PSEUDOSPECTRA AND OPTIMIZATION

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Technion: Lecture 2 Joint work with J. Burke and M. Overton
1. OUTLINE

• Dynamics and the spectral radius
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- Transient peaks and pseudospectra:
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- Transient peaks and pseudospectra: the Kreiss Matrix Theorem
- Visualizing, computing, and optimizing pseudospectra
- Lipschitz properties
- Distance to uncontrollability: Milnor and von-Neumann-Wigner
2. THE SPECTRAL RADIUS

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**Example**

The spectral radius of

$$A(t) = \begin{bmatrix} k & 1 \\ t & k - t \end{bmatrix}$$

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(with $k$ slightly less than 1) is minimized at $t = 0$. 
3. ROBUSTNESS AND TRANSIENT PEAKS

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One difficulty is the **multiple eigenvalue**. But this is **typical** at optimal solutions of spectral radius minimization problems.
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where the smallest singular value

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**Kreiss Matrix Theorem (1962)**

\[ A^n < K \rho^n \text{ for all } n, \text{ with } K \text{ not too large } \iff \max\{|\lambda| : \lambda \in \Lambda_\epsilon(A)\} < \rho, \text{ with } \epsilon \text{ not too small.} \]
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Analogously, in **continuous time**, \( e^{At} \to 0 \) with peaks not too large when \( \Lambda_\epsilon(A) \) lies in the left halfplane for \( \epsilon \) not too small.
5. **EXAMPLES**

Pseudospectra for a random $5 \times 5$ triangular complex matrix, plotted by [T. Wright’s EigTool](#):
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![Pseudospectra plot](image)
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Demmel’s example: $A = \begin{bmatrix} 1 & 5 & 5^2 & 5^3 & 5^4 \\ 0 & 1 & 5 & 5^2 & 5^3 \\ 0 & 0 & 1 & 5 & 5^2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
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dim = 6
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![Graph showing transient behavior of $\|e^{At}\|$](image)
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Hence a globally and quadratically convergent criss-cross algorithm for $\alpha_\epsilon$. 
Example Where does $\Lambda_\epsilon(A)$ intersects the imaginary axis?
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So, we just need to check $(\ast)$ for each imaginary eigenvalue of a Hamiltonian matrix.

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The criss-cross algorithm for $\alpha_\epsilon$ (available in eigtool) is fast, accurate and robust, and also returns $\nabla \alpha_\epsilon$ (when it exists).

Hence nonsmooth gradient sampling for optimizing $\alpha_\epsilon$. 
Pseudospectra resolve **Difficulty II**: the Kreiss Theorem shows

- avoiding transient peaks in \( \{ A^n \} \)
- ensuring \( \Lambda_\epsilon(A) \) lies in the unit disk (for reasonable \( \epsilon \))

are equivalent.
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where the Hausdorff distance between \( U, V \subset \mathbb{C} \) is

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d(U, V) = \max \left\{ \sup_{u \in U} \inf_{v \in V} |u - v|, \sup_{v \in V} \inf_{u \in U} |v - u| \right\}.
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What about the pseudospectral map \( A \mapsto \Lambda_\epsilon(A) \)?
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**Theorem**  Typically (Arnold), all eigenspaces of \( A \) are one-dimensional.
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\[ r = \frac{\sqrt{5} - 1}{2}. \]

For small \( s \geq 0 \),

\[ i \Omega(\sqrt{s}) \in \Lambda_{r+s}(A) = \bigcup_{\|X-A\| \leq s} \Lambda_r(X) \]

so \( \exists A_s \) with \( \|A_s - A\| \leq s \) and \( i \Omega(\sqrt{s}) \in \Lambda_r(A_s) \). Hence

\[ d(\Lambda_r(A_s), \Lambda_r(A)) \geq d(i \Omega(\sqrt{s}), \Lambda_r(A)) = \Omega(\sqrt{s}). \]

**Theorem** Typically (Arnold), all eigenspaces of \( A \) are one-dimensional. Then, \( \Lambda_\epsilon \) is Lipschitz around \( A \) for all small \( \epsilon > 0 \).
11. CONTROLLABILITY

A control system with state \( x \in \mathbb{C}^m \) and control \( u \),

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Computing \( \delta \) is tractable but impractical: \( O(m^6) \) (Gu 2000).
12. CONNECTED COMPONENTS

Question (Trefethen)  How many components can the rectangular pseudospectrum

\[ \{z \in \mathbb{C} : \sigma_{\min}[A - zI, B] \leq \epsilon \} \]

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(We’ll prove a slightly weaker version... )
13. **THE TYPICAL CASE**

**Theorem (Milnor 1964)**

If \( p : \mathbb{R}^2 \to \mathbb{R} \) is a polynomial of degree \( d \), then

\[
\# \{(x, y) : p(x, y) = 0\} \leq d(2d - 1).
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This assumes \( \lambda_{\min}((P + zQ)(P + zQ)^*) \) simple \( \forall z \in \mathbb{C} \).
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In the space of $m$-by-$m$ Hermitian matrices,

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- Use **lower semicontinuity** of \(\#(\cdot)\) on compact sets.
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- Optimizing pseudospectra is feasible computationally, and avoids these difficulties, by the Kreiss Matrix Theorem.
- The distance to uncontrollability can be computed polynomially by globally minimizing a bivariate function with simple level sets.