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Clarke critical values of subanalytic Lipschitz continuous functions

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To the memory of Stanisław Łojasiewicz

Abstract. The main result of this note asserts that for any subanalytic locally Lipschitz function the set of its Clarke critical values is locally finite. The proof relies on Pawłucki's extension of the Puiseux lemma. In the last section we give an example of a continuous subanalytic function which is not constant on a segment of "broadly critical" points, that is, points for which we can find arbitrarily short convex combinations of gradients at nearby points.

1. Introduction. Several Sard-type results are known in the literature using various notions of a critical point. For example, Yomdin's classical paper [18] addresses this issue for *near-critical* points and gives an evaluation of the Kolmogorov metric entropy for the set of near-critical values. In a recent work, Kurdyka–Orro–Simon [12] show that the set of *asymptotically-critical* values of a C^p-semialgebraic mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ has dimension less than k provided that $p \geq \max\{1, n-k+1\}$. Concerning nondifferentiable functions, Rifford [15], extending a previous result of Itoh–Tanaka [10], establishes that the set of *Clarke critical values* of the distance function to a closed submanifold of a complete Riemannian manifold has Lebesgue measure zero.

Our work relies on concepts of generalized critical points in the sense of nonsmooth analysis that we now proceed to describe. We say that x^* is a *limiting subgradient* for the continuous function f on $\mathbb{R}^n$ at x, and we write $x^* \in \partial f(x)$ (the *limiting subdifferential* of f), if there exist sequences $x_n \to x$ and $x_n^* \to x^*$ such that, for n fixed,

$$\liminf_{y \to x_n,\, y \neq x_n} \frac{f(y) - f(x_n) - \langle x_n^*, y - x_n\rangle}{\|y - x_n\|} \geq 0,$$

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that is, each x_n^* is a *Fréchet subgradient* of f at x_n (see also Definition 1(i), (ii)). Clearly, for C^1 functions the notion of limiting subgradient coincides with the usual gradient ∇f of f, while in general the operator $x \mapsto \partial f(x)$ is multivalued. A *limiting-critical point* of f is therefore a point for which there exists a zero subgradient, that is, $\partial f(x) \ni 0$. Concerning nonsmooth analysis and related problems of subdifferentiation, see the introductory books of Clarke [6], Clarke–Ledyaev–Stern–Wolenski [7] or Rockafellar–Wets [16].

In a recent work [4, Theorem 13], we show that any continuous subanalytic function f on $\mathbb{R}^n$ is constant on each connected component of the set of its limiting-critical points. The main motivation for proving this Sard-type result for subanalytic continuous functions was to derive a generalized Łojasiewicz inequality which in turn was used in the asymptotic analysis of subgradient-like dynamical systems [3, Theorem 3.1]. These dynamics occur frequently in various domains such as optimization, mechanics and PDE's.

With this line of research in mind we adopt here a different viewpoint. The assumptions on f are strengthened—namely, f is assumed to be locally Lipschitz continuous—while the definition of a critical point is weakened. As above, this alternative notion relies on a concept of subdifferentiation: we say that x^* is a *Clarke-subgradient* of f at x if

$$x^* \in \partial^\circ f(x) := \overline{\mathrm{co}}\, \partial f(x),$$

where $\overline{\mathrm{co}}\, \partial f(x)$ is the closed convex hull of $\partial f(x)$ (see also Definition 1(iii)). Accordingly, a point x is said to be *Clarke critical* if $\partial^\circ f(x) \ni 0$. This turns out to be equivalent to the following property:

$$(\mathcal{CR}) \qquad 0 \in \overline{\mathrm{co}} \Big\{ \bigcup_{z \in B(x,\varepsilon)} \widehat{\partial} f(z) \Big\} \quad \text{for every } \varepsilon > 0$$

(see Proposition 9 or [5]), which reflects the idea that a point is Clarke critical if we can find short convex combinations of gradients at nearby points ([1]). For instance, $x = 0$ is a Clarke critical point for the function $x \mapsto -\|x\|$, but it is not limiting-critical, since $\partial f(0) = S^{n-1}$ (the unit sphere of $\mathbb{R}^n$), while $\partial^\circ f(x) = B_{\mathbb{R}^n}(0,1)$ (the unit ball of $\mathbb{R}^n$).

Our main result asserts that any locally Lipschitz continuous subanalytic function f defined on some open subset of $\mathbb{R}^n$ is constant on each connected component of the set of its Clarke critical points. Since the latter is subanalytic, it follows directly that the set of Clarke critical values of f is locally finite. The proof of this result is based on a "path-perturbation" lemma [4, Lemma 12], which itself relies heavily on Pawłucki's extension of the Puiseux Lemma [14, Proposition 2].

([1]) This is no longer true for continuous functions: a point satisfying $(\mathcal{CR})$ need not be Clarke critical.

An alternative notion of subdifferential, namely the convex-stable subdifferential, has been introduced by Burke, Lewis and Overton [5]. The corresponding critical points are precisely the points which comply with $(\mathcal{CR})$. As pointed out above, if f is a Lipschitz continuous function, one recovers exactly the notion of a Clarke critical point; however for general continuous functions the convex-stable subdifferential appears to be larger than the usual Clarke subdifferential, giving rise to another concept of a critical point: the "broadly critical points". In the last section we show that a continuous subanalytic function may fail to have the Sard property on the broadly critical set. We indeed exhibit a function $f : \mathbb{R}^3 \to \mathbb{R}$ which is not constant on some segment of points satisfying $(\mathcal{CR})$.

2. Preliminaries. In this section we recall several definitions and results necessary for further developments. For basic and fundamental results of subanalytic geometry see Bierstone–Milman [2], Łojasiewicz [13], van der Dries–Miller [9] or Shiota [17]. Concerning nonsmooth analysis some general references are Clarke [6], Clarke–Ledyaev–Stern–Wolenski [7] or Rockafellar–Wets [16].

In the first two sections, we are interested in locally Lipschitz functions: accordingly, we state the definitions and theorems of nonsmooth analysis that we use specifically for this case. The case of continuous functions is treated in Section 4.

Consequently, throughout Sections 2 and 3 we make the following standing assumption:

$$\boxed{\begin{array}{l} U \text{ is a nonempty open subset of } \mathbb{R}^n \text{ and} \\ f : U \to \mathbb{R} \text{ is locally Lipschitz continuous.} \end{array}}$$

We shall essentially deal with the following three notions of subdifferentiation.

Definition 1. For any $x \in U$ let us define

(i) the *Fréchet subdifferential* $\widehat{\partial} f(x)$ of f at x:

$$\widehat{\partial} f(x) = \Big\{ x^* \in \mathbb{R}^n : \liminf_{y \to x,\, y \neq x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|} \geq 0 \Big\},$$

(ii) the *limiting subdifferential* $\partial f(x)$ of f at x:

$$x^* \in \partial f(x) \ \Leftrightarrow\ \exists x_n \in U,\ \exists x_n^* \in \widehat{\partial} f(x_n) : x_n \to x,\ x_n^* \to x^* \text{ as } n \to \infty,$$

(iii) the *Clarke subdifferential* $\partial^\circ f(x)$ of f at x:

$$\partial^\circ f(x) = \overline{\mathrm{co}}\, \partial f(x), \tag{1}$$

where $\overline{\mathrm{co}}\, \partial f(x)$ is the closed convex hull of $\partial f(x)$.

For every function f and every $x \in \operatorname{dom} f$ we obviously have:

$$\widehat{\partial} f(x) \subset \partial f(x) \subset \partial^\circ f(x).$$

REMARK 1. (a) If $T : U \rightrightarrows \mathbb{R}^n$ is a point-to-set mapping, its domain and its graph are respectively defined by $\operatorname{dom} T := \{x \in U : T(x) \neq \emptyset\}$ and $\operatorname{Graph} T := \{(x, y) \in U \times \mathbb{R}^n : y \in T(x)\}$. Clearly $\operatorname{dom} \widehat{\partial} f \subset \operatorname{dom} \partial f \subset \operatorname{dom} \partial^\circ f$. A well known result of variational analysis asserts that $\operatorname{dom} \widehat{\partial} f$ is a dense subset of U (see [6], for example).

(b) Since f is locally Lipschitz continuous, the point-to-set mapping $U \ni x \mapsto \partial^\circ f(x)$ is bounded on compact subsets of U.

(c) If f is differentiable at x, then $\widehat{\partial} f(x) = \{\nabla f(x)\}$.

(d) If f is a subanalytic function all the subdifferential mappings defined above have a subanalytic graph (see [4, Proposition 2.13]).

The notion of a Clarke critical point is then defined naturally.

DEFINITION 2. A point $a \in U$ is called *Clarke critical* for a locally Lipschitz function f if

$$\partial^\circ f(a) \ni 0,$$

or equivalently, if relation $(\mathcal{CR})$ holds (see Proposition 9).

REMARK 2. Let us recall that a locally Lipschitz function f is called *subdifferentially regular* if

$$\widehat{\partial} f = \partial f,$$

or equivalently if

$$\widehat{\partial} f = \partial^\circ f.$$

For subdifferentially regular functions, the sets of Fréchet critical and of Clarke critical points coincide and one can obtain easily the conclusion of our main result via an elementary argument (see Remark 3 for details).

Let us recall the chain rule for subdifferentials (see [16, Theorem 10.6, p. 427], for example).

PROPOSITION 3 (subdifferential chain rule). *Let V be an open subset of $\mathbb{R}^m$ and $G : V \to U$ a C^1 mapping. Define $g : V \to \mathbb{R}$ by $g(x) = f(G(x))$ for all $x \in V$. Then*

$$\widehat{\partial g}(x) \supset \nabla G(x)^T \widehat{\partial} f(G(x)), \tag{2}$$

$$\partial g(x) \subset \nabla G(x)^T \partial f(G(x)), \tag{3}$$

where $\nabla G(x)^T$ denotes the transpose of the Jacobian matrix of G at x. If in addition G is a diffeomorphism the above inclusions become equalities, thus

$$\partial g(x) = \nabla G(x)^T \partial f(G(x)), \quad \partial^\circ g(x) = \nabla G(x)^T \partial^\circ f(G(x)), \quad \forall x \in V. \tag{4}$$

The following lemma, based on a result of Pawłucki [14], plays a key role in the proof of both Theorem 5 and Theorem 7.

LEMMA 4 (path perturbation lemma, [4, Lemma 12]). *Let F be a nonempty subanalytic subset of $\mathbb{R}^n$, $\gamma : [0,1] \to \operatorname{cl} F$ a one-to-one continuous subanalytic path and $\eta > 0$. Then there exists a continuous subanalytic path $z : [0,1] \to \operatorname{cl} F$ such that*

(i) $\|\dot{z}(t) - \dot{\gamma}(t)\| < \eta$ *for almost all* $t \in (0,1)$,

(ii) *the (subanalytic) set*

$$\varDelta := \{t \in [0,1] : z(t) \in \operatorname{cl} F \setminus F\} \tag{5}$$

has Lebesgue measure less than η,

(iii) $z(t) = \gamma(t)$ *for all* $t \in \varDelta \cup \{0,1\}$.

Let us recall the following Sard-type result concerning the limiting-critical points of continuous subanalytic functions.

THEOREM 5 (Sard theorem for limiting-critical points, [4, Theorem 13]). *Let $g : U \to \mathbb{R}$ be a subanalytic continuous function. Then f is constant on each connected component of the set of its limiting-critical points*

$$(\partial f)^{-1}(0) := \{x \in U : \partial f(x) \ni 0\}.$$

Unless the function is subdifferentially regular, the above theorem is obviously not appropriate for the study of locally Lipschitz functions with the Clarke subdifferential. Typical examples are given by functions whose epigraphs have "inward corners", such as for instance $f(x) = -\|x\|$. Sharp saddle points also provide some elementary illustrations. For example if one sets

$$f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \ni (x,y,z) \mapsto \|x\| - \|y\|,$$

then points of the type $(0,0,z)$ are Clarke critical but not limiting-critical. Indeed, by straightforward computations, $\partial f(0,0,z) = B_{\mathbb{R}^m}(0,1) \times S^{n-1} \times \{0\}_p$ and $\partial^\circ f(0,0,z) = B_{\mathbb{R}^m}(0,1) \times B_{\mathbb{R}^n}(0,1) \times \{0\}_p$.

3. A Sard theorem for subanalytic Lipschitz continuous functions. For the proof of the central result of this note we will need the following lemma.

LEMMA 6. *Set $e := (1,0,\ldots,0) \in \mathbb{R}^n$ and assume that $[0,1]e \subset U$, with $\partial^\circ f(te) \ni 0$ for all $t \in [0,1]$. Then f is constant on $[0,1]e$.*

Proof. Let us provisionally set $S_L := \{x \in [0,1]e : 0 \in \partial f(x)\}$, where ∂f denotes the limiting subdifferential of f (Definition 1(ii)). By Remark 1(d), the set S_L is subanalytic, thus, being a (closed) subset of $[0,1]e$, it is a finite union of segments. By using Theorem 5 we conclude that f is constant on each one of them. Owing to the continuity of f, it is therefore sufficient to prove that f is also constant on each nontrivial segment of $[0,1]e \setminus S_L$. This shows that there is no loss of generality to assume that S_L is empty, that is:

$$0 \notin \partial f(te), \quad t \in [0,1].$$

Now fix some $\delta > 0$ and define

$$\Gamma_\delta = \{x \in [0,1]e : \forall x^* \in \partial f(x),\ |\langle x^*, e\rangle| > \delta\}. \tag{6}$$

We observe that (6) defines a subanalytic subset of $\mathbb{R}^n$. Let us prove by contradiction that this set is finite.

Indeed, if this were not the case, then by using the subanalyticity of Γ_δ, there would exist $a < b$ in $[0,1]$ such that $(a,b)e \subset \Gamma_\delta$. Let V be an open bounded subset of U such that $[0,1]e \subset V \subset \operatorname{cl} V \subset U$ and define

$$\widehat{\Gamma}_\delta^+ = \{x \in \operatorname{cl} V : \exists x^* \in \widehat{\partial} f(x),\ \langle x^*, e\rangle > \delta\},$$
$$\widehat{\Gamma}_\delta^- = \{x \in \operatorname{cl} V : \exists x^* \in \widehat{\partial} f(x),\ \langle x^*, e\rangle < -\delta\},$$

where $\widehat{\partial} f$ denotes the Fréchet subdifferential of f (Definition 1(i)). Since $0 \in \partial^\circ f(x) = \overline{\text{co}}\, \partial f(x)$ for every $x \in \Gamma_\delta$, we have

$$\max\{\langle x^*, e\rangle : x^* \in \partial f(x)\} > \delta \quad \text{and} \quad \min\{\langle x^*, e\rangle : x^* \in \partial f(x)\} < -\delta.$$

So using the definition of the limiting subdifferential we conclude that $(a,b)e \subset \operatorname{cl} \widehat{\Gamma}_\delta^+$ and $(a,b)e \subset \operatorname{cl} \widehat{\Gamma}_\delta^-$.

Set $l = b-a$ and $M := \sup\{\|x^*\| : x^* \in \partial^\circ f(x),\ x \in \operatorname{cl} V\}$. The finiteness of M comes from the Lipschitz continuity property of f (see Remark 1(b) for instance) and the compactness of $\operatorname{cl} V$. The function $t \mapsto f(te)$ is subanalytic and continuous, hence absolutely continuous ([4, Lemma 5]). Thus by using relation (3) of Proposition 3 (subdifferential chain rule), we infer for all $0 \le u \le v \le 1$ that

$$\int_u^v \left|\frac{d}{dt} f(te)\right| dt \le (v-u) \sup\{|\langle e, x^*\rangle| : t \in [u,v],\ x^* \in \partial f(te)\} \le (v-u)M.$$

Take $\eta > 0$ and apply Lemma 4 (path perturbation lemma) for $F = \widehat{\Gamma}_\delta^+$, and $\gamma(t) = te$, $t \in (a,b)$. Since $\dot\gamma(t) = e$ for all $t \in [0,1]$, it follows that there exists a subanalytic continuous curve $z : [a,b] \to \operatorname{cl} \widehat{\Gamma}_\delta^+$ such that

- $\|\dot z(t) - e\| < \eta$ for almost all $t \in (a,b)$,
- the (subanalytic) set $\Delta := \{t \in [a,b] : z(t) \in \operatorname{cl} \widehat{\Gamma}_\delta^+ \setminus \widehat{\Gamma}_\delta^+\}$ has Lebesgue measure less than η,
- $z(t) = \gamma(t)$ for all $t \in \Delta \cup \{a,b\}$.

The continuous function $g(t) = f(z(t))$ is also subanalytic, so for all but finitely many t's in $(a,b) \setminus \Delta$ we conclude from relation (2) of Proposition 3 and Remark 1(c) that

$$\{g'(t)\} = \widehat{\partial} g(t) \supset \langle \dot z(t), \widehat{\partial} f(z(t))\rangle \supset \{\langle \dot z(t), z_+^*(t)\rangle\},$$

where $z_+^*(t) \in \widehat{\partial} f(z(t))$ can be chosen in order to satisfy $\langle e, z_+^*(t)\rangle > \delta$ (since $z(t) \in \widehat{\Gamma}_\delta^+$). Thus for almost all t in $[a,b] \setminus \Delta$ we have

$$g'(t) = \langle e, z_+^*(t)\rangle + \langle \dot z(t) - e, z_+^*(t)\rangle \geq \delta - \|\dot z(t) - e\| M \geq \delta - \eta M,$$

so that

$$\begin{aligned} f(be) - f(ae) &= \int_a^b \frac{d}{dt} f(z(t))\,dt \geq \int_{[a,b]\setminus\Delta} g'(t)\,dt - \int_\Delta \left|\frac{d}{dt} f(z(t))\right| dt \\ &\geq (l-\eta)(\delta - \eta M) - \eta M. \end{aligned}$$

By choosing η small enough, the above quantity can be made positive so that $f(be) > f(ae)$. It suffices to repeat the argument with $\widehat{\Gamma}_\delta^-$ to obtain $f(be) < f(ae)$, which yields a contradiction.

Thus the set Γ_δ is finite. We further set

$$\Gamma_0 = \{x \in [0,1]e : \exists x^* \in \partial f(te),\ \langle x^*, e\rangle = 0\}.$$

It follows easily from Definition 1(ii) that the limiting subdifferential ∂f has closed values. Thus, the set $\partial f(te)$ is closed for every $t \in [0,1]$, which yields

$$[0,1]e = \Gamma_0 \cup \bigcup_{i\geq 1} \Gamma_{1/i}.$$

Note that $\bigcup_{i\geq1} \Gamma_{1/i}$ is countable and equal to the subanalytic set $[0,1]e \setminus \Gamma_0$. It follows that $\bigcup_{i\geq1} \Gamma_{1/i}$ is finite and so $\{t \in [0,1] : te \in \Gamma_0\}$ is a finite union of intervals with a finite complement in $[0,1]$. Using the continuity of f, it suffices to prove that f is constant on each segment of Γ_0.

Let $(a,b)e \subset \Gamma_0$ with $0 \leq a < b \leq 1$. For any $\varepsilon > 0$ we define

$$\widehat{\Gamma}_0^\varepsilon := \{x \in \operatorname{cl} V : \exists x^* \in \widehat{\partial} f(x),\ |\langle x^*, e\rangle| < \varepsilon\}.$$

By definition of the limiting subdifferential, $(a,b)e \subset \operatorname{cl} \widehat{\Gamma}_0^\varepsilon$. Applying Lemma 4 for the set $\widehat{\Gamma}_0^\varepsilon$, for $\eta < \varepsilon$ and for the path $\gamma(t) = te$, we obtain a curve $z : [a,b] \to \widehat{\Gamma}_0^\varepsilon$ and a set $\Delta \subset [a,b]$ satisfying (i)–(iii) of Lemma 4. Set $h(t) = f(z(t))$. As before, for all but finitely many t's in $[a,b] \setminus \Delta$ we have $\{h'(t)\} = \{\langle \dot z(t), z_\varepsilon^*(t)\rangle\}$, where $z_\varepsilon^*(t) \in \widehat{\partial} f(z(t))$ can be taken such that $|\langle z_\varepsilon^*(t), e\rangle| < \varepsilon$. Therefore for almost all t in $[a,b] \setminus \Delta$ we have

$$|h'(t)| = |\langle e, z_+^*(t)\rangle + \langle \dot z(t) - e, z_+^*(t)\rangle| \leq \varepsilon + \eta M,$$

so that

$$\begin{aligned} |f(be) - f(ae)| &\leq \int_a^b \left|\frac{d}{dt} f(z(t))\right| dt \leq \int_{[a,b]\setminus\Delta} |h'(t)|\,dt + \int_\Delta \left|\frac{d}{dt} f(z(t))\,dt\right| \\ &\leq (l-\eta)(\varepsilon + \eta M) + \eta M. \end{aligned}$$

Taking ε (and thus η) sufficiently small, we see that the function f is constant on $[0,1]e$ and the proof is complete. ∎

THEOREM 7 (main result). *Let U be a nonempty open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a locally Lipschitz subanalytic mapping. Let S denote the set of Clarke critical points of f, that is,*

$$S := \{x \in U : \partial^\circ f(x) \ni 0\}.$$

Then f is constant on each connected component of S.

Proof. Let x, y belong to the same connected component of S. We have to prove that $f(x) = f(y)$. Since $S = (U \times \{0\}_n) \cap \operatorname{Graph} \partial^\circ f$, we conclude by Remark 1(d) that it is a subanalytic set, so every connected component of S is also path-connected (see [1], [2] or [8], for example). Thus, there exists a continuous subanalytic path $\gamma : [0,1] \to S$ joining x to y. To prove that $f(x) = f(y)$ it suffices to prove that f is constant on $\gamma(0,1)$. By using the subanalyticity of γ together with the continuity of f, we can assume that:

- $\gamma(0,1)$ is a subanalytic submanifold of U.
  [Indeed, since $\gamma(0,1)$ is a finite union of subanalytic manifolds, we can deal with each one separately, establishing (as will be described below) that f is constant on each such manifold. Then the same conclusion will follow for $\gamma(0,1)$ by a continuity argument.]
- There exists a subanalytic diffeomorphism G from a neighbourhood V of $\gamma(0,1)$ into an open subset of $\mathbb{R}^n$ such that $G(\gamma(0,1)) = (0,1)e$; see [2] for instance.

In view of relation (4) of Proposition 3 we have

$$\gamma(0,1) \subset (\partial^\circ f)^{-1}(0) \quad \text{if and only if} \quad (0,1)e \subset [\partial^\circ (f \circ G^{-1})]^{-1}(0).$$

This is indeed a consequence of the equivalence

$$\partial^\circ f(x) \ni 0 \;\Leftrightarrow\; \partial^\circ [f \circ G^{-1}](G(x)) \ni 0, \qquad \text{for all } x \in V.$$

As a consequence f is constant on $\gamma(0,1)$ if and only if $f \circ G^{-1}$ is constant on $(0,1)e$. The conclusion then follows from Lemma 6. ∎

COROLLARY 8 (Sard theorem for Clarke critical points). *Under the assumptions of Theorem 7 the set $f(S)$ of the Clarke critical values of f is countable (and hence has measure zero).*

Proof. This follows from Theorem 7 and the fact that the set S, being subanalytic, has at most a countable number of connected components (a finite number on each compact subset of U). ∎

Let us finally conclude with the following remark.

REMARK 3 (case of subdifferential regularity). If f is assumed to be subdifferentially regular (see Remark 1(b)) then Theorem 7 follows via a straightforward application of [16, Theorem 10.6]. Let us recall this simple argument (see also [3, Remark 3.2]). Assume that x, y are in the same connected component of S. Let $z : [0,1] \to S$ be a continuous subanalytic path with $z(0) = x$ and $z(1) = y$ and define the subanalytic function $h(t) = (f \circ z)(t)$. From the "monotonicity lemma" (see [9, Fact 4.1], or [11, Lemma 2], for example) we get $h'(t) = 0$ for all $t \in [0,1] \setminus F$ where F is a finite set. Since $0 \in \widehat{\partial} f(z(t))$ for all $t \in [0,1]$, using the chain rule for the Fréchet subdifferential we obtain

$$\{h'(t)\} = \widehat{\partial} h(t) \supseteq z'(t)\widehat{\partial} f(z(t)) \supseteq \{0\}$$

for all $t \in [0,1] \setminus F$. It follows that h is constant on $[0,1]$, whence $f(x) = f(y)$.

4. An example of a continuous subanalytic function which is not constant on the set of its broadly critical points. In this section we assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. In that case the definition of the Clarke subdifferential (1) of f at $x \in \mathbb{R}^n$ is as follows:

$$\partial^\circ f(x) = \overline{\mathrm{co}}\{\partial f(x) + \partial^\infty f(x)\} \tag{7}$$

where $\partial^\infty f(x)$ is the *asymptotic limiting subdifferential* of f at x, that is, the set of all $y^* \in \mathbb{R}^n$ such that there exists $\{t_n\}_n \subset \mathbb{R}_+$ with $\{t_n\} \searrow 0_+$, $\{y_n\}_n \subset \mathbb{R}^n$, $y_n^* \in \widehat{\partial} f(y_n)$ such that $y_n \to x$ and $t_n y_n^* \to y^*$. When f is locally Lipschitz continuous, the local boundedness of the limiting subgradients (Remark 1(b)) implies $\partial^\infty f(x) = 0$, and so the above definition is—of course—compatible with Definition 1(iii).

Following the terminology of [5], let us now introduce the convex-stable subdifferential. For every $x \in \mathbb{R}^n$ set

$$T_f(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\Big\{ \bigcup_{x \in B(x_0,\varepsilon)} \widehat{\partial} f(x) \Big\}. \tag{8}$$

A point $x_0 \in \mathbb{R}^n$ is called a *broadly critical point* for f if

$$0 \in T_f(x_0). \tag{9}$$

The proof of the following proposition can be found in [5].

PROPOSITION 9. *Let U be nonempty open subset of $\mathbb{R}^n$.*

(i) *For any continuous function $f : U \to \mathbb{R}$ we have*

$$\partial^\circ f(x) \subset T_f(x) \quad \text{for all } x \in U.$$

(ii) *If $f : U \to \mathbb{R}$ is a locally Lipschitz function, then*

$$\partial^\circ f(x) = T_f(x) \quad \text{for all } x \in U.$$

Consequently, for locally Lipschitz functions, Clarke critical and broadly critical points coincide.

We now provide an example showing that the conclusion of Theorem 7 (main result) is no more valid for the set of broadly critical points of a continuous subanalytic function. More precisely (see Facts 1–3 below):

There exists a continuous subanalytic function $f : \mathbb{R}^3 \to \mathbb{R}$ which is not constant on a segment of broadly critical points.

Construction of the example. Consider the function $\theta_0 : [0,\pi) \to [0,\pi/2]$ defined by

$$\theta_0(z) := \begin{cases} z & \text{if } 0 \le z \le \pi/2, \\ \pi - z & \text{if } \pi/2 < z < \pi. \end{cases}$$

We extend the domain of θ_0 from $[0,\pi)$ to $\mathbb{R}$ in the following way:

$$z \mapsto \widetilde{\theta}_0(z) := \theta_0(z \,(\mathrm{mod}\ \pi)).$$

Then for every $(\theta, z) \in [0,\pi/2] \times \mathbb{R}$ we define

$$\sigma(\theta, z) := \begin{cases} 1 & \text{if } \theta \ge \widetilde{\theta}_0(z), \\ -1 & \text{if } \theta < \widetilde{\theta}_0(z). \end{cases}$$

Finally, for every $(\varrho, \theta, z) \in \mathbb{R}_+^* \times [0,\pi/2] \times \mathbb{R}$ we set

$$\varPhi_1(\varrho,\theta,z) = \begin{cases} (2/\pi)\widetilde{\theta}_0(z) + \sigma(\theta,z)\varrho & \text{if } \varrho \le (2/\pi)|\theta - \widetilde{\theta}_0(z)|, \\ (2/\pi)\theta & \text{if } \varrho > (2/\pi)|\theta - \widetilde{\theta}_0(z)|. \end{cases} \tag{10}$$

Now for $(\varrho, \theta, z) \in \mathbb{R}_+^* \times [0,\pi) \times \mathbb{R}$ we set

$$\varPhi_2(\varrho,\theta,z) = \begin{cases} \varPhi_1(\varrho,\theta,z) & \text{if } 0 \le \theta \le \pi/2, \\ \varPhi_1(\varrho,\pi-\theta,z) & \text{if } \pi/2 < \theta \le \pi. \end{cases}$$

Finally, we define $\varPhi : \mathbb{R}_+^* \times [0,2\pi) \times \mathbb{R} \to [0,1]$ by

$$\varPhi(\varrho,\theta,z) = \begin{cases} \varPhi_2(\varrho,\theta,z) & \text{if } 0 \le \theta \le \pi, \\ \varPhi_2(\varrho,\theta-\pi,z) & \text{if } \pi < \theta < 2\pi. \end{cases} \tag{11}$$

Define $f : \mathbb{R}^3 \to [0,1]$ as the function whose graph in cartesian coordinates is the one of $\varPhi$ in cylindrical coordinates. For instance, for any $x, y > 0$ we have

$$f(x,y,z) = \varPhi(\sqrt{x^2+y^2}, \arctan(y/x), z).$$

FACT 1. *The function f is continuous and subanalytic.*

[For the subanalyticity of f it is crucial that the function $t \mapsto \arctan(1/t)$, $t > 0$, extends to an analytic function in a neighbourhood of $t = 0$. To see this, note that $\arctan(1/t) = \pi/2 - \arctan t$ for all $t > 0$.]

FACT 2. *The restriction of f to the set $Z = \{(0,0,z) : z \in \mathbb{R}\}$ is not constant.*

FACT 3. *Every point of Z is broadly critical, that is, $Z \subset \{u \in \mathbb{R}^3 : T_f(u) \ni 0\}$.*

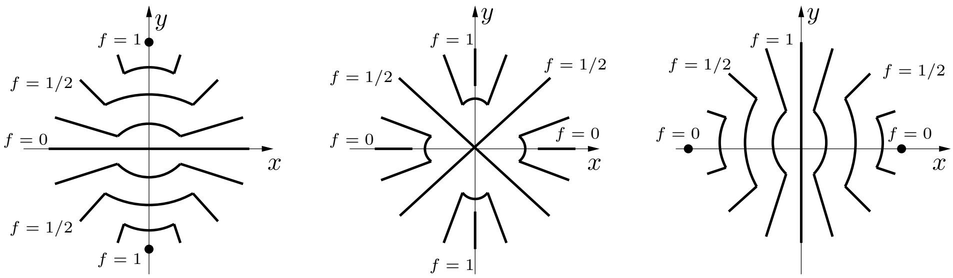

Fig. 1. Level sets of the function $f(\cdot,\cdot,z)$ for $z=0,\pi/4,\pi/2$

Proof of Fact 3. It is sufficient to prove that if $0<z_0<\pi/2$, then $0\in\partial^\circ f((0,0,z_0))$.

To this end, set $u_0=(0,0,z_0)$, $\theta_0=\widetilde{\theta}_0(z_0)$ (so that $0<\theta_0<\pi/2$) and let

$$\theta_n=\theta_0+\frac{\pi}{2^{n+2}} \tag{12}$$

(so that $\theta_n\searrow\theta_0$). Then set $a_n=\tan\theta_n$ and

$$x_n=\frac{1}{2^n\sqrt{1+a_n^2}} \tag{13}$$

(so that $x_n\searrow 0$), $y_n:=a_nx_n$ and thus

$$\varrho_n=\sqrt{x_n^2+y_n^2}=(\sqrt{1+a_n^2})x_n=\frac{1}{2^n}. \tag{14}$$

For every $n\ge 1$ we define

$$u_n:=(x_n,y_n,z_0)\quad\text{and}\quad\overline{u}_n=(-x_n,-y_n,z_0).$$

In view of (10), (13) and (14), the sequences $\{u_n\}_{n\ge1}$, $\{\overline{u}_n\}_{n\ge1}\subset\mathbb{R}^3$ converge to u_0 and satisfy

$$f(u_n)=f(\overline{u}_n)=\Phi(\varrho_n,\theta_n,z_0)=(2/\pi)\theta_n.$$

By (11) and (10) it is easily seen that f is differentiable at u_n (respectively, at $\overline{u}_n$). Precisely, we have

$$\frac{\partial\Phi}{\partial\varrho}(u_n)=\frac{\partial\Phi}{\partial z}(u_n)=0$$

and

$$\frac{\partial\Phi}{\partial\theta}(u_n)=\frac{2}{\pi},$$

so we conclude that

$$\nabla f(u_n)=\frac{2}{\pi}\left(\frac{-y_n}{x_n^2+y_n^2},\frac{x_n}{x_n^2+y_n^2},0\right).$$

Repeating the above for the sequence $\{\overline{u}_n\}_{n\geq 1}$ we obtain

$$\nabla f(\overline{u}_n) = -\nabla f(u_n),$$

or in other words,

$$0 \in \bigcap_{\varepsilon>0} \mathrm{co}\{\nabla f(u) : u \in B(u_0,\varepsilon) \cap D_f\}$$

where D_f denotes the points of differentiability of f. This shows that the point u_0 is broadly critical. ∎

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