Weak-Star Convergence of Convex Sets

S. Fitzpatrick
A. S. Lewis

ORIE, Cornell University, Ithaca, NY 14853, USA
aslewis@orie.cornell.edu  www.orie.cornell.edu/~aslewis

Dedicated to the memory of Simon Fitzpatrick.

Received: March 23, 2005

We show that if a Banach space $X$ is weakly compactly generated and $C, C_n$ are weak-star-closed bounded convex nonempty subsets of the dual space $X^*$, then the support functionals $\delta^*_C$ converge to $\delta^*_C$ pointwise on $X$ if and only if the sequence $(C_n)$ is uniformly bounded with weak-star limit $C$.

Author’s note. The authors were visiting Dalhousie University in 1988 during a seventeen-day labor dispute that left the Mathematics Department empty. During this period they occupied themselves writing the present paper, on a natural topic in variational analysis known elsewhere as “scalar convergence” [22, 16, 4, 17, 3, 18]. Although referenced in the literature [19, 2], this work was never published. The second author reproduces the 1988 manuscript here, essentially unedited, in tribute to the insightful and elegant mathematical vision of Simon Fitzpatrick: it was a privilege to work with Simon.

Keywords: Scalar convergence, weak-star convergence, set convergence, weakly compactly generated

1. Introduction

In any normed space it is possible to define various types of convergence for sequences of convex sets (see for example [16] for a review in the finite-dimensional case). One particularly useful idea was introduced in [12]. A sequence of sets $(C_n)$ converges Mosco to a set $C$ in a normed space $E$ if

\[
\text{for each } x \in C \text{ there exist, for large } n, \ x_n \in C_n, \\
\text{such that } x_n \text{ converges in norm to } x,
\]

(1)

and

\[
\text{for any subsequence } x_{n_j} \in C_{n_j} \text{ with } x_{n_j} \text{ converging weakly to } x, \ x \in C.
\]

(2)

Another way of expressing this is

\[
\text{weak- lim sup } C_n \subset C \subset \text{strong- lim inf } C_n.
\]

(3)

In [13] this notion of convergence is related to the convergence of the corresponding support functionals,

\[
\delta^*_C(x^*) = \sup \{ \langle x^*, x \rangle : x \in C_n \} \quad (x^* \in E^*).
\]
If $F$ is a normed space and $f, f_n : F \rightarrow [-\infty, +\infty]$, then we say $f_n$ converges Mosco to $f$ if for each $x \in F$,

$$\limsup f_n(x_n) \leq f(x) \leq \liminf f_n(y_j) \quad \text{as } n, j \rightarrow \infty$$

for some sequence $x_n$ converging to $x$ in norm, and for any sequence $y_j$ converging weakly to $x$ and any subsequence $(f_n)$. It is shown in the above paper that if $E$ is reflexive and $C$ and all $C_n$ are closed, convex and nonempty, then $C_n$ converges Mosco to $C$ if and only if $\delta_{C_n} \rightarrow \delta_C$ pointwise.

Unfortunately $\delta_{C_n}^*$ can converge pointwise to $\delta_C^*$ without necessarily converging Mosco. For example, take $E = l_2$, $C_n = \{e_n\}$ (the $n$th unit vector), and $C = \{0\}$. Then $\delta_{C_n}^*(x^*) = (x^*)_n$ and $\delta_C^*(x^*) = 0$ for all $x^* \in l_2$, so clearly $\delta_{C_n}^* \rightarrow \delta_C^*$ pointwise. However, the sequence $(-e_n)$ converges to 0 weakly, and $\delta_{C_n}^*(-e_n) = -1$ for all $n$, so

$$0 = \delta_{C_n}(0) > \liminf \delta_{C_n}(-e_n) = -1,$$

and thus $\delta_{C_n}^*$ does not converge Mosco to $\delta_C^*$. It is also easy to see that $C_n$ does not converge Mosco to $C$ (since $e_n$ does not converge to 0 in norm). The question of when $\delta_{C_n}^* \rightarrow \delta_C^*$ pointwise thus remains open. This type of convergence is called $\ast$-convergence in [16], where it is shown that in finite dimensions, when $C$ and all $C_n$ are compact, convex and nonempty, it coincides with classical Kuratowski convergence:

$$\limsup C_n \subset C \subset \liminf C_n.$$

Our purpose here is to generalize this result to the infinite-dimensional case.

It is worth remarking that convergence of support functionals is closely connected with another type of convergence introduced in [21, 22]. We define $d(x, C) = \inf\{\|x - c\| : c \in C\}$ (and $d(x, \emptyset) = +\infty$). Then $C_n$ converges Wijsman to $C$ if $d(\cdot, C_n) \rightarrow d(\cdot, C)$ pointwise. In [5] it is shown that Mosco and Wijsman convergence coincide for sequences of convex sets if and only if the underlying space is reflexive with the dual norm having the Kadec property.

Consider the conjugate of the distance function:

$$d^*(x^*, C) = \sup_{x \in E} \{\langle x^*, x \rangle - d(x, C)\} = \sup_{x \in E} \{\langle x^*, x \rangle - \inf_{y \in C} \|x - y\|\}$$

$$= \sup_{y \in C} \{\langle x^*, x \rangle - \|x - y\|\} = \sup_{y \in C} \{\langle x^*, y + z \rangle - \|z\|\}$$

$$= \sup_{y \in C} \{\langle x^*, y \rangle + \sup_{z \in E} \{\langle x^*, z \rangle - \|z\|\}\}.$$ 

Now it is easy to check that

$$\sup_{x \in E} \{\langle x^*, z \rangle - \|z\|\} = \begin{cases} 0 & (\|x^*\| \leq 1) \\ +\infty & (\text{otherwise}) \end{cases}$$

$$= \delta_B^*(x^*),$$

the indicator function of $B^*$, the unit ball in $E^*$. Thus

$$d^*(x^*, C) = \delta_C^*(x^*) + \delta_B^*(x^*) \quad (x^* \in E^*).$$

Since $\delta_C^*$ is always positively homogeneous it follows that $\delta_{C_n} \rightarrow \delta_C^*$ pointwise if and only if $d^*(\cdot, C_n) \rightarrow d^*(\cdot, C)$ pointwise on $B^*$. 

2. Weak-star convergence

Throughout the remainder of this paper $X$ will be a Banach space and $C$ and $C_n$ ($n = 1, 2, \ldots$) will be weak-star-closed, bounded, nonempty, convex subsets of the dual space $X^*$. We say $C_n$ converges weak-star to $C$ if

$$w^*\text{-lim sup } C_n \subset C \subset w^*\text{-lim inf } C_n,$$

where the weak-star semilimits are taken in the sequential sense. More precisely, we make the following definition.

**Definition 2.1.** The sequence of sets $C_n$ converges weak-star to $C$ if

for each $x^* \in C$ there exist, for large $n$, $x^*_n \in C_n$, such that $x^*_n$ converges weak-star to $x^*$, (4)

and

if, for some subsequence $x^*_{n_j} \in C_{n_j}$, $x^*_{n_j}$ converges weak-star to $x^*$, then $x^* \in C$. (5)

We shall also need the following definition.

**Definition 2.2.** The Banach space $X$ is weakly compactly generated (WCG) if there exists a weakly compact subset of $X$ whose linear span is dense in $X$.

In particular, if $X$ is either separable or reflexive then it is WCG (see for example [7]). We shall need the following two properties of WCG spaces.

**Theorem 2.3 (Amir and Lindenstrauss [1]).** If the Banach space $X$ is a subspace of a WCG space then the unit ball in $X^*$ is weak-star sequentially compact.

**Theorem 2.4 (Davis, Figiel, Johnson and Pelczynski [6]).** A Banach space $X$ is WCG if and only if there is a reflexive space $Y$ and a one-to-one continuous linear operator $T : Y \to X$ with $T(Y)$ dense in $X$.

We are now ready to proceed with one direction in our main result.

**Theorem 2.5.** Suppose that $X$ is WCG. Suppose further that the $C_n$ are uniformly bounded and converge weak-star to $C$. Then $\delta_{C_n} \to \delta_C$ pointwise on $X$.

**Proof.** Suppose for some $x \in X$, $\delta_{C_n}(x) \neq \delta_C(x)$. Then for some $\epsilon > 0$, and for some subsequence $(n_i)$, either

$$\delta_{C_{n_i}}(x) - \delta_C^*(x) > \epsilon \text{ for each } i$$

(6)

or

$$\delta_C^*(x) - \delta_{C_{n_i}}(x) > \epsilon \text{ for each } i.$$  

(7)

Suppose first that (6) holds. Then there exist $x^*_n \in C_n$, for each $i$ satisfying

$$\langle x^*_n, x \rangle > \epsilon + \delta_C^*(x).$$

(8)
Now by assumption $\cup_{i}C_{n_{i}}$ is bounded, and by Theorem 2.3, any closed bounded set in $X^{\ast}$ is weak-star sequentially compact. Thus there exists a subsequence $x_{n_{j}}^{\ast}$ converging weak-star to some $x_{\infty}^{\ast}$, and by (5), $x_{\infty}^{\ast} \in C$. However, from (8),

$$\langle x_{\infty}^{\ast}, x \rangle = \lim_{j} \langle x_{n_{j}}^{\ast}, x \rangle \geq \epsilon + \delta_{C}^{\ast}(x),$$

so $x_{\infty}^{\ast} \not\in C$, which is a contradiction.

On the other hand, suppose (7) holds. Then for some $x^{\ast} \in C$,

$$\langle x^{\ast}, x \rangle > \epsilon + \delta_{C_{n_{i}}}^{\ast}(x) \quad \text{for each } i. \quad (9)$$

By (4) there exists a sequence $x_{n}^{\ast} \in C_{n}$ such that $x_{n}^{\ast}$ converges weak-star to $x^{\ast}$. But then

$$\langle x_{n}^{\ast}, x \rangle \leq \delta_{C_{n_{i}}}^{\ast}(x) < \langle x^{\ast}, x \rangle - \epsilon \quad \text{for each } i,$$

which gives a contradiction as $i \to \infty$.

Notice that in fact this result will hold whenever the unit ball in $X^{\ast}$ is weak-star sequentially compact—for example if $X$ is a subspace of a WCG space, or is smoothly normed or is a weak Asplund space (see [8]).

The following example shows that uniform boundedness of the $C_{n}$ is necessary in Theorem 2.5.

**Example 2.6.** Let $X = l_{2}$, $C = \{0\}$, and $C_{n} = \{\lambda e_{n} : 0 \leq \lambda \leq n\}$, where $e_{n}$ is the $n$th unit vector. $X$ is reflexive, so it is certainly WCG. We claim $C_{n}$ converges weak-star (in fact Mosco) to $C$.

Certainly (4) holds, since $0 \in C_{n}$ for each $n$. Suppose the sequence $(\lambda_{n}, e_{n})$ is weak-star (and so weakly) convergent. It follows from the uniform boundedness principle [10, p. 135] that the sequence is bounded in norm, and so $(\lambda_{n})$ is a bounded sequence. Thus clearly $\lambda_{n}e_{n}$ converges weak-star to $0$, as required. So $C_{n}$ converges weak-star to $C$. Clearly $\delta_{C}^{\ast}(x) = 0$ for all $x \in l_{2}$. On the other hand,

$$\delta_{C_{n}}^{\ast}(x^{\ast}) = \sup\{\langle x, \lambda e_{n} \rangle : 0 \leq \lambda \leq n\} = \max\{nx_{n}, 0\}.$$

Thus if we define $x \in l_{2}$ by $x_{n} = 1/n$, then $\delta_{C}^{\ast}(x) = 0$, whereas $\delta_{C_{n}}^{\ast}(x) = 1$, for all $n$. Thus $\delta_{C_{n}}^{\ast} \not\sim \delta_{C}^{\ast}$ pointwise.

Theorem 2.5 can also fail if the unit ball in $X^{\ast}$ fails to be weak-star sequentially compact.

**Example 2.7.** Consider the subsets of $l_{1}$ defined by $C = \{0\}$, and $C_{n} = \{\lambda e_{n} : -1 \leq \lambda \leq 1\}$ for each $n$, where $e_{n}$ is the $n$th unit vector. Then $C$ and $C_{n}$ are nonempty, convex, uniformly bounded, and both norm and weak-star closed (regarding $l_{1}$ as the dual of $c_{0}$). It is easily checked that $\delta_{C}^{\ast}(x) = 0$ and $\delta_{C_{n}}^{\ast}(x) = |x_{n}|$, for all $x \in l_{\infty}$. Thus $\delta_{C_{n}}^{\ast} \to \delta_{C}^{\ast}$ pointwise on $c_{0}$ but not on $l_{\infty}$. It follows from Theorem 2.5 (since $c_{0}$ is WCG) that $C_{n}$ converges weak-star to $C$ in $l_{1}$, and this may easily be checked directly.

Suppose some subsequence of $(\lambda_{n}e_{n})_{1}^{\infty}$ converges weakly in $l_{1}$, where $-1 \leq \lambda_{n} \leq 1$ for each $n$. Denote the subsequence by $(\lambda_{n_{i}}e_{n_{i}})$, and define $x \in l_{\infty}$ by

$$x_{n} = \left\{ \begin{array}{ll}
(-1)^{i}\text{sign}(\lambda_{n_{i}}), & (n = n_{i}) \\
0, & \text{(otherwise)}
\end{array} \right.$$
Then \( \langle x, \lambda_n e_n \rangle = (-1)^i |\lambda_n| \) for each \( i \). Since this sequence is convergent, \( \lambda_n \to 0 \), so the subsequence converges to 0. Thus \( C_n \to C \) weakly (in fact Mosco) in \( l_1 \). However, as we saw, \( \delta_{C_n} \not\to \delta_C \) pointwise on \( l_\infty \). Thus Theorem 2.5 can fail in the weak case.

Now regard the sets \( C \) and \( C_n (n = 1, 2, \ldots) \) as subsets of \( l_\infty^* \). Thus they are nonempty, convex, uniformly bounded, and weak-star closed. If any subsequence of \( (\lambda_n e_n)_{n=1}^\infty \) converges weak-star in \( l_\infty^* \), where \(-1 \leq \lambda_n \leq 1\) for each \( n \), then the same argument shows it converges to 0. Thus \( C_n \to C \) weak-star in \( l_\infty^* \), and yet as we know, \( \delta_{C_n} \not\to \delta_C \) pointwise on \( l_\infty \). Thus Theorem 2.5 fails in this case, precisely because the unit ball in \( l_\infty^* \) is not weak-star sequentially compact.

3. The reflexive case

We now turn our attention to the converse of Theorem 2.5. The first step is an easy application of the uniform boundedness principle.

**Proposition 3.1.** Suppose \( \delta_{C_n} \to \delta_C \) pointwise on \( X \). Then the \( C_n \) are uniformly bounded.

**Proof.** For any \( x \in X \), \( \delta_{C_n}(x) \to \delta_C(x) \), so \( (\delta_{C_n}(x)) \) is a bounded sequence, and thus

\[
\sup\{\langle x^*, x \rangle : x^* \in \bigcup_n C_n \} = \sup_n \delta_{C_n}(x) < +\infty.
\]

The uniform boundedness principle then implies that \( \bigcup_n C_n \) is bounded in \( X^* \) [10, p. 135] \( \Box \)

Checking the second half (5) of the definition of weak-star convergence follows from the Hahn-Banach Theorem.

**Proposition 3.2.** Suppose \( \delta_{C_n} \to \delta_C \) pointwise on \( X \). If, for some subsequence \( x_{n_j}^* \in C_{n_j}, x_{n_j}^* \to x^* \) weak-star, then \( x^* \in C \).

**Proof.** Suppose \( x^* \not\in C \). Then by the Hahn-Banach Theorem [10, p. 64], we can strongly separate \( x^* \) from \( C \): there exists \( x \in X \) and \( \epsilon > 0 \) with \( \langle x^*, x \rangle > \delta_C(x) + \epsilon \). However, \( \langle x_{n_j}^*, x \rangle \leq \delta_{C_{n_j}}(x) \) for each \( j \), and letting \( j \to \infty \) gives \( \langle x^*, x \rangle \leq \delta_C(x) \), which is a contradiction. \( \Box \)

We will first prove the converse of Theorem 2.5 in the case where \( X \) is reflexive, and then apply Theorem 2.4 to extend to the WCG case. We will need the following lemma. We denote the closed unit ball in the normed space \( Y \) by \( B_Y \), and the weak topology on \( Y \) by \( \sigma(Y, Y^*) \).

**Lemma 3.3.** Suppose the Banach space \( Y \) is reflexive and \( (y_n)_{i=1}^\infty \subset Y \). Then \( (B_Y \cap \text{cl span}(y_n), \sigma(Y, Y^*)) \) is metrizable.

**Proof.** Define \( Z = \text{cl span}(y_n) \). Then \( Z \) is separable, so \( Z^* \) is weak-star separable [10, Ex. 2.22(a)]. Furthermore \( Z \) is a closed subspace of a reflexive space, so is reflexive [10, p. 126]. Thus \( Z^* \) is weakly separable, so separable [10, Ex. 2.22(b)], so \( B_{Z^*} \) is weak-star metrizable [10, p. 72], or in other words \( B_Z \) is weakly metrizable. Now \( B_Z = Z \cap B_Y \), and it is easy to check that \( \sigma(Z, Z^*) \) is exactly \( \sigma(Y, Y^*) \) restricted to \( Z \), since any element of \( Z^* \) has an extension to an element of \( Y^* \). The result follows. \( \Box \)
The next step is to show that the set of all weak limits of $x_n^* \in C_n$ is closed.

**Lemma 3.4.** Suppose $X$ is reflexive and $\delta_{C_n} \to \delta_{C}$ pointwise on $X$. Then the set

$$S = \left\{ x^* \in X^* : \exists x_n^* \in C_n \text{ with } x_n^* \to x^* \text{ weakly} \right\}$$

is closed, convex and bounded.

**Proof.** Since each $C_n$ is convex, so is $S$, and the $C_n$ are uniformly bounded (by $K$ say) by Proposition 3.1, and $KBX^*$ is weakly closed [10, p. 66], $S \subset KBX^*$. It remains to show $S$ is norm closed.

Suppose $x_m^* \in S$ ($m = 1, 2, \ldots$) and $\|x_m^* - x_{\infty}^*\| \to 0$ as $m \to \infty$. By definition, for each $m$ there exist $x_{m,n}^* \in C_n$ ($n = 1, 2, \ldots$) with $x_{m,n}^* \to x_m^*$ weakly as $n \to \infty$. Define $Z = \text{cl span}\{x_{m,n}^* : m, n = 1, 2, \ldots\}$. By Lemma 3.3, $(KBX, \sigma(X^*, X))$ is metrizable, by the metric $\rho$, and note that $x_{m,n}^*$ and $x_{\infty}^*$ lie in $KBX \cap Z$ for each $m, n$.

Now $x_m^* \to x_{\infty}^*$ in norm as $m \to \infty$, so certainly $x_m \to x_{\infty}$ weakly and thus $\rho(x_m^*, x_{\infty}^*) \to 0$ as $m \to \infty$. Thus for each $j$ we can pick $m_j$ so that $\rho(x_{m_j}^*, x_{\infty}^*) < 2^{-j}$. Now pick an increasing sequence $n_j$ ($j = 1, 2, \ldots$) so that for all $n \geq n_j$, $\rho(x_{m_j,n}^*, x_{m_j}^*) < 2^{-j}$: this is possible because $x_{m_j,n}^* \to x_{m_j}^*$ weakly as $n \to \infty$. It now follows by the triangle inequality that $\rho(x_{m_j,n}^*, x_{\infty}^*) < 2^{-j+1}$ for all $n \geq n_j$, and for all $j$. Finally, for each $n$ define $x_n^* = x_{m_j,n}^*$ whenever $n_j \leq n < n_{j+1}$. Then $x_n^* \in C_n$ and $\rho(x_n^*, x_{\infty}^*) < 2^{-j+1}$ for all $n \geq n_j$ and all $j$, so $x_n^* \to x_{\infty}^*$ weakly. Thus $x_{\infty}^* \in S$ as required.

We are now ready to prove the converse of Theorem 2.5 in the reflexive case.

**Theorem 3.5.** Suppose $X$ is reflexive. If $\delta_{C_n} \to \delta_{C}$ pointwise on $X$ then the sequence $(C_n)$ is uniformly bounded with weak-star limit $C$.

**Proof.** After Propositions 3.1 and 3.2, it only remains to show that (4) holds in the definition of weak-star convergence. If we denote by $S$ the set in (10) again, what we need to show is $C \subset S$. (Notice in fact Proposition 3.2 already shows $S \subset C$.)

Suppose that $x^*$ is an exposed point of $C$. In other words, for some $x \in X$, $\langle x^*, x \rangle = \delta_{C}(x)$, but for any $y^* \in C \setminus \{x^*\}$, $\langle y^*, x \rangle < \delta_{C}(x)$. Since $\delta_{C_n}(x) \to \delta_{C}(x)$, there exists a sequence $x_n^* \in C_n$ with $\langle x_n^*, x \rangle \to \delta_{C}(x) = \langle x^*, x \rangle$. We claim $x_n^* \to x^*$ weakly.

Suppose not. Then for some $y \in X$, some $\epsilon > 0$, and some subsequence $(n_i)$, we have $\langle x_{n_i}^*, y \rangle \leq \langle x^*, y \rangle - \epsilon$, for each $i$. Now by Proposition 3.1, the sequence $(x_{n_i}^*)$ is bounded in the reflexive space $X^*$, so has a weakly convergent subsequence, $x_{m_j}^* \to x_{\infty}^*$ weakly as $j \to \infty$. From the above we know therefore that $\langle x_{\infty}^*, x \rangle = \langle x^*, x \rangle$, and $x_{\infty}^* \in C$ by Proposition 3.2, so $x_{\infty}^* = x^*$. But we also have $\langle x_{\infty}^*, y \rangle \leq \langle x^*, y \rangle - \epsilon$, which is a contradiction.

Thus we have shown that the exposed points of $C$ are contained in $S$. Since $S$ is closed and convex by Lemma 3.4, it follows that the closed convex hull of the exposed points of $C$ is contained in $S$. Now, using results that may be found, for example, in [9], since $X$ is reflexive, $X^*$ has the Radon-Nikodym property, and thus $C$ is the closed convex hull of its strongly exposed points. Since each strongly exposed point is exposed, we have $C \subset S$, as required.

\[\square\]
The results in this section may be compared to those in [4], where it is shown in the reflexive case that if \( \delta_{C_n} \to \delta_C \) pointwise on \( X \) then the \( C_n \) are uniformly bounded and \( C \) is the closed convex hull of \( w\)-lim sup \( C_n \).

4. The WCG case

We have now developed the tools to prove our main result. The idea is to use the Davis-Figiel-Johnson-Pelczyński factorization theorem (2.4) to extend Theorem 3.5 from the reflexive case to the WCG case.

**Theorem 4.1.** Suppose \( X \) is WCG and \( C, C_n (n = 1, 2, \ldots) \) are weak-star-closed, bounded, convex, nonempty subsets of \( X^* \). Then \( \delta_{C_n} \to \delta_C \) pointwise on \( X \) if and only if the sequence \( (C_n) \) is uniformly bounded with weak-star limit \( C \).

**Proof.** One direction has already been proved in Theorem 2.5. In the other direction, in view of Propositions 3.1 and 3.2, it only remains to show (4) holds in the definition of weak-star convergence: given \( x^* \in C \), we need to find a sequence \( x^*_n \in C_n \) with \( x^*_n \to x^* \) weak-star.

By Theorem 2.4 there is a reflexive space \( Y \) and a one-to-one continuous linear operator \( T: Y \to X \) with \( T(Y) \) dense in \( X \). Denote the adjoint map by \( T^*: X^* \to Y^* \). Then \( T^* \) is continuous, linear and one-to-one [15, p. 95]. It is easy to see that \( T^* \) is weak-star-to-weak continuous: if a net \( x^*_\alpha \to x^* \) weak-star in \( X^* \) then for any \( y \in Y \),

\[
\langle T^* x^*_\alpha, y \rangle = \langle x^*_\alpha, Ty \rangle \to \langle x^*, Ty \rangle = \langle T^* x^*, y \rangle,
\]

so \( T^* x^*_\alpha \to T^* x^* \) weak-star in \( Y^* \).

Now define \( D, D_n (n = 1, 2, \ldots) \) by \( D = T^* C \), \( D_n = T^* C_n \). The sets \( D \) and \( D_n \) are convex (by the linearity of \( T^* \)), bounded (since \( C \) and \( C_n \) are bounded and \( T^* \) is continuous) and closed (since \( C \) and \( C_n \) are weak-star compact and \( T^* \) is weak-star-to-weak continuous, so \( D \) and \( D_n \) are weakly closed and convex, so closed). Furthermore, by Proposition 3.1, \( C \cup \bigcup_n C_n \) is bounded, so

\[
\text{weak-star-cl} \left( C \cup \bigcup_n C_n \right)
\]

is weak-star compact by the Alaoglu-Bourbaki theorem [10, p. 70]. It now follows by, for example, [20, 3.83], that

\[
T^* : \text{weak-star-cl} \left( C \cup \bigcup_n C_n \right) \to \text{weak-cl} \left( D \cup \bigcup_n D_n \right) \tag{11}
\]

is a weak-star-to-weak homeomorphism. Specifically, it is one-to-one and weak-star-to-weak continuous as above, and it is easily seen to be onto (as defined). Since its range is further weakly compact, it is a weak-star-to-weak homeomorphism.

Now by definition, for \( y \in Y \),

\[
\delta_D^* (y) = \sup \{ \langle y^*, y \rangle : y^* \in D \} = \sup \{ \langle T^* x^*, y \rangle : x^* \in C \} = \sup \{ \langle x^*, Ty \rangle : x^* \in C \} = \delta_C^* (Ty),
\]

and similarly \( \delta_{D_n}^* (y) = \delta_{C_n}^* (Ty) \). By assumption therefore, \( \delta_{D_n}^* \to \delta_D^* \) pointwise on \( Y \).
Since $Y$ is reflexive we can now apply Theorem 3.5. Since $T^*x^* \in D$, there exists a sequence $x^*_n \in C_n$ with $T^*x^*_n \to T^*x^*$ in $Y^*$. Since $T^*$ is a weak-star-to weak homeomorphism, it follows from (11) that $x^*_n \to x^*$ weak-star in $X^*$, as required.

A more general question than that of this paper would seek to relate the convergence of a sequence of functions $f_n$ to $f$ with the convergence of $f^*_n$ to $f^*$. In [13] it is shown that if the underlying space $X$ is reflexive and $f, f_n : X \to (-\infty, +\infty]$ are closed, convex and proper, then $f_n$ converges Mosco to $f$ if and only if $f^*_n$ converges Mosco to $f^*$ (see also [11]). The question of when $f^*_n \to f^*$ pointwise appears to be harder, although it is proved in [14] that under the same conditions, if $f_n \to f$ uniformly on all bounded sets then $f^*_n(x^*) \to f^*(x^*)$ for all $x^* \in \text{int}(\text{dom } f^*)$. This condition seems difficult to check in practice.

References