}^1_{[j]}, \frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4}, \frac{\delta}{4})T}\big)$ and store in A^2_j

Compute the minimum value of $\mathbf{A}^2$ and store in A^0_i

Generate an ind. call to $\mathcal{B}^k(t, \gamma, \frac{\epsilon}{2}, \frac{\delta}{2})$ and store as variable A^3

Return $A^3 - \big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} A^0_i$

5.1.2. Formal analysis of $\mathcal{B}^k$. We now formally analyze $\mathcal{B}^k$, proving in an appropriate sense that it is indeed a "good" algorithm for approximating $Z^{k+1}_t(\gamma)$. First, we recall a standard result from probability theory used often to prove concentration for estimators.

LEMMA 15 **(Hoeffding's inequality)**. *Suppose that for some $n \geq 1$ and $U > 0$, $\{X_i, i \in [1, n]\}$ are i.i.d., and $P(X_1 \in [0, U]) = 1$. Then $\mathbb{P}\left(\left|n^{-1}\sum_{i=1}^n X_i - E[X_1]\right| \geq \eta\right) \leq 2\exp\left(-\frac{2\eta^2 n}{U^2}\right)$.*

Recall that

$$f_k(\epsilon, \delta) = 10^{2(k-1)^2} \times \epsilon^{-2(k-1)} \times (T+2)^{k-1} \times \big(1 + \log(\frac{1}{\delta}) + \log(\frac{1}{\epsilon}) + \log(T)\big)^{k-1}.$$

We will also need the following auxiliary lemma, which demonstrates that f_k satisfies certain recursions corresponding to our algorithms' performance, and whose proof we defer to the technical appendix.

LEMMA 16. *For all $\epsilon, \delta \in (0, 1)$ and $k \geq 1$,*

$$f_{k+1}(\epsilon, \delta) \geq \big(N(\frac{\epsilon}{4}, \frac{\delta}{4}) + 1\big) \times (T+2) \times f_k\big(\frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4}, \frac{\delta}{4})T}\big).$$

With Lemmas 15 and 16 in hand, we now prove the following result, which certifies that $\mathcal{B}^k$ is indeed a "good" algorithm.

LEMMA 17. *For all $k \geq 1$, $t \in [1, T]$, $\gamma \in \aleph^t$, $\epsilon, \delta \in (0, 1)$, algorithm $\mathcal{B}^k$ achieves the following when evaluated on $t, \gamma, \epsilon, \delta$. In total computational time at most $(C + G + 1) f_k(\epsilon, \delta)$, and with only access to randomness at most $f_k(\epsilon, \delta)$ calls to the base simulator $\mathcal{B}$, returns a random number X satisfying $P\big(|X - Z^k_t(\gamma)| \geq \epsilon\big) \leq \delta$.*

Proof : We proceed by induction. The base case $k=1$ is trivial. Now, suppose the induction is true for some $k \geq 1$. We first prove that $\mathcal{B}^{k+1}$ satisfies the desired probabilistic error bounds. Let $\{X_i, i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]\}$ be an i.i.d. sequence of r.v.s, each distributed as $\min_{i\in[1,T]} Z_i^k(\mathbf{Y}(\gamma)_{[i]})$, where the same realization of $\mathbf{Y}(\gamma)$ is used for all $i \in [1,T]$. Then it follows from our inductive hypothesis, the Lipschitz property of the min function, a union bound over all $i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]$ and $j \in [1,T]$, and some straightforward algebra, that we may construct $\{X_i, i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]\}$ and $\{A_i^0, i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]\}$ on a common probability space s.t. with probability at least $1-\frac{\delta}{4}$, $|X_i - A_i^0| < \frac{\epsilon}{4}$ for all $i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]$. Applying Lemma 15 to $\{X_i, i \in [1, N(\frac{\epsilon}{4}, \frac{\delta}{4})]\}$, with parameters $\eta = \frac{\epsilon}{4}, U = 1, n = N(\frac{\epsilon}{4}, \frac{\delta}{4})$, we conclude (after some straightforward algebra) that on the same probability space,

$$P\Big(\big|\big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} X_i - E[X_1]\big| < \frac{\epsilon}{4}\Big) \geq 1 - \frac{\delta}{4}.$$

Here we note that in the above we apply Lemma 15 with $U=1$, since $X_i \in [0,1]$ for all $i \geq 1$. Noting that the event $\Big\{|X_i - A_i^0| < \frac{\epsilon}{4} \text{ for all i}\Big\}$ implies the event

$$\Big\{\Big|\big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} A_i^0 - \big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} X_i\Big| < \frac{\epsilon}{4}\Big\},$$

we may combine the above with a union bound and the triangle inequality to conclude that on the same probability space (and hence in general),

$$P\Big(\Big|\big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} A_i^0 - E[X_1]\Big| < \frac{\epsilon}{2}\Big) \geq 1 - \frac{\delta}{2}.$$

As the inductive hypothesis ensures that $P\big(\big|A^3 - Z_t^k(\gamma)\big| \geq \frac{\epsilon}{2}\big) \leq \frac{\delta}{2}$, we may again apply a union bound and the triangle inequality, along with the definition of $Z_t^k(\gamma)$ and X_1, to conclude that

$$P\Big(\Big|A^3 - \big(N(\frac{\epsilon}{4}, \frac{\delta}{4})\big)^{-1} \sum_{i=1}^{N(\frac{\epsilon}{4}, \frac{\delta}{4})} A_i^0 - Z_t^{k+1}(\gamma)\Big| \geq \epsilon\Big) \leq \delta \quad \text{as desired.} \tag{6}$$

We next prove that $\mathcal{B}^{k+1}$ satisfies the desired computational and sample complexity bounds. The only access to randomness for $\mathcal{B}^{k+1}$ is through its $N(\frac{\epsilon}{4}, \frac{\delta}{4})$ direct calls to $\mathcal{B}(t,\gamma)$ (whose output is each time stored in $\mathbf{A}^1$), its $N(\frac{\epsilon}{4}, \frac{\delta}{4})T$ calls to $\mathcal{B}^k\big(j, A^1_{[j]}, \frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4}, \frac{\delta}{4})T}\big)$, and its one final call to $\mathcal{B}^k(t, \gamma, \frac{\epsilon}{2}, \frac{\delta}{2})$ (whose output is stored in A^3). It thus follows from the inductive hypothesis, and several easily verified monotonicities of N and f_k, that the number of calls to the base simulator made by $\mathcal{B}^{k+1}(t,\gamma,\epsilon,\delta)$ is at most

$$\begin{aligned} & N(\frac{\epsilon}{4}, \frac{\delta}{4}) + N(\frac{\epsilon}{4}, \frac{\delta}{4}) \times T \times f_k\big(\frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4}, \frac{\delta}{4})T}\big) + f_k(\frac{\epsilon}{2}, \frac{\delta}{2}) \\ \leq \quad & N(\frac{\epsilon}{4}, \frac{\delta}{4}) \times (T+2) \times f_k\big(\frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4}, \frac{\delta}{4})T}\big). \end{aligned} \tag{7}$$

We next focus on computational costs. In each of the $N(\frac{\epsilon}{4},\frac{\delta}{4})$ iterations of the outer for loop (indexed by i), first one direct call is made to $\mathcal{B}(t,\gamma)$ (at computational cost C); then T calls are made to $\mathcal{B}^k\big(j, A^1_{[j]}, \frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T}\big)$ (for different values of j), each at computational cost at most $(C+G+1)\times f_k\big(\frac{\epsilon}{4}, \frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T}\big)$; then the minimum of a length-T vector is computed (at computational cost T). One additional call is then made to $\mathcal{B}^k(t,\gamma,\frac{\epsilon}{2},\frac{\delta}{2})$, at computational cost at most $(C+G+1)\times f_k(\frac{\epsilon}{2},\frac{\delta}{2})$; and finally the average of the $N(\frac{\epsilon}{4},\frac{\delta}{4})$ elements of $\mathbf{A}^0$ is computed and subtracted from A^3, at computational cost $N(\frac{\epsilon}{4},\frac{\delta}{4})+1$. It thus follows from the inductive hypothesis, and several easily verified monotonicities of N and f_k, that the computational cost of $\mathcal{B}^{k+1}(t,\gamma,\epsilon,\delta)$ is at most

$$\begin{aligned} & N(\frac{\epsilon}{4},\frac{\delta}{4})C + N(\frac{\epsilon}{4},\frac{\delta}{4})T(C+G+1)f_k\big(\frac{\epsilon}{4},\frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T}\big) \\ & + \; T + (C+G+1)f_k(\frac{\epsilon}{2},\frac{\delta}{2}) + N(\frac{\epsilon}{4},\frac{\delta}{4}) + 1 \\ & \leq \quad (C+G+1)\times\big(N(\frac{\epsilon}{4},\frac{\delta}{4})+1\big)\times(T+2)\times f_k\big(\frac{\epsilon}{4},\frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T}\big). \end{aligned} \tag{8}$$

Combining with Lemma 16 completes the proof. *Q.E.D.*

5.2. Proof of Theorem 6 and Corollary 2

With Lemma 17 in hand, we now complete the proof of our main algorithmic result Theorem 6, and Corollary 2. First, let us formally define the associated algorithm $\hat{\mathcal{B}}^k$, which will use $\mathcal{B}^k$ and simulation to approximate H_k.

Algorithm $\hat{\mathcal{B}}^k(\epsilon,\delta)$:

Create a length-$N(\frac{\epsilon}{2},\frac{\delta}{2})$ vector $\mathbf{A}^0$
For i = 1 to $N(\frac{\epsilon}{2},\frac{\delta}{2})$
  Generate an ind. call to $\mathcal{B}(0,\emptyset)$ and store in D by T matrix $\mathbf{A}^1$
  Create a length-T vector $\mathbf{A}^2$
  For j = 1 to T
    Generate an ind. call to $\mathcal{B}^k\big(j, A^1_{[j]}, \frac{\epsilon}{2}, \frac{\delta}{2N(\frac{\epsilon}{2},\frac{\delta}{2})T}\big)$ and store in A^2_j
  Compute the minimum value of $\mathbf{A}^2$ and store in A^0_i
Return $\big(N(\frac{\epsilon}{2},\frac{\delta}{2})\big)^{-1}\sum_{i=1}^{N(\frac{\epsilon}{2},\frac{\delta}{2})} A^0_i$

Proof of Theorem 6 : In light of Lemma 17, the result follows from a union bound, an application of Lemmas 15 and 16 nearly identical to that used in our previous proofs, and some straightforward algebra in which we bound $N(\frac{\epsilon}{2},\frac{\delta}{2})$ by $N(\frac{\epsilon}{4},\frac{\delta}{4})$, and bound $f_k(\frac{\epsilon}{2},\frac{\delta}{2N(\frac{\epsilon}{2},\frac{\delta}{2})T})$ by $f_k(\frac{\epsilon}{4},\frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T})$, and we omit the details. The accounting for computational and sample complexity is also nearly identical, and we similarly omit the details. *Q.E.D.*

We defer the proof of Corollary 2 to the technical appendix.

5.3. Proof of Corollary 3

In this section, we discuss how to use our simulation-based approach to implement good approximate stopping strategies, proving Corollary 3. We begin with the following lemma, relating the value achieved by a single stopping strategy across different stopping problems (defined by $\mathbf{Z}^k$).

LEMMA 18. *For all (possibly randomized) integer-valued stopping times τ adapted to $\mathcal{F}$ which w.p.1 belong to $[1,T]$, and all $k \geq 1$, $E[Z_\tau] = E[Z_\tau^k] + \sum_{i=1}^{k-1} E[\min_{t\in[1,T]} Z_t^i]$.*

Proof : We prove only for the case of non-randomized stopping times, as the general setting then follows from a straightforward conditioning argument. We proceed by induction. The base case $k=1$ follows from definitions. Now, suppose the induction is true for some $k \geq 1$. Then again from definitions and optional stopping,

$$E[Z_\tau^{k+1}] \quad = \quad E\big[Z_\tau^k - E[\min_{t\in[1,T]} Z_t^k | \mathcal{F}_\tau]\big] \quad = \quad E[Z_\tau^k] - E[\min_{t\in[1,T]} Z_t^k],$$

itself (by induction) equal to $E[Z_\tau] - \sum_{i=1}^{k} E[\min_{t\in[1,T]} Z_t^i]$, which after rearranging completes the proof. *Q.E.D.*

Combining Lemma 18 with Lemma 8 and Theorem 1, we are led to the following corollary.

COROLLARY 4. *For $k \geq 1$, let τ_k denote the stopping time that stops the first time that $Z_t^k \leq \frac{1}{k}$, where we note that by Lemma 8 such a time exists w.p.1. Then $E[Z_{\tau_k}] - OPT \leq \frac{1}{k}$.*

Now, we would be done, if not for the fact that we cannot compute Z_t^k exactly in an efficient manner. However, in light of Lemma 17, it is clear how to proceed. In every time period t, we will use simulation to estimate $Z_t^k(Y_{[t]})$ (for appropriate k) for the given history $Y_{[t]}$ observed so far, and do so with sufficient accuracy and high enough probability to make sure all bounds go through. Let us now make this precise. For any given $\epsilon > 0$, we begin by defining an appropriate (randomized) stopping time τ_ϵ. Namely, τ_ϵ is defined as follows. At time 1, after seeing $Y_{[1]}$, make an independent call to $\mathcal{B}^{\lceil 4\epsilon^{-1}\rceil}\left(1, Y_{[1]}, \frac{\epsilon}{4}, \frac{\epsilon}{4T}\right)$. If the value returned is at most $\frac{1}{2}\epsilon$, stop. If not, continue. We define the future behavior inductively as follows. Suppose that for some $t \in [1, T-2]$, we have not yet stopped by the end of period t. At time $t+1$, after observing Y_{t+1}, make an independent call to $\mathcal{B}^{\lceil 4\epsilon^{-1}\rceil}\left(t+1, Y_{[t+1]}, \frac{\epsilon}{4}, \frac{\epsilon}{4T}\right)$. If the value returned is at most $\frac{1}{2}\epsilon$, stop. If not, continue. Finally, if we have not yet stopped by period T, stop in period T. It is easily verified that for any $\epsilon \in (0,1)$, τ_ϵ is a well-defined, appropriately adapted, randomized stopping time. We now use τ_ϵ to complete the proof of Corollary 3.

Proof of Corollary 3 : Let $k_\epsilon \triangleq \lceil \frac{4}{\epsilon} \rceil$. Let $\mathcal{G}_{1,\epsilon}$ denote the event

$$\left\{ \left| \mathcal{B}^{\lceil 4\epsilon^{-1} \rceil}\left(t, Y_{[t]}, \frac{\epsilon}{4}, \frac{\epsilon}{4T}\right) - Z_t^{k_\epsilon}(Y_{[t]}) \right| \leq \frac{\epsilon}{4} \quad \forall \ \ t \in [1,T] \right\};$$

$\mathcal{G}_{2,\epsilon}$ denote the event

$$\left\{ \exists \ t \ \in [1,T] \text{ such that } \mathcal{B}^{\lceil 4\epsilon^{-1} \rceil}\left(t, Y_{[t]}, \frac{\epsilon}{4}, \frac{\epsilon}{4T}\right) \leq \frac{1}{2}\epsilon \right\};$$

and $\mathcal{G}_{3,\epsilon}$ denote the event $\left\{ Z_{\tau_\epsilon}^{k_\epsilon} \leq \frac{3}{4}\epsilon \right\}$. Observe that Lemma 8, Lemma 17, definitions, and several straightforward union bounds and applications of the triangle inequality ensure that : 1. $P(\mathcal{G}_{1,\epsilon}) \geq 1 - \frac{\epsilon}{4}$; 2. $P(\mathcal{G}_{2,\epsilon} | \mathcal{G}_{1,\epsilon}) = 1$; and 3. $P(\mathcal{G}_{3,\epsilon} | \mathcal{G}_{1,\epsilon} \bigcap \mathcal{G}_{2,\epsilon}) = 1$. It follows that $\left(\mathcal{G}_{3,\epsilon}^c\right) \leq \frac{\epsilon}{4}$, and thus since by our assumptions and monotonicity $P(Z_t^{k_\epsilon} \leq 1) = 1$ for all $t \in [1,T]$,

$$\begin{aligned} E\left[Z_{\tau_\epsilon}^{k_\epsilon}\right] &= E\left[Z_{\tau_\epsilon}^{k_\epsilon} I(\mathcal{G}_{3,\epsilon})\right] + E\left[Z_{\tau_\epsilon}^{k_\epsilon} I(\mathcal{G}_{3,\epsilon}^c)\right] \\ &\leq \frac{3}{4}\epsilon + E\left[I(\mathcal{G}_{3,\epsilon}^c)\right] \quad \leq \quad \epsilon. \end{aligned}$$

Combining with Lemma 18, Lemma 8, and Theorem 1, completes the proof of the first part off the lemma. The second part follows directly from Lemma 17. *Q.E.D.*

We note that under different assumptions, e.g. outside the normalized setting, it may be possible to derive more efficient policies by alternate means. For example, the tools from our analysis of Robbins' problem in the proof of Theorem 7 can be combined with the above results to develop efficient algorithms and policies in that setting. Although we do not formalize such extensions here due to space considerations, we leave such generalizations and refinements as interesting directions for future research.

6. Conclusion

In this work we developed a new pure-dual methodology for the fundamental problem of optimal stopping and options pricing with high-dimensionality and full path-dependence. In contrast to most past approaches in the literature, our (completely data-driven) algorithms allow one to gracefully trade-off between the desired level of approximation and run-time / sample-complexity / level of nesting in the associated conditional expectations. Indeed, a key insight of our results is that even in the presence of full path-dependence and high-dimensionality, for any given error parameter ϵ one can obtain an ϵ-approximation in time polynomial in T, and depending on the dimension (and state-space more generally) only through the cost of simulating individual sample paths, where only a polynomial number of such simulations are needed. Our approach also brings to light new connections with network flows and other results from the literature.

Our work leaves many interesting directions for future research.

Implementation and testing on real data and examples from finance. At this point, our results and analysis are a proof-of-concept that such a trade-off between accuracy and computational / sample complexity is theoretically possible. Testing the approach on real data and instances, understanding how to combine our approach with other heuristics (e.g. from ADP and simulation) to improve speed and accuracy, and rigorously comparing to past approaches, will of course be crucial for moving from the proof-of-concept stage to a useful tool for options pricing.

Better theoretical understanding of convergence. We provided several bounds on the rate of converegence of our approach, in various settings. We suspect that in many cases our analysis will be loose, and one may get quite accurate results using only a few terms of the relevant series. It is interesting to understand this phenonema both in the general setting of options pricing, as well as for more structured optimal stopping problems such as Robbins' problem. We also note that if one suspected that for any particular instance the expansion was converging more rapidly than suggested by our theoretical results, one could derive tighter upper bounds by simply exhibiting a stopping time τ for which $E[Z_\tau^{k+1}]$ was small, and formalizing such a procedure may also be interesting to consider.

Better algorithms and analysis using advanced simulation techniques. It seems likely that by combining our approach with more sophisticated tools and analysis from simulation, one could derive faster algorithms and tighter bounds. For example, we have made no effort to optimize our use of samples, and better-understanding how to allocate samples between the different "levels" of our nested simulations, and/or how to reuse and recombine samples more intelligently, could lead to significant speedups. The same goes for applying techniques from multi-level Monte-carlo. In addition, we suspect that techniques from importance sampling and change-of-measure more generally could be quite helpful here, as e.g. our study of several 2-period problems in Section 4 suggests a source of algorithmic slow-down may be the presence of zeroes and/or very small values (which cause one to make slow progress), and biasing/conditioning to avoid such paths may enable faster progress.

In-depth comparison to other dual formulations. The ability of our method to yield fast approximations seems connected to the fact that our approach does not have the sure-optimal property (Schoenmakers et al. (2013)), a property shared by most previous approaches (although not the so-called multiplicative dual). Developing a better understanding of how our approach relates, both conceptually and technically/computationally, to past dual approaches remains an

interesting open question.

Generalization to stochastic control broadly. We believe that our methodology can be extended to a broad family of stochastic control problems. The first step here would be the extension to multiple stopping, which follows almost directly from our current analysis in light of the well-known (recursive) relation between multiple stopping and optimal stopping (Bender et al. (2015)). Indeed, there has been much recent progress on understanding the Bayesian regret for such multiple stopping problems (Arlotto and Gurvich (2017), Bumpensanti and Wang (2018), Vera and Banerjee (2018)), and drawing connections to our own work remains an interesting direction for future research. Of course, there is also the broader question of how far our methodology can be extended to general stochastic control problems, while maintaining the relevant notions of tractability. Using reductions similar to those of Bender et al. (2015), it seems plausible that control problems with few actions, in which one cannot change action too many times (in an appropriate sense), may be a good starting point here. There are also several technical directions in which the work can be extended, e.g. the setting of continuous-time and associated stochastic processes, infinite-horizon problems, etc.

Lower bounds, randomization, and computational complexity. An interesting set of questions revolve around proving lower bounds on the computational and sample complexity for the problems studied, e.g. path-dependent optimal stopping. There has been much interesting recent work laying out a theory of computational complexity (with positive and negative results) in the settings of stochastic control and Markov decision processes (Halman et al. (2014, 2015), Chen and Wang (2017), Sidford et al. (2018)), and the pricing of complex financial products (Bertsimas et al. (2002), Van Roy (2009), Arora et al. (2011), Braverman et al. (2014)). Better understanding the connection between our approach and those works remains an interesting direction for future research. A key question here centers around the use of randomization and different notions of approximation, as well as questions such as the interaction between computational complexity and sample complexity.

Application of other tools from network flow theory. As our Lemma 6 shows that general optimal stopping may be expressed as a massive max-flow problem on a tree network, this opens the door to utilizing the vast algorithmic toolkit developed over the last century in the theory of network flows (Ahuja et al. (2014)). Furthermore, combinatorial insights from the theory of network flows may be helpful in identifying structural properties under which our methods converge more quickly.

Alternate expansions and preprocessing. As we saw in Section 4, for some problems it may be helpful to consider modifications of our approach. Indeed, it may be shown that in principle many other explicit expansions also converge to the optimal value, e.g. (for the case that $P(Z_t \in [0,1]) = 1$ for all t) defining $Z_t'^{k+1} = Z_t'^k - E[\prod_{i=1}^T Z_i'^k | \mathcal{F}_t]$. One could also attempt to incorporate certain changes-of-measure into the expansion itself, e.g. by defining $Z_t'^{k+1} = Z_t'^k - E\big[\min_{i\in[1,T]} Z_i'^k I\big(\min_{i\in[1,T]} Z_i'^k > \epsilon\big) | \mathcal{F}_t\big]$. On a related note, one method that was quite successful in accelerating known algorithms is to start with a good initial approximation. For our method, this would be equivalent to performing an initial round in which one compensated $\mathbf{Z}$ by a martingale $\mathbf{M}$ which (due perhaps to problem-specific features or the use of some other algorithms/approximations) one suspected to well-approximate the optimal dual martingale, and then performed our expansion with this compensated process as the base process.

Formulation of new prophet inequalities. Our approach may open the door to the formulation of new prophet inequalities (Hill and Kertz (1992)), as it provides a new way to express the value of optimal stopping problems. We note that in some sense, truncating our expansion after more than one term can be interpreted as a type of "higher-order" prophet inequality, expressing the value which can be attained by some kind of intermediate adversary (i.e. one who is not omnipotent, yet who need not behave according to an adapted policy).

Application to other theoretical problems in optimal stopping, sequential hypothesis testing, and machine learning. Our approach may also be helpful in shedding new insight into several well-studied theoretical problems in the optimal stopping literature, such as Robbins' problem and the calculation of the Gittins' index. Here our results may be useful not just as a computational tool, but in the sense that they provide new purely analytical expansions for the optimal value which may yield novel theoretical and structural insights. Indeed, our original motivation was attempting to develop a deeper understanding of the regret in Bayesian multi-arm bandit problems, and we believe our results may prove very helpful along these lines.

Implications for robust optimal stopping. Our approach may also be helpful in shedding new insight into problems in so-called robust optimal stopping (Bayraktar et al. (2014), Nutz et al. (2015), Goldenshluger and Zeevi (2017)), as our expansions are general and do not depend (structurally) on the particular distribution under consideration.

Application to problems in pricing and mechanism design. Another area where optimal

stopping, prophet inequalities, and other such tools have proven useful is in the domain of mechanism design and pricing problems (Abholhassani et al. (2017), Correa et al. (2017), Lucier (2017)). Extending our results to this setting, and more generally bridging this literature with that on dual martingale approaches to optimal stopping, remains an interesting direction for future research.

7. Technical Appendix

7.1. Proof of Lemma 7

Proof of Lemma 7 : First we prove that $\mathrm{MAR}_t = 0$ for all $t \in [1,3]$ w.p.1. Indeed, we have $\min_{1\le t\le 3} Z_t^1 = \min_{1\le t\le 3} Y_t = Y_1 = 0$ w.p.1. Monotonicity and non-negativity then yields $\min_{1\le t\le 3} Z_t^k = 0$ w.p.1. Hence $S = \sum_{k=1}^{\infty} \min_{1\le t\le 3} Z_t^k = 0$ w.p.1. We thus conclude that MAR_t, defined as the Doob martingale $E[S|\mathcal{F}_t]$, must equal zero for all $t \in [1,3]$ w.p.1 by the basic properties of conditional expectation.

We next show that the unique (0-mean) surely-optimal dual martingale M satisfies $M_1(0) = M_2(0,1) = 0, M_3(0,1,1) = \frac{1}{4}, M_3(0,1,\frac{1}{2}) = -\frac{1}{4}$. Indeed, that $M_1(0) = M_2(0,1) = 0$ follows directly from the fact that Y_1 and Y_2 are constants w.p.1, that M_t is adapted to $\mathcal{F}_t$ for all $t \in [1,3]$, and the 0-mean assumption on M. Recall that the surely-optimal property for $i = 2$ requires $P\Big(\min_{t\in[2,3]}\big(Z_t - M_t + M_2\big) = \inf_{\tau\in\mathcal{T}^{2,3}} E[Z_\tau|\mathcal{F}_2]\Big) = 1$, which (after simplifying and using the fact that Y_1 and Y_2 are constants) is equivalent to $P\Big(\min\big(1, Z_3 - M_3\big) = \inf_{\tau\in\mathcal{T}^{2,3}} E[Z_\tau]\Big) = 1$. However, as $P(Z_3 \le Z_2) = 1$, we find that $\inf_{\tau\in\mathcal{T}^{2,3}} E[Z_\tau] = E[Z_3] = \frac{3}{4}$. Thus any (0-mean) surely-optimal martingale M must satisfy $\min\big(1, Z_3 - M_3\big) = \frac{3}{4}$ w.p.1. It follows that $1 - M_3(0,1,1) = \frac{3}{4}$, and $\frac{1}{2} - M_3(0,1,\frac{1}{2}) = \frac{3}{4}$. Combining the above completes the proof. *Q.E.D.*

7.2. Proof of Theorem 7

We begin by proving a more general result regarding our modified expansion, separate from the particular problems of i.i.d. U[0,1] stopping or Robbins' problem. Recall that for $\eta \in (0,1)$ and $t \in [1,T]$, $Z_{\eta,t}^1 = Z_t^1$. For $T \ge 1$ and $\eta \in (0,1)$, let $t_\eta(T) \triangleq \lceil (1-\eta)T \rceil$. Then recall that for $k \ge 1, \eta \in (0,1), t \in [1,T]$, $Z_{\eta,t}^{k+1} = Z_{\eta,t}^k - E\big[\min_{i\in[1,t_\eta(T)]} Z_{\eta,i}^k | \mathcal{F}_t\big]$. Recall that for $1 \le t_1 \le t_2 \le T$, $\mathcal{T}^{t_1,t_2}$ denotes the set of all integer-valued stopping times τ, adapted to $\mathcal{F}$, s.t. w.p.1 $t_1 \le \tau \le t_2$. Also, let $\mathrm{OPT}_\eta(T) \triangleq \inf_{\tau\in\mathcal{T}^{1,t_\eta(T)}} E[Z_\tau]$. Note that $\mathrm{OPT}_\eta(T)$ represents the best you can do if restricted to stop by time $\lceil (1-\eta)T \rceil$. Further recall that $H_k(\eta) = E[\min_{1\le t\le t_\eta(T)} Z_{\eta,t}^k]$, and $E_k(\eta) = \sum_{i=1}^k H_i(\eta)$. We first make a few observations regarding our modified expansion, which follow from definitions, the basic properties of conditional expectation, monotonicity, and proofs nearly identical to that of Theorem 1 (and we omit the details).

LEMMA 19. *For all $\eta \in (0,1), k \geq 1$, and $t \in [1, t_\eta(T)]$, $Z^k_{\eta,t} \geq 0$. For all $\eta \in (0,1)$ and $t \in [1,T]$, $\{Z^k_{\eta,t}, k \geq 1\}$ is monotone decreasing. Also, for all $\eta \in (0,1)$, $OPT_\eta(T) = \sum_{k=1}^{\infty} H_k(\eta)$. Furthermore, for all $k \geq 1$, $H_k(\eta) \leq \frac{1}{k} \times OPT_\eta(T)$.*

We note that some care will have to be taken, as it is however possible that $Z^k_{\eta,t} < 0$ for $t > t_\eta(T)$. For $T \geq 1$ and $\eta \in (0,1)$, let $\overline{\mathrm{OPT}}_\eta(T) \triangleq \inf_{\tau \in \mathcal{T}^{t_\eta(T),T}} E[(Z_\tau)^2]$. Note that $\overline{\mathrm{OPT}}_\eta(T)$ represents the best you can do (with respect to the square of Z) if restricted to stop after time $t_\eta(T)$. It follows from standard results in the theory of optimal stopping, e.g. Chow and Robbins (1963), that the optimal value of the optimal stopping problem $\inf_{\tau \in \mathcal{T}^{t_\eta(T),T}} E[(Z_\tau)^2]$ is attained (as opposed to only being approached as the limit of some sequence of stopping times), and we denote some optimal stopping time to this problem by $\overline{\tau}_\eta(T)$. Thus $\overline{\mathrm{OPT}}_\eta(T) = E[(Z_{\overline{\tau}_\eta(T)})^2]$. To make the dependence on T explicit, we also denote OPT by $\mathrm{OPT}(T)$. As a notational convenience (allowing us to divide by certain quantities), we exclude the degenerate setting that $\mathrm{OPT}(T) = 0$. Then our general auxiliary result is as follows.

LEMMA 20. *For all $\eta \in (0,1)$ and $k \geq 1$,*

$$\frac{\left|OPT(T) - E_k(\eta)\right|}{OPT(T)} \leq \max\left(\frac{OPT_\eta(T) - OPT(T)}{OPT(T)}, 3 \times \frac{\left(\overline{OPT}_\eta(T) \times OPT_\eta(T)\right)^{\frac{1}{3}}}{(k+1)^{\frac{1}{3}} \times OPT(T)}\right).$$

Proof : First we show that

$$\frac{E_k(\eta) - \mathrm{OPT}(T)}{\mathrm{OPT}(T)} \leq \frac{\mathrm{OPT}_\eta(T) - \mathrm{OPT}(T)}{\mathrm{OPT}(T)}. \tag{9}$$

By Lemma 19, we conclude that $\mathrm{OPT}_\eta(T) \geq E_k(\eta)$. It follows that

$$\begin{aligned}\mathrm{OPT}(T) &= \mathrm{OPT}_\eta(T) + \big(\mathrm{OPT}(T) - \mathrm{OPT}_\eta(T)\big)\\ &\geq E_k(\eta) + \big(\mathrm{OPT}(T) - \mathrm{OPT}_\eta(T)\big),\end{aligned}$$

from which (9) follows.

Next, we prove that

$$\frac{OPT(T) - E_k(\eta)}{OPT(T)} \leq 3 \times \frac{\left(\overline{\mathrm{OPT}}_\eta(T) \times \mathrm{OPT}_\eta(T)\right)^{\frac{1}{3}}}{(k+1)^{\frac{1}{3}} \times \mathrm{OPT}(T)}. \tag{10}$$

To prove (10), we first prove that for all $k \geq 1$,

$$OPT(T) - E_k(\eta) = \inf_{\tau \in \mathcal{T}} E[Z^{k+1}_{\eta,\tau}]. \tag{11}$$

Indeed, by definitions and a straightforward induction, for all $t \in [1,T]$ and $k \geq 1$, $Z^{k+1}_{\eta,t} = Z^1_{\eta,t} - E\big[\sum_{j=1}^k \min_{i\in[1,t_\eta(T)]} Z^j_{\eta,i}|\mathcal{F}_t\big]$. (11) then follows from optional stopping and definitions.

Thus to prove (10), it suffices to prove that for all $k \geq 1$,

$$\inf_{\tau\in\mathcal{T}} E[Z^{k+1}_{\eta,\tau}] \leq 3 \times \left(\frac{\overline{\mathrm{OPT}}_\eta(T) \times \mathrm{OPT}_\eta(T)}{k+1}\right)^{\frac{1}{3}}. \tag{12}$$

Let $x_{\eta,k}(T) \triangleq \left(\frac{\overline{\mathrm{OPT}}_\eta(T)\times\mathrm{OPT}_\eta(T)}{4(k+1)}\right)^{\frac{1}{3}}$. Let $\tau_{\eta,k}(T)$ denote the following stopping time. It stops the first time in $[1,t_\eta(T)]$ that $Z^{k+1}_{\eta,t} \leq x_{\eta,k}(T)$, if such a time t exists in $[1,t_\eta(T)]$. If not, it stops according to the stopping time $\overline{\tau}_\eta(T)$. Note that all relevant r.v.s may be constructed on a common probability space s.t. w.p.1,

$$\begin{aligned} Z^{k+1}_{\eta,\tau_{\eta,k}(T)} &\leq x_{\eta,k}(T) + I\big(\min_{i\in[1,t_\eta(T)]} Z^{k+1}_{\eta,i} > x_{\eta,k}(T)\big) Z^{k+1}_{\eta,\overline{\tau}_\eta(T)} \\ &\leq x_{\eta,k}(T) + I\big(\min_{i\in[1,t_\eta(T)]} Z^{k+1}_{\eta,i} > x_{\eta,k}(T)\big) Z_{\overline{\tau}_\eta(T)}, \end{aligned}$$

the final inequality following from Lemma 19 and the fact that (by definition, w.p.1) $Z^1_{\eta,t} = Z_t$ for all $t \in [1,T]$. Combining with Markov's inequality, Lemma 19, Cauchy-Schwartz, and some straightforward algebra, we conclude that

$$E\big[Z^{k+1}_{\eta,\tau_{\eta,k}(T)}\big] \quad \leq \quad x_{\eta,k}(T) + \left(\frac{\frac{1}{k+1}\times \mathrm{OPT}_\eta(T)}{x_{\eta,k}(T)} \times \overline{\mathrm{OPT}}_\eta(T)\right)^{\frac{1}{2}} \quad = \quad 3 \times x_{\eta,k}(T).$$

Combining the above completes the proof. *Q.E.D.*

We now present an auxiliary result, which will be useful in several of our proofs.

Lemma 21. *Consider the sequence $\{y_T, T \geq 1\}$ defined by $y_1 = \frac{1}{2}$, and $y_{T+1} = y_T - \frac{1}{2}y_T^2$ for all $T \geq 1$. Then $\{T \times y_T, T \geq 1\}$ is monotone increasing, with limit equal to 2.*

Proof : For $T \geq 1$, let $x_T \triangleq T \times y_T$. Trivially $x_1 = \frac{1}{2}$. It follows from the definition of y_T that

$$x_{T+1} = \frac{T+1}{T}x_T - \frac{1}{2} \times \frac{T+1}{T^2}x_T^2. \tag{13}$$

Thus as it may be easily verified that $\{x_T, T \geq 1\}$ is strictly positive for all $T \geq 1$, we find that $x_{T+1} > x_T$ if and only if $\frac{T+1}{T}x_T - \frac{1}{2} \times \frac{T+1}{T^2}x_T^2 > x_T$, equivalently (after some straightforward algebra) $x_T < \frac{2T}{T+1}$. Thus to prove the desired monotonicity, it suffices to prove that $x_T < \frac{2T}{T+1}$ for all $T \geq 1$. Let us proceed by induction. The cases $T = 1$ and $T = 2$ are easily verified, as $x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{4}$. Now, suppose the induction is true for some $T \geq 2$. As it is a straightforward exercise in calculus to verify that the function $f(x) = \frac{T+1}{T}x - \frac{1}{2} \times \frac{T+1}{T^2}x^2$ is increasing in x on $[0,2]$ for all $T \geq 2$, to complete the induction it suffices to verify that

$$\frac{T+1}{T} \times (\frac{2T}{T+1}) - \frac{1}{2} \times \frac{T+1}{T^2} \times (\frac{2T}{T+1})^2 < \frac{2(T+1)}{T+2}. \tag{14}$$

As some straightforward algebra demonstrates that the left-hand-side of (14) equals $\frac{2T}{T+1}$, the induction then follows from the fact that $f(x)=\frac{x}{x+1}$ is increasing in x on $[0,\infty)$. That $\lim_{T\to\infty} x_T = 2$ follows from the results of Gilbert and Mosteller (1966).

We now present the desired bound for the i.i.d. uniform setting.

LEMMA 22. *In the i.i.d. uniform setting, i.e. when* $\mathbf{Y}=\mathbf{Y}^1$ *and* $g_t = g_t^U$, *for all* $T\geq 1, \eta\in(0,1)$, *and* $k\geq 1$, $\frac{\left|E_k(\eta)-OPT(T)\right|}{OPT(T)} \leq \max\left(\frac{\eta}{1-\eta}, 12\times\eta^{-\frac{2}{3}}\times(1-\eta)^{-\frac{1}{3}}\times(k+1)^{-\frac{1}{3}}\right)$.

Proof : We first prove that for all $T\geq 1$ and $\eta\in(0,1)$,

$$\frac{\mathrm{OPT}_\eta(T)-\mathrm{OPT}(T)}{\mathrm{OPT}(T)} \leq \frac{\eta}{1-\eta}. \tag{15}$$

For $T\geq 1$, let $x_T \triangleq T\times\mathrm{OPT}(T)$. Note that for $T\geq 1$,

$$\mathrm{OPT}(T+1) \quad = \quad E\big[\min\big(U,\mathrm{OPT}(T)\big)\big] \quad = \quad \mathrm{OPT}(T)-\frac{1}{2}\big(\mathrm{OPT}(T)\big)^2. \tag{16}$$

It thus follows from Lemma 21 that $\{x_T, T\geq 1\}$ is monotone increasing with limit 2. We will also need the fact that for all $T\geq 1$ and $\eta\in(0,1)$, it holds that

$$\mathrm{OPT}_\eta(T) = \mathrm{OPT}\big(t_\eta(T)\big), \tag{17}$$

which follows from the i.i.d. property and a straightforward probabilistic argument. Combining with the demonstrated monotonicity of $\{x_T, T\geq 1\}$, we conclude that $\frac{\mathrm{OPT}_\eta(T)-\mathrm{OPT}(T)}{\mathrm{OPT}(T)}$ equals

$$\frac{\mathrm{OPT}\big(t_\eta(T)\big)}{\mathrm{OPT}(T)}-1 \quad = \quad \frac{T}{t_\eta(T)}\times\frac{t_\eta(T)\times\mathrm{OPT}\big(t_\eta(T)\big)}{T\times\mathrm{OPT}(T)}-1 \quad \leq \quad \frac{T}{t_\eta(T)}-1 \quad \leq \quad \frac{\eta}{1-\eta},$$

completing the proof of (15).

We next prove that for all $T\geq 1$ and $\eta\in(0,1)$,

$$\frac{\left(\overline{\mathrm{OPT}}_\eta(T)\times\mathrm{OPT}_\eta(T)\right)^{\frac{1}{3}}}{\mathrm{OPT}(T)} \leq 4\times\eta^{-\frac{2}{3}}\times(1-\eta)^{-\frac{1}{3}}. \tag{18}$$

Let $t'_\eta(T) \triangleq T-t_\eta(T)+1$. It again follows from the i.i.d. property and a straightforward probabilistic argument that $\overline{\mathrm{OPT}}_\eta(T) = \inf_{\tau\in\mathcal{T}^{1,t'_\eta(T)}} E[(Z_\tau)^2]$, i.e. the best you can do if observing a length-$t'_\eta(T)$ sequence of i.i.d. squared uniforms. For $t\geq 1$, let $y_t \triangleq \inf_{\tau\in\mathcal{T}^{1,t}} E[(Z_\tau)^2]$. It follows from logic nearly identical to that used to prove (16) that $y_1=\frac{1}{3}$, and for $t\geq 1$,

$$y_{t+1} \quad = \quad E\big[\min\big(U^2,y_t\big)\big] \quad = \quad y_t-\frac{2}{3}y_t^{\frac{3}{2}}. \tag{19}$$

Using (19), we now prove that $y_t\leq\frac{9}{t^2}$ for all $t\geq 1$. Noting that $y_t\leq 1$ for all $t\geq 1$, the cases $t=1,2,3$ follow trivially. Thus suppose for induction that $y_t\leq\frac{9}{t^2}$ for some $t\geq 3$. Noting that $f(x)=x-\frac{2}{3}x^{\frac{3}{2}}$

is increasing on $[0,1]$, and that $\frac{9}{t^2} \le 1$ for all $t \ge 3$, to complete the proof by induction it thus suffices to verify that $\frac{9}{t^2} - \frac{2}{3} \times (\frac{9}{t^2})^{\frac{3}{2}} - \frac{9}{(t+1)^2} \le 0$, which we now show.

$$\begin{aligned} & \frac{9}{t^2} - \frac{2}{3} \times (\frac{9}{t^2})^{\frac{3}{2}} - \frac{9}{(t+1)^2} \\ = \quad & \frac{9t(t+1)^2 - 18(t+1)^2 - 9t^3}{t^3(t+1)^2} \quad = \quad \frac{-27t-18}{t^3(t+1)^2} \quad < \quad 0, \end{aligned}$$

which completes the proof by induction that $y_t \le \frac{9}{t^2}$ for all $t \ge 1$. It follows that for all $T \ge 1$ and $\eta \in (0,1)$,

$$\overline{\mathrm{OPT}}_\eta(T) \quad \le \quad \frac{9}{(T - \lceil (1-\eta)T \rceil + 1)^2} \quad \le \quad \frac{9}{\eta^2 T^2}.$$

Again applying the monotonicity of $\{x_T, T \ge 1\}$ and (17), we conclude that $\frac{\left(\overline{\mathrm{OPT}}_\eta(T) \times \mathrm{OPT}_\eta(T)\right)^{\frac{1}{3}}}{\mathrm{OPT}(T)}$ is at most

$$\begin{aligned} & 9^{\frac{1}{3}} \eta^{-\frac{2}{3}} T^{-\frac{2}{3}} \times \frac{\Big(t_\eta(T) \times \mathrm{OPT}\big(t_\eta(T)\big)\Big)^{\frac{1}{3}}}{\big(t_\eta(T)\big)^{\frac{1}{3}} \times \mathrm{OPT}(T)} \\ = \quad & 9^{\frac{1}{3}} \eta^{-\frac{2}{3}} T^{-\frac{2}{3}} \times \left(\frac{t_\eta(T) \times \mathrm{OPT}\big(t_\eta(T)\big)}{T \times \mathrm{OPT}(T)}\right)^{\frac{1}{3}} \times \big(\frac{T}{t_\eta(T)}\big)^{\frac{1}{3}} \times \big(\mathrm{OPT}(T)\big)^{-\frac{2}{3}} \\ \le \quad & 9^{\frac{1}{3}} \eta^{-\frac{2}{3}} \big(T \times \mathrm{OPT}(T)\big)^{-\frac{2}{3}} \times \big(1-\eta\big)^{-\frac{1}{3}} \\ \le \quad & 9^{\frac{1}{3}} \eta^{-\frac{2}{3}} \times \big(\frac{1}{2}\big)^{-\frac{2}{3}} \times \big(1-\eta\big)^{-\frac{1}{3}} \quad \le \quad 4 \times \eta^{-\frac{2}{3}} \times (1-\eta)^{-\frac{1}{3}}. \end{aligned}$$

Combining with Lemma 20 and (15) completes the proof. *Q.E.D.*.

We now present the desired bound for the setting of Robbins' problem.

LEMMA 23. *In the setting of Robbins' problem, i.e. when* $\mathbf{Y} = \mathbf{Y}^1$ *and* $g_t = g_t^R$, *there exists an absolute constant* C^3 *(independent of* T, η, k*) s.t. for all* $T \ge 1, \eta \in (0,1)$, *and* $k \ge 1$, $\frac{\left|E_k(\eta) - OPT(T)\right|}{OPT(T)} \le C^3 \times \max\left(\frac{\eta}{1-\eta}, \eta^{-\frac{2}{3}} \times (1-\eta)^{-\frac{1}{3}} \times (k+1)^{-\frac{1}{3}}\right)$.

Proof : We first prove that there exists a universal constant C^1, independent of T and η, s.t. for all $T \ge 1$ and $\eta \in (0,1)$,

$$\frac{\mathrm{OPT}_\eta(T) - \mathrm{OPT}(T)}{\mathrm{OPT}(T)} \le C^1 \times \frac{\eta}{1-\eta}. \tag{20}$$

For $T \ge 1$, let $\tau(T)$ denote an optimal stopping time for Robbins' problem when the horizon is T, i.e. $\mathrm{OPT}(T) = E[Z_{\tau(T)}]$, where existence again follows from general results in optimal stopping (Chow and Robbins (1963)). Note that since $\tau\big(t_\eta(T)\big) \in \mathcal{T}^{1,t_\eta(T)}$ for all $T \ge 1$ and $\eta \in (0,1)$, it holds that

$$\mathrm{OPT}_\eta(T) = \inf_{\tau \in \mathcal{T}^{1,t_\eta(T)}} E\big[\sum_{i=1}^{\tau} I(Y_i \le Y_\tau) + (T-\tau)Y_\tau\big]$$

$$
\begin{aligned}
&= \inf_{\tau\in\mathcal{T}^{1,t_\eta(T)}} E\big[\sum_{i=1}^{\tau} I(Y_i \le Y_\tau) + \big(t_\eta(T)-\tau\big)Y_\tau + \big(T-t_\eta(T)\big)Y_\tau\big] \\
&\le E\Bigg[\sum_{i=1}^{\tau\left(t_\eta(T)\right)} I(Y_i \le Y_{\tau\left(t_\eta(T)\right)}) + \Big(t_\eta(T)-\tau\big(t_\eta(T)\big)\Big)Y_{\tau\left(t_\eta(T)\right)} + \big(T-t_\eta(T)\big)Y_{\tau\left(t_\eta(T)\right)}\Bigg] \\
&= \mathrm{OPT}\big(t_\eta(T)\big) + \frac{T-t_\eta(T)}{t_\eta(T)} \times \Big(t_\eta(T) \times E\big[Y_{\tau\left(t_\eta(T)\right)}\big]\Big) \\
&\le \mathrm{OPT}\big(t_\eta(T)\big) + \frac{\eta}{1-\eta} \times \Big(t_\eta(T) \times E\big[Y_{\tau\left(t_\eta(T)\right)}\big]\Big). \qquad (21)
\end{aligned}
$$

As it is proven in Gnedin and Iksanov (2011) that $C^2 \triangleq \sup_{T\ge 1}\big(T \times E[Y_{\tau(T)}]\big) < \infty$, and that $\{\mathrm{OPT}(T), T \ge 1\}$ is monotone increasing to a finite limit, the desired result (20) then follows from the fact that trivially $\mathrm{OPT}(1) = 1$.

Next, we prove that for all $T \ge 1$ and $\eta \in (0,1)$,

$$
\overline{\mathrm{OPT}}_\eta(T) \le 10^{10} \times \eta^{-2}. \qquad (22)
$$

Let τ denote the following stopping time. Stop at the first time $t \ge t_\eta(T)$ s.t. $Y_t \le \frac{3}{T-t+3}$. Note that such a time always exists, as $\frac{3}{T-T+3} = 1$, and $P(Y_T \le 1) = 1$. Note that w.p.1, for all $t \in [1,T]$, $g_t^R(Y_{[t]}) \le \sum_{i=1}^T I(Y_i \le Y_t) + TY_t$. It then follows from definitions and some straightforward algebra, including the fact that $E[X] \le 1 + E[X^2]$ for all non-negative r.v.s X, that

$$
\begin{aligned}
\overline{\mathrm{OPT}}_\eta(T) &\le E\big[\big(\sum_{i=1}^{T} I(Y_i \le Y_\tau) + TY_\tau\big)^2\big] \\
&= E\big[\big(1 + \sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau) + TY_\tau\big)^2\big] \\
&= 1 + 2 \times E\big[\sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau) + TY_\tau\big] + E\Bigg[\Bigg(\sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau) + TY_\tau\Bigg)^2\Bigg] \\
&\le 3 + 3 \times E\Bigg[\Bigg(\sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau) + TY_\tau\Bigg)^2\Bigg],
\end{aligned}
$$

itself at most

$$
\begin{aligned}
&3 + 3 \times E\Bigg[\Bigg(\sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau)\Bigg)^2\Bigg] \qquad (23) \\
&\quad + 6 \times T \times E\Bigg[\sum_{i\in[1,T]\setminus\tau} I(Y_i \le Y_\tau) \times Y_\tau\Bigg] + 3 \times T^2 \times E[Y_\tau^2].
\end{aligned}
$$

We now examine each term of (23). For $k \in [t_\eta(T), T]$, let $c_k \triangleq \frac{3}{T-k+3}$. For $k \in [t_\eta(T), T]$ and $i \in [1,T] \setminus k$, let $c_{k,i}$ equal $\frac{3}{T-i+3}$ if $i \in [t_\eta(T), k]$, and 0 otherwise. Then it follows from some

straightforward algebra, the basic properties of (conditional) expectation, the i.i.d. structure of $\mathbf{Y}$, and a straightforward probabilistic argument that

$$\begin{aligned}
E\Big[\Big(\sum_{i\in[1,T]\setminus\tau} I(Y_i\le Y_\tau)\Big)^2\Big] &= E\Big[\sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k}\sum_{j\in[1,T]\setminus k} I(Y_i\le Y_k)I(Y_j\le Y_k)I(\tau=k)\Big]\\
&= \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} E\big[I(Y_i\le Y_k)I(\tau=k)\big]\\
&\quad + \ 2\sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k}\sum_{j\in[i+1,T]\setminus k} E\big[I(Y_i\le Y_k)I(Y_j\le Y_k)I(\tau=k)\big]\\
&\le \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} P(\tau=k)\times P\big(Y_i\le c_k|\tau=k\big)\\
&\quad + \ 2\sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k}\sum_{j\in[i+1,T]\setminus k} P(\tau=k)\times P\big(Y_i\le c_k, Y_j\le c_k|\tau=k\big)\\
&= \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} P(\tau=k)\times P\big(Y_i\le c_k|Y_i> c_{k,i}\big)\\
&\quad + \ 2\sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k}\sum_{j\in[i+1,T]\setminus k} P(\tau=k)\times P\big(Y_i\le c_k, Y_j\le c_k|Y_i> c_{k,i}, Y_j> c_{k,j}\big)\\
&\le \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} P(\tau=k)\times P(Y_i\le c_k)\\
&\quad + \ 2\sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k}\sum_{j\in[i+1,T]\setminus k} P(\tau=k)\times P\big(Y_i\le c_k, Y_j\le c_k\big)\\
&\le T\times\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k + 2\times T^2\times\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2; \qquad (24)
\end{aligned}$$

$$\begin{aligned}
E\Big[\sum_{i\in[1,T]\setminus\tau} I(Y_i\le Y_\tau)\times Y_\tau\Big] &= \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} E\big[I(Y_i\le Y_k)\times Y_k\times I(\tau=k)\big]\\
&\le \sum_{k=t_\eta(T)}^{T}\sum_{i\in[1,T]\setminus k} P(\tau=k)\times c_k\times P(Y_i\le c_k|Y_i> c_{k,i})\\
&\le T\times\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2. \qquad (25)
\end{aligned}$$

As similar arguments yield the inequality $E[Y_\tau^2]\le\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2$, we may combine (23) with (24) - (25) and the fact that $E[c_\tau]\le\big(E[c_\tau^2]\big)^{\frac{1}{2}}$ to conclude that

$$\overline{\mathrm{OPT}}_\eta(T)\le 3+3\times T\times\Big(\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2\Big)^{\frac{1}{2}} + 15\times T^2\times\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2. \qquad (26)$$

Now, we bound $\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2$, with all arguments following from some straightforward algebra, the i.i.d. structure of $\mathbf{Y}$, several Taylor-series approximations, and the fact that for all $n\geq 1$, the nth harmonic number $\sum_{i=1}^{n}\frac{1}{i}$ satisfies $|\sum_{i=1}^{n}\frac{1}{i}-\log(n)|\leq 2$. Then $\sum_{k=t_\eta(T)}^{T} P(\tau=k)\times c_k^2$ equals

$$\begin{aligned}
&\sum_{k=t_\eta(T)}^{T}\Bigg(\prod_{i=t_\eta(T)}^{k-1}\big(1-\frac{3}{T-i+3}\big)\Bigg)\times\big(\frac{3}{T-k+3}\big)^3\\
&\leq\quad 30\times\sum_{k=t_\eta(T)}^{T}\exp\big(-3\times\sum_{i=t_\eta(T)}^{k-1}\frac{1}{T-i+3}\big)\times\frac{1}{(T-k+3)^3}\\
&\leq\quad 30\times\exp(12)\times\sum_{k=t_\eta(T)}^{T}\frac{\exp\Big(-3\times\big(\log(T-t_\eta(T)+3)-\log(T-k+4)\big)\Big)}{(T-k+3)^3}\\
&\leq\quad 240\times\exp(12)\times(T-t_\eta(T)+3)^{-2}\quad\leq\quad 10^8\times\eta^{-2}\times T^{-2}.
\end{aligned}$$

Combining with (26) completes the proof of (22), and the lemma then follows by combining with (15), Lemma 20, the result of Gnedin and Iksanov (2011) that $\{\mathrm{OPT}(T), T\geq 1\}$ is monotone increasing to a finite limit, and some straightforward algebra. *Q.E.D.*

7.2.1. Proof of Theorem 7.

Proof of Theorem 7 : The proof follows immediately from Lemmas 22 and 23 by setting $\eta=(k+1)^{-\frac{1}{5}}$. *Q.E.D.*

7.3. Proof of Lemma 12

Proof of Lemma 12 : Let us define the following sequence. $c_1=1$ and for $k\geq 1$, $c_{k+1}=c_k\exp(-c_k)$. We now prove by induction that for all $k\geq 1$, $P\big(Z_1^k=c_k\big)=1$, and $Z_2^k=_d B(c_k)\times X$. The base case $k=1$ is trivial. Now, suppose the induction is true for some $k\geq 1$. Then since $Var[Z_1^k]=0$, it follows that

$$\begin{aligned}
Z_1^{k+1}&=c_k-E\big[\min\big(c_k,B(c_k)\times X\big)\big]\\
&=c_k-c_k\times E[\min(c_k,X)]\quad=\quad c_k\times\Bigg(1-\int_0^{c_k}y\exp(-y)dy-c_k\exp(-c_k)\Bigg)\quad=\quad c_{k+1}.
\end{aligned}$$

Also, by the memoryless property,

$$\begin{aligned}
Z_2^{k+1}&=\;Z_2^k-\min(Z_1^k,Z_2^k)\\
&=_d\max(0,B(c_k)\times X-c_k)\quad=_d\quad B(c_k)\times\max(0,X-c_k)\quad=_d\quad B(c_{k+1})\times X.
\end{aligned}$$

Combining the above completes the induction. As $\mathbf{Z}^k$ is a martingale for all $k\geq 1$, by Lemma 1 it thus suffices to prove the desired bounds for $\{c_{k+1}, k\geq 1\}$. First, we prove by induction that for all $k\geq 1$, $k\times c_k\leq 1$. The base case $k=1$ is trivial. Now, suppose the induction is true for some $k\geq 1$.

Then as $\exp(-x) \leq (x+1)^{-1}$ for all $x > 0$, and $f(x) = \frac{x}{x+1}$ is increasing in x on $[0,1]$, it holds that $(k+1)c_{k+1}$ equals

$$(k+1)c_k \exp(-c_k) \quad \leq \quad (k+1)\frac{c_k}{c_k+1} \quad \leq \quad (k+1) \times \frac{\frac{1}{k}}{\frac{1}{k}+1} \quad = \quad 1,$$

completing the induction and proof that $c_{k+1} \leq \frac{1}{k+1}$ for all $k \geq 1$. Next we prove by induction that $\{kc_k, k \geq 2\}$ is monotonically increasing. This is equivalent to demonstrating that $(k+1)c_k \exp\{-c_k\} \geq kc_k$ for all $k \geq 2$; equivalently that $c_k \leq \log(1+\frac{1}{k})$ for all $k \geq 2$. The base case $k = 2$ may be verified by a straightforward and direct calculation, and we omit the details. Now suppose the induction is true for some $k \geq 2$. Using the defining recursion of c_{k+1}, the inductive hypothesis, and the easily verified facts that $w^1(x) \triangleq x\log(1+1/x)$ is strictly increasing on $(0,\infty)$ and $w^2(x) \triangleq x\exp(-x)$ is strictly increasing on $[0,1]$, we have

$$\begin{aligned}
\log(1+\frac{1}{k+1}) &\geq \frac{k}{k+1}\log(1+\frac{1}{k}), \\
&= \log(1+\frac{1}{k})\exp\{-\log(1+\frac{1}{k})\}, \\
&\geq c_k \exp\{-c_k\} \quad = \quad c_{k+1},
\end{aligned}$$

which completes the induction. Thus $\{kc_k, k \geq 2\}$ is monotone increasing, with limit c at most 1. We now prove that $c = 1$, and begin by proving by induction that for all $k \geq 2$, $c_k = \exp\big(-\sum_{i=1}^{k-1} c_i\big)$. The base case $k = 2$ is trivial. Thus suppose the induction holds for some $k \geq 2$. Then by the recursive definition of c_{k+1}, it holds that c_{k+1} equals

$$c_k \exp(-c_k) \quad = \quad \exp\big(-\sum_{i=1}^{k-1} c_i\big)\exp(-c_k) \quad = \quad \exp\big(-\sum_{i=1}^{k} c_i\big),$$

completing the induction. Let $y_k \triangleq kc_k$. Then the facts that: 1. $c_k = \exp\big(-\sum_{i=1}^{k-1} c_i\big)$, and 2. $\{y_k, k \geq 2\}$ increases monotonically to c, together imply that for all $k \geq 2$,

$$\log(y_k) \quad = \quad \log(k) - \sum_{i=1}^{k-1}\frac{y_i}{i} \quad \geq \quad \log(k) - c\sum_{i=1}^{k-1}\frac{1}{i},$$

and thus $c \times \frac{\sum_{i=1}^{k-1}\frac{1}{i}}{\log(k)} \geq 1 - \frac{\log(c)}{\log(k)}$. As it is easily verified (and well-known from the basic properties of the logarithm and harmonic numbers) that $\lim_{k\to\infty}\frac{\sum_{i=1}^{k-1}\frac{1}{i}}{\log(k)} = 1$, we may take limits in the above to conclude that $c \geq 1$. Combining the above completes the proof. *Q.E.D.*

7.4. Proof of Lemma 13

Proof of Lemma 13: First, we claim that for all $k \geq 1$, $\inf_{\tau\in\mathcal{T}} E[Z_\tau^k] = E[Z_1^k]$. Indeed, it follows from a straightforward induction and definitions that for all $k \geq 1$, $Var[Z_1^k] = 0$, and $E[Z_1^k] < E[Z_2^k]$. The desired claim then follows from basic results in the theory of optimal stopping (i.e. its solution

by backwards induction, see Chow and Robbins (1963)) and a straightforward contradiction, and we omit the details.

Next, let us define the following sequences. $z_1 = \frac{1}{2}, p_1 = 1$. For $k \geq 1$, $z_{k+1} = p_k \exp(-z_k) + z_k - p_k$, and $p_{k+1} = p_k \exp(-z_k)$. We now prove by induction that for all $k \geq 1$, $P\big(Z_1^k = z_k\big) = 1$, and $Z_2^k =_d B(p_k) \times X$. The base case $k = 1$ is trivial. Now, suppose the induction is true for some $k \geq 1$. Then since $Var[Z_1^k] = 0$, it follows that

$$\begin{aligned} Z_1^{k+1} &= z_k - E\big[\min\big(z_k, B(p_k) \times X\big)\big] \\ &= z_k - p_k \times E[\min(z_k, X)] \quad = \quad z_k - p_k \times \Big(\int_0^{z_k} y \exp(-y) dy + z_k \exp(-z_k)\Big) \quad = \quad z_{k+1}. \end{aligned}$$

Also, by the memoryless property,

$$\begin{aligned} Z_2^{k+1} &= Z_2^k - \min(Z_1^k, Z_2^k) \\ &=_d \max(0, B(p_k) \times X - z_k) \quad =_d \quad B(p_k) \times \max(0, X - z_k) \quad =_d \quad B(p_{k+1}) \times X. \end{aligned}$$

Combining the above completes the induction.

It follows that for all $k \geq 1$, $p_{k+1} - p_k = z_{k+1} - z_k$, and thus $p_k = z_k + (p_1 - z_1) = z_k + \frac{1}{2}$. Plugging back into the original definitions of p_k and z_k, we find that $z_{k+1} = (z_k + \frac{1}{2}) \exp(-z_k) - \frac{1}{2}$. The fact that $\inf_{\tau \in \mathcal{T}} E[Z_\tau^k] = E[Z_1^k]$, combined with Lemma 1 and Theorem 2, implies that $\{z_k, k \geq 1\}$ converges monotonically to 0. It then follows from a straightforward application of L'Hopital's rule that

$$\lim_{k \to \infty} \frac{z_{k+1}}{z_k} \quad = \quad \lim_{k \to \infty} \frac{(z_k + \frac{1}{2}) \exp(-z_k) - \frac{1}{2}}{z_k} \quad = \quad \lim_{z \downarrow 0} \frac{(z + \frac{1}{2}) \exp(-z) - \frac{1}{2}}{z} \quad = \quad \frac{1}{2},$$

from which the desired result follows. *Q.E.D.*

7.5. Proof of Lemma 14

Proof of Lemma 14: Let us define the following sequences. Let $a_1 = 2$, and $b_1 = 1$. For $k \geq 1$, let $a_{k+1} = a_k - \frac{1}{2} a_k b_k$, and $b_{k+1} = b_k(1 - \frac{b_k}{2})$. We now prove by induction that for all $k \geq 1$, $P\big(Z_1^k = \frac{a_k b_k}{2}\big) = 1$, and $Z_2^k =_d B(b_k) \times U(a_k)$. The base case $k = 1$ is trivial. Now, suppose the induction is true for some $k \geq 1$. Then since $Var[Z_1^k] = 0$, and a straightforward induction demonstrates that $b_k \in (0, 1)$ and $a_k > 0$, it follows from a straightforward probabilistic argument that

$$Z_2^{k+1} \quad =_d \quad \max\big(0, B(b_k) \times U(a_k) - \frac{a_k b_k}{2}\big) \quad =_d \quad B\big(b_k(1 - \frac{b_k}{2})\big) U(a_k - \frac{a_k b_k}{2}),$$

completing the proof that $Z_2^{k+1} =_d B(b_{k+1}) \times U(a_{k+1})$. The desired induction then follows from the martingale property.

Note that the definition of the recursions for a_{k+1} and b_{k+1} implies that $\frac{a_{k+1}}{a_k} = \frac{b_{k+1}}{b_k} = 1 - \frac{b_k}{2}$ for all $k \geq 1$. It follows that for all $k \geq 1$, $\frac{a_k}{b_k} = \frac{a_1}{b_1} = 2$, and hence $a_k = 2b_k$. It follows from Lemma 21 that $\{k \times b_k, k \geq 2\}$ is monotone increasing, with limit 2. As $Z_1^k = \frac{a_k \times b_k}{2} = b_k^2$, it follows that $\{k^2 Z_1^k, k \geq 2\}$ is monotone increasing with limit 4. Combining the above with the fact that $\mathbf{Z}^k$ is a martingale and Lemma 1 completes the proof. *Q.E.D.*

7.6. Proof of Lemma 16

Proof of Lemma 16 : Recall that

$$f_k(\epsilon,\delta) = 10^{2(k-1)^2}\epsilon^{-2(k-1)}(T+2)^{k-1}\big(1+\log(\frac{1}{\delta})+\log(\frac{1}{\epsilon})+\log(T)\big)^{k-1},$$

and $N(\epsilon,\delta) = \lceil \frac{1}{2\epsilon^2}\log(\frac{2}{\delta})\rceil$. As it is easily verified that $1 < \frac{\log(8)}{2}$, and thus for all $\epsilon,\delta \in (0,1)$, $N(\frac{\epsilon}{4},\frac{\delta}{4}) \le \frac{8.5}{\epsilon^2}\log(\frac{8}{\delta})$ and $N(\frac{\epsilon}{4},\frac{\delta}{4})+1 \le \frac{9}{\epsilon^2}\log(\frac{8}{\delta})$, we argue as follows. $\big(N(\frac{\epsilon}{4},\frac{\delta}{4})+1\big)\times\big(T+2\big)\times f_k\big(\frac{\epsilon}{4},\frac{\delta}{4N(\frac{\epsilon}{4},\frac{\delta}{4})T}\big)$ is at most

$$\begin{aligned}
&\frac{9}{\epsilon^2}\times\log(\frac{8}{\delta})\times(T+2)\times f_k(\frac{\epsilon}{4},\frac{\delta}{\frac{34}{\epsilon^2}\log(\frac{8}{\delta})T})\\
&\le \frac{9}{\epsilon^2}\times\log(\frac{8}{\delta})\times(T+2)\\
&\qquad\times 10^{2(k-1)^2}(T+2)^{k-1}(\frac{\epsilon}{4})^{-2(k-1)}\left(1+\log\left(\frac{34}{\epsilon^2}\log(\frac{8}{\delta})\frac{T}{\delta}\right)+\log\big(\frac{4}{\epsilon}\big)+\log(T)\right)^{k-1}\\
&\le 9\times\log(\frac{8}{\delta})\times(T+2)^k\times\epsilon^{-2k}\times 10^{2(k-1)^2}\times 16^{k-1}\\
&\qquad\times\left(1+\log(34)+2\log(\frac{1}{\epsilon})+\log(\frac{8}{\delta})+\log(\frac{T}{\delta})+\log(4)+\log(\frac{1}{\epsilon})+\log(T)\right)^{k-1}\\
&\le 9\times\log(\frac{8}{\delta})\times(T+2)^k\times\epsilon^{-2k}\times 10^{2(k-1)^2}\times 16^{k-1}\times\left(10+3\log(\frac{1}{\epsilon})+2\log(\frac{1}{\delta})+2\log(T)\right)^{k-1}\\
&\le 9\times(T+2)^k\times\epsilon^{-2k}\times 10^{2(k-1)^2}\times 16^{k-1}\times\left(10+3\log(\frac{1}{\epsilon})+2\log(\frac{1}{\delta})+2\log(T)\right)^{k}\\
&\le 10^{k+1}\times(T+2)^k\times\epsilon^{-2k}\times 10^{2(k-1)^2}\times 10^{2(k-1)}\times\left(1+\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\right)^{k},
\end{aligned}$$

which is further bounded by $f_{k+1}\big(\epsilon,\delta\big)$ since $2k^2-\big(k+1+2(k-1)^2+2(k-1)\big)=k-1\ge 0$. *Q.E.D.*

7.7. Proof of Corollary 2:

Proof of Corollary 2: In light of Theorems 6 and 2, by a union bound and the triangle inequality it suffices to individually approximate each of the first $\lceil\frac{2}{\epsilon}\rceil$ of the H_i, each to within additive error $\frac{\epsilon}{2}\big(\lceil\frac{2}{\epsilon}\rceil\big)^{-1}$ with probability $1-\delta\big(\lceil\frac{2}{\epsilon}\rceil\big)^{-1}$. After accounting for the cost of averaging the $\lceil\frac{2}{\epsilon}\rceil$ values, and combining with the monotonicities of f_k and some straightforward algebra, we find that the the computational cost divided by $C+G+1$ is at most

$$\begin{aligned}
&\sum_{i=1}^{\lceil\frac{2}{\epsilon}\rceil} f_{i+1}\left(\frac{\epsilon}{2}\big(\lceil\frac{2}{\epsilon}\rceil\big)^{-1},\delta\big(\lceil\frac{2}{\epsilon}\rceil\big)^{-1}\right)+\lceil\frac{2}{\epsilon}\rceil+1\\
&\le \lceil\frac{2}{\epsilon}\rceil f_{\lceil\frac{2}{\epsilon}\rceil+1}(\frac{\epsilon^2}{6},\frac{\delta\epsilon}{3})+\lceil\frac{2}{\epsilon}\rceil+1\\
&\le 6\epsilon^{-1}f_{\lceil\frac{2}{\epsilon}\rceil+1}(\frac{\epsilon^2}{6},\frac{\delta\epsilon}{3})
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 6\epsilon^{-1}10^{2(3\epsilon^{-1})^2}(\frac{\epsilon^2}{6})^{-2(3\epsilon^{-1})}(T+2)^{3\epsilon^{-1}}\big(1+\log(\frac{3}{\epsilon\delta})+\log(\frac{6}{\epsilon^2})+\log(T)\big)^{3\epsilon^{-1}} \\
&= \quad 10^{18\epsilon^{-2}}6^{6\epsilon^{-1}+1}\epsilon^{-12\epsilon^{-1}-1}(T+2)^{3\epsilon^{-1}}\big(1+\log(3)+\log(6)+3\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&\leq \quad 10^{18\epsilon^{-2}}\exp\Big(\log(6)(7\epsilon^{-1})+\log(\frac{1}{\epsilon})(13\epsilon^{-1})\Big)(T+2)^{3\epsilon^{-1}} \\
&\qquad \times \quad \big(1+\log(3)+\log(6)+3\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&\leq \quad 10^{18\epsilon^{-2}}\exp\big(14\epsilon^{-2}+13\epsilon^{-2}\big)(T+2)^{3\epsilon^{-1}}\big(1+\log(3)+\log(6)+3\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&= \quad \exp\Big(\big(27+18\log(10)\big)\epsilon^{-2}\Big)(T+2)^{3\epsilon^{-1}}\big(1+\log(3)+\log(6)+3\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&\leq \quad \exp(80\epsilon^{-2})27^{\epsilon^{-1}}T^{3\epsilon^{-1}}\big(5+3\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&\leq \quad \exp(80\epsilon^{-2})(5000)^{\epsilon^{-1}}T^{3\epsilon^{-1}}\big(1+\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}} \\
&\leq \quad \exp(100\epsilon^{-2})T^{3\epsilon^{-1}}\big(1+\log(\frac{1}{\epsilon})+\log(\frac{1}{\delta})+\log(T)\big)^{3\epsilon^{-1}}.
\end{aligned}
$$

The analysis for the number of calls to the base simulator follows nearly identically, and we omit the details. Combining the above completes the proof. *Q.E.D.*

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