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Variational Analysis of the Abscissa Mapping for Polynomials via the Gauss-Lucas Theorem

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Abstract. Consider the linear space $\mathcal{P}^n$ of polynomials of degree n or less over the complex field. The abscissa mapping on $\mathcal{P}^n$ is the mapping that takes a polynomial to the maximum real part of its roots. This mapping plays a key role in the study of stability properties for linear systems. Burke and Overton have shown that the abscissa mapping is everywhere subdifferentially regular in the sense of Clarke on the manifold $\mathcal{M}^n$ of polynomials of degree n. In addition, they provide a formula for the subdifferential. The result is surprising since the abscissa mapping is not Lipschitzian on $\mathcal{M}^n$. A key supporting lemma uses a proof technique due to Levantovskii for determining the tangent cone to the set of stable polynomials. This proof is arduous and opaque. It is a major obstacle to extending the variational theory to other functions of the roots of polynomials. In this note, we provide an alternative proof based on the Gauss-Lucas Theorem. This new proof is both insightful and elementary.

1. Introduction

Stability theory for linear systems continues to be one of the most important and challenging topics in systems engineering and control. A fundamental result states that every solution to the linear system $u' = Au$ is stable if and only if all of the eigenvalues of A lie in the left half of the complex plane. For this reason, a matrix is said to be *stable* if its spectrum lies in the left half plane.

The maximum real part of the spectrum of A is called the *spectral abscissa* of A. The goal is to understand the variational behavior of this mapping in the context of optimization. Although there exists an extensive classical literature on thevariational properties of eigenvalues, these results alone are inadequate for the development of satisfactory necessary and sufficient conditions for optimization problems involving eigenvalues, as well as for understanding the sensitivity of optimal solutions to perturbations. As an illustration, consider the optimization problem

$$\min\{\alpha(A) \mid A \in M\}, \tag{1}$$

where $\alpha:\mathbb{C}^{n\times n}\to\mathbb{R}$ is the spectral abscissa mapping,

$$\alpha(A)=\max\{\operatorname{Re}\lambda\,|\,\det(\lambda I-A)=0\}, \tag{2}$$

and M is a manifold in the space of all complex $n\times n$ matrices $\mathbb{C}^{n\times n}$. In the presence of smoothness, a standard first-order optimality condition for (1) is $\nabla\alpha(A)\perp M$. Unfortunately, this condition is woefully inadequate for most problems involving the spectral abscissa mapping since the optimal solution typically occurs at points where the mapping α is not differentiable. The reason for this is familiar, and comes from the underlying *minimax* nature of the problem. As one pushes down on the spectral abscissa, the eigenvalues get squeezed further and further into the left half of the complex plane so that at an optimal solution there are typically several eigenvalues whose real parts attain the optimal value [2]. Often these *active* eigenvalues are multiple. At such points the spectral abscissa mapping is generically not Lipschitzian. Until quite recently there did not exist a tractable general purpose set of necessary conditions for problems of this type. In [5], Burke and Overton employ the techniques of modern nonsmooth analysis [7, 11] and derive optimality conditions for this problem using the notion of a generalized subdifferential. In [1] and [2] this theory is used to study the sensitivity of solutions to optimization problems of the type (1).

The study of the variational properties of the spectral abscissa mapping begins in the more elementary setting of polynomials. Polynomial results are then extended to matrices by applying them to the characteristic polynomial. The polynomial results are developed in [4] and are based on a proof technique due to Levantovskii [8] for computing the tangent cone to the set of *stable polynomials*. Unfortunately, this proof technique is extremely arduous and opaque, giving little insight into the underlying geometry of the problem. In this note we introduce a completely different approach based on the Gauss-Lucas theorem. The approach simplifies the analysis in [4] and provides wonderful insight into the underlying variational geometry. In addition, the results in [8] also follow as an elementary consequence. For the purpose of illustration, we focus on the application of the Gauss-Lucas theorem to the computation of the tangent cone to the epigraph of the abscissa mapping at the polynomial $p(\lambda)=\lambda^n$.

2. The Abscissa Mapping for Polynomials

Let $\mathcal{P}^n$ be the linear space of polynomials of degree n or less over the complex field, and let $\mathcal{M}^n$ be the open dense subset of $\mathcal{P}^n$ of polynomials of degree n. Define the abscissa mapping $a:\mathcal{P}^n\to\mathbb{R}$ to be the mapping that takes a polynomial to the maximum real part of its roots:

$$a(p)=\max\{\operatorname{Re}\lambda\,|\,p(\lambda)=0\}. \tag{3}$$

The abscissa mapping is continuous on $\mathcal{M}^n$, but not on $\mathcal{P}^n$. Indeed, a is unbounded in a neighborhood of every polynomial not in $\mathcal{M}^n$. To see this one need only consider the polynomial $q_\epsilon(\lambda)=(1-\epsilon\lambda)p(\lambda)$ for any polynomial p of degree less than n. Clearly, $q_\epsilon \to p$ as $\epsilon \downarrow 0$ while for ϵ sufficiently small and positive $a(q_\epsilon)=\epsilon^{-1}$.

Further note that although the abscissa mapping is continuous on $\mathscr{M}^n$ it is not Lipschitz continuous there. An easy of illustration of this is obtained by considering the family of polynomials $p_\epsilon(\lambda)=\lambda^n-\epsilon$ for ϵ real and positive and noting that $a(p_\epsilon)=\epsilon^{1/n}$. Classical methods of differential analysis are inadequate for the analysis of this kind of variational behavior. Hence we turn to modern methods of nonconvex variational analysis. These methods have been extensively developed over the last 30 years beginning with the seminal work of Clarke [6]. We employ the terminology and notation developed in the recent book by Rockafellar and Wets [11]. With these tools Burke and Overton [4] establish the following remarkable result on the subdifferential geometry of the abscissa mapping for polynomials.

THEOREM 2.1. *[4, Theorem 3.2] The abscissa mapping for polynomials is everywhere subdifferentially regular on $\mathscr{M}^n$. Equivalently, epi(a) is everywhere Clarke regular.*

In order to appreciate this result we need to recall a few basic definitions from variational analysis. The *dual* variational objects defined below require the notion of an inner product on $\mathscr{P}^n$. First define the real inner product on $\mathbb{C}$ in the usual way by setting by

$$\langle w,z\rangle=\mathrm{Re}\,\overline{w}z.$$

The real inner product on $\mathscr{P}^n$ is then given by

$$\langle q,p\rangle=\sum_{j=0}^{n}\langle a_j,b_j\rangle,$$

where

$$q(\lambda)=\sum_{j=0}^{n}a_j\lambda^{n-j} \quad \text{and} \quad p(\lambda)=\sum_{j=0}^{n}b_j\lambda^{n-j}\ .$$

This induces the inner product

$$\langle(q,\omega),(p,\eta)\rangle=\langle q,p\rangle+\omega\eta$$

on $\mathscr{P}^n\times\mathbb{R}$. The *polar* of a set S in a real inner product space X is given by

$$S^\circ=\{z\,|\,\langle z,x\rangle\leqslant 1\ \forall\ x\in S\}.$$

DEFINITION 2.2. *Let a be the abscissa mapping defined in (3).*

1. *Epigraph: The epigraph of a is the subset of $\mathscr{P}^n\times\mathbb{R}$ given by*

$$epi(a)=\{(p,\mu)\,|\,a(p)\leqslant\mu\}.$$

2. *Tangent Cone*: *The tangent cone to epi(a) at $(p,\mu)\in epi(a)$ is the cone*

$$T_{epi(a)}(p,\mu)=\left\{(z,\eta)\;\middle|\;\begin{array}{c}\exists\,\{(p^k,\mu_k)\}\subset epi(a) \text{ and } \{t_k\}\subset\mathbb{R}_+\\ \text{such that}\\ (p^k,\mu_k)\to(p,\mu),\ t_k\searrow 0, \text{ and}\\ t_k^{-1}((p^k,\mu_k)-(p,\mu))\to(z,\eta)\end{array}\right\}.$$

3. *Regular Normals*: *The cone of regular normals to epi(a) at the point $(p,\mu)\in$ epi(a) is the convex cone $\widehat{N}_{epi(a)}(p,\mu)$ given by*

$$\widehat{N}_{epi(a)}(p,\mu)=T_{epi(a)}(p,\mu)^{\circ}.$$

4. *Normal Cone*: *The cone of normals to epi(a) at the point $(p,\mu)\in epi(a)$ is the cone*

$$N_{epi(a)}(p,\mu)=\left\{(z,\omega)\;\middle|\;\begin{array}{c}(p_k,\mu_k)\to(p,\mu), \text{ and } (z_k,\omega_k)\to(z,\omega)\\ (z_k,\omega_k)\in\widehat{N}_{epi(a)}(p_k,\mu_k)\ \forall k\end{array}\right\}.$$

5. *Subdifferential*: *The subdifferential of a at any point $p\in\mathcal{P}^n$ is given by*

$$\partial a(p)=\left\{z\;\middle|\;(z,-1)\in N_{epi(a)}(p,a(p))\right\}.$$

6. *Subdifferential Regularity*: *The function a is said to be subdifferentially regular at the point $p\in\mathcal{P}^n$ if*

$$\widehat{N}_{epi(a)}(p,a(p))=N_{epi(a)}(p,a(p)).$$

The proof of Theorem 2.1 is built up in a series of steps that gradually reveal the structure of the normal cone to the epigraph and the associated subdifferential. The first step in this process is to characterize the tangent cone to epi(a) at the the point $(\lambda^n,0)$. It is the proof of this result that is the most difficult technical hurdle in [4]. The purpose of this paper is to present an elementary proof based on the Gauss-Lucas theorem.

3. The Gauss–Lucas Theorem

A statement of the Gauss–Lucas theorem first appears in a letter written by Gauss in 1830. However, it appears that Gauss was aware of the result several years earlier [10]. In 1879 Lucas [9] published his paper on the theorem refining the result in certain special cases. We present the result as it was first used by Gauss.

THEOREM 3.1. (The Gauss–Lucas Theorem). *All critical points of a non-constant polynomial lie in the convex hull of the set of roots of the polynomial.*

Proof. Let $p\in\mathcal{P}^n$ have representation

$$p(\lambda)=\prod_{j=1}^{k}(\lambda-\lambda_j)^{m_j},$$

where $m_1+m_2+\cdots+m_k \leqslant n$. Then the complex conjugate of the logarithmic derivative of p defines the function

$$F(\lambda) = \overline{\left[\frac{p'(\lambda)}{p(\lambda)}\right]} = \sum_{j=1}^{k} \frac{m_j}{|\lambda-\lambda_j|^2}(\lambda-\lambda_j).$$

If λ_0 is a root of p' that is not already a root of p, then $F(\lambda_0)=0$. That is,

$$0=\sum_{j=1}^{k} \frac{m_j}{|\lambda_0-\lambda_j|^2}(\lambda_0-\lambda_j),$$

or equivalently,

$$\lambda_0=\sum_{j=1}^{k} \mu_j \lambda_j,$$

where

$$0 \leqslant \mu_s = \left[\sum_{j=1}^{k} \frac{m_j}{|\lambda_0-\lambda_j|^2}\right]^{-1} \frac{m_s}{|\lambda_0-\lambda_s|^2} \leqslant 1$$

for $s=1,\ldots,k$. □

If we define the multifunction $\mathcal{R}:\mathcal{P}^n \to \mathbb{C}$ by

$$\mathcal{R}(q)=\{\lambda \,|\, q(\lambda)=0\},$$

then Theorem 3.1 states that for any non-constant polynomial p

$$\mathcal{R}(p') \subset \operatorname{conv}\mathcal{R}(p).$$

With this notation we have $(p,\mu)\in \operatorname{epi}(a)$ if and only if $\mathcal{R}(p)$ lies in the closed half space $\{\lambda \,|\, \langle 1,\lambda\rangle \leqslant \mu\}$. Consequently, the Gauss-Lucas theorem tells us that the roots of the derivatives of the polynomial $p\in\mathcal{M}^n$ are related by

$$\mathcal{R}(p^{(n-1)}) \subset \operatorname{conv}\mathcal{R}(p^{(n-2)}) \subset \cdots \subset \operatorname{conv}\mathcal{R}(p) \subset \{\lambda \,|\, \langle 1,\lambda\rangle \leqslant \mu\},$$

whenever $(p,\mu)\in\operatorname{epi}(a)$. Hence,

$$a(p^{(n-1)}) \leqslant a(p^{(n-2)}) \leqslant \ldots \leqslant a(p) \leqslant \mu.$$

This system of inequalities is used to derive the structure of the tangent cone $T_{epi(a)}(\lambda^n,0)$.

THEOREM 3.2. *[4, Theorem 1.2] We have* $(v,\eta)\in T_{epi(a)}(\lambda^n,0)$, *with*

$$v(\lambda)=b_0\lambda^n+b_1\lambda^{n-1}+b_2\lambda^{n-2}+\cdots+b_n,$$

if and only if

$$-\frac{\operatorname{Re} b_1}{n} \leqslant \eta, \tag{4}$$

$$\operatorname{Re} b_2 \geqslant 0,\ \operatorname{Im} b_2 = 0,\ \textit{and} \tag{5}$$

$$b_k = 0,\ \textit{for}\ k = 3, \ldots, n. \tag{6}$$

Proof. Let us first assume that $(v,\eta) \in T_{epi(a)}(\lambda^n, 0)$ and verify that (4)–(6) hold. From the definition of the tangent cone we know that there exists

$$t_j \downarrow 0 \quad \text{and} \quad \{(p_j, \mu_j)\} \in \operatorname{epi}(a)$$

such that

$$t_j^{-1}((p_j, \mu_j) - (\lambda^n, 0)) \to (v, \eta).$$

Consequently, there exists

$$\{(a_0^j, a_1^j, \ldots, a_n^j)\} \in \mathbb{C}^{n+1}$$

such that

$$p_j(\lambda) = \sum_{k=0}^{n} a_k^j \lambda^{n-k}$$

with

$$t_j^{-1}\mu_j \to \eta, \qquad t_j^{-1}(a_0^j - 1) \to b_0,$$
$$t_j^{-1} a_k^j \to b_k,\ k = 1, \ldots, n,$$

where

$$v(\lambda) = \sum_{k=0}^{n} b_k \lambda^{n-k}.$$

Since $a_0^j \to 1$, we may assume with no loss of generality that $a_0^j \neq 0$ for all $j = 1, 2, \ldots$. By the Gauss–Lucas Theorem we have

$$\mathcal{R}(p_j^{(n-1)}) \subset \operatorname{conv} \mathcal{R}(p_j^{(n-2)}) \subset \ldots \operatorname{conv} \mathcal{R}(p_j) \subset \left\{ \zeta \,\middle|\, \langle 1, \zeta \rangle \leqslant \mu_j \right\},$$

for each $j = 1, 2, 3, \ldots$. Thus, for $j = 1, 2, 3, \ldots$ and $\ell = 1, 2, \ldots, n-1$

$$\mu_j \geqslant \max \left\{ \operatorname{Re} \zeta \,\middle|\, p_j^{(\ell)}(\zeta) = 0 \right\}, \tag{7}$$

where

$$p_j^{(\ell)}(\lambda) \;=\; \ell! \sum_{k=0}^{n-\ell} \binom{n-k}{\ell} a_k^j \lambda^{n-(\ell+k)}$$

is the ℓth derivative of p_j.

The argument now proceeds by taking limits in the expression (7) for the different possible choices for ℓ. In each of these limits we use the fact that the zeroes of a polynomial are continuous over a region of constant degree. Henceforth, we take this fact as given and do not mention it each time it is used.

For $\ell = n-1$, (7) yields

$$\begin{aligned}\mu_j &\geqslant \max\left\{\operatorname{Re}\lambda \,\middle|\, n!a_0^j\lambda + (n-1)!a_1^j = 0\right\}\\ &= -\frac{1}{n}\operatorname{Re}\frac{a_1^j}{a_0^j}.\end{aligned}$$

Hence

$$\frac{\mu_j}{t_j} \geqslant -\frac{1}{n}\operatorname{Re}\frac{a_1^j}{t_j a_0^j}.$$

Taking the limit in j yields

$$\eta \geqslant -\frac{\operatorname{Re} b_1}{n},$$

whereby (4) is verified. For $\ell = n-2$ we get

$$\begin{aligned}\mu_j &\geqslant \max\left\{\operatorname{Re}\lambda \,\middle|\, \frac{n!}{2}a_0^j\lambda^2 + (n-1)!a_1^j\lambda + (n-2)!a_2^j = 0\right\}\\ &\geqslant -\frac{1}{n}\operatorname{Re}\left[\frac{a_1^j}{a_0^j} \pm \sqrt{\left(\frac{a_1^j}{a_0^j}\right)^2 - \frac{2n}{(n-1)}\frac{a_2^j}{a_0^j}}\,\right].\end{aligned}$$

Divide through by $\sqrt{t_j}$ and take the limit in j to get

$$0 \geqslant \pm\frac{1}{n}\sqrt{\frac{2n}{n-1}}\operatorname{Re}\sqrt{-b_2}.$$

Consequently, $0 = \operatorname{Re}\sqrt{-b_2}$, or equivalently,

$$\operatorname{Im} b_2 = 0 \quad \text{and} \quad \operatorname{Re} b_2 \geqslant 0\,,$$

thus verifying (5).

For $\ell = n-s$ with $3 \leqslant s \leqslant n$, we have

$$p_j^{(n-s)}(\lambda) = (n-s)!\sum_{k=0}^{s}\binom{n-k}{n-s}a_k^j\lambda^{s-k}.$$

Setting $\lambda = t_j^{1/s}\gamma$, we get $p_j^{(n-s)}(t_j^{1/s}\gamma) =$

$$\begin{aligned}(n-s)!t_j\Bigg[\binom{n}{n-s}a_0^j\gamma^s &+ \binom{n-1}{n-s}a_1^j t_j^{-1/s}\gamma^{s-1} + \dots\\ &+ \binom{n+1-s}{n-s}a_{s-1}^j t_j^{(1-s)/s}\gamma + a_s^j t_j^{-1}\Bigg].\end{aligned}$$

Therefore,

$$\mu_j \geqslant \max\left\{\mathrm{Re}\,\lambda \,\middle|\, p_j^{(n-s)}(\lambda)=0\right\}$$

implies that

$$\frac{\mu_j}{t_j^{1/s}} \geqslant \max\left\{\mathrm{Re}\, t_j^{-1/s}\lambda \,\middle|\, p_j^{(n-s)}(\lambda)=0\right\}$$
$$= \max\left\{\mathrm{Re}\,\gamma \,\middle|\, p_j^{(n-s)}(t_j^{1/s}\gamma)=0\right\}.$$

Taking the limit in j yields

$$0 \geqslant \max\left\{\mathrm{Re}\,\gamma \,\middle|\, \binom{n}{n-s}\gamma^s + b_s = 0\right\} \tag{8}$$

which implies that $b_s=0$ for $s=3,\ldots,n$. Hence (6) also holds.

As in [4], we complete the proof by showing that any pair (v,η) satisfying (4)–(6) must be an element of $T_{epi(a)}(\lambda^n,0)$. We do this by establishing the existence of a trajectory in $\mathrm{epi}(a)$ passing through the point $(\lambda^n,0)$ and whose limiting tangential direction at $(\lambda^n,0)$ is the pair (v,η). Consider the following family of polynomials in λ:

$$\begin{aligned}
p_\xi(\lambda) &= (1+b_0\xi)\left(\lambda+\frac{b_1\xi}{n}\right)^{n-2} \\
&\quad\cdot\left(\lambda+\sqrt{-1}(b_2\xi)^{\frac12}+\frac{b_1\xi}{n}\right)\left(\lambda-\sqrt{-1}(b_2\xi)^{\frac12}+\frac{b_1\xi}{n}\right) \\
&= (1+b_0\xi)\left(\lambda^{n-2}+(n-2)\frac{b_1\xi}{n}\lambda^{n-3}+o(\xi)\right) \\
&\quad\cdot\left(\lambda^2+2\frac{b_1\xi}{n}\lambda+b_2\xi+o(\xi)\right) \\
&= \lambda^n+b_0\xi\lambda^n+b_1\xi\lambda^{n-1}+b_2\xi\lambda^{n-2}+o(\xi) \\
&= \lambda^n+\xi v(\lambda)+o(\xi)\ .
\end{aligned}$$

Since b_2 is real and non-negative, we have, for all sufficiently small ξ real and positive, that

$$a(p_\xi) = -\mathrm{Re}\,\frac{b_1}{n}\xi \leqslant \eta\xi.$$

By taking the limit along this trajectory as $\xi\downarrow 0$ we find that $(v,\eta)\in T_{epi(a)}(\lambda^n,0)$. □

This new proof of Theorem 3.2 is far simpler and much more illuminating than the proof given in [4]. In particular, the derivation of (8) illustrates in an elementary and accessible way how each of the coefficients b_k corresponds to a splitting of the roots of order $1/k$ for $k=1,2,\ldots,n$. This observation can be used to understand the variational geometry of other functions of the roots of a polynomial. An alternative approach based on Puiseux–Newton series which also reveals this splitting behavior is developed in [3]. However, the approach in [3] is unable to obtain the result in its full generality because of its reliance on analyticity assumptions.

Having characterized the tangent cone $T_{epi(a)}(\lambda^n,0)$ in Theorem 3.2, it follows easily from the relation $\widehat{N}_{epi(a)}(\lambda^n,0)=T_{epi(a)}(\lambda^n,0)^\circ$ that

$$\widehat{N}_{epi(a)}(\lambda^n,0)=\left\{\left(\sum_{k=0}^{n}\mu_k\lambda^{n-k},\omega\right)\middle|\begin{array}{c}\mu_0=0,\ \mu_1=\frac{\omega}{n},\ \omega\leqslant 0,\\ \text{and } \mathrm{Re}\mu_2\leqslant 0\end{array}\right\}.$$

This representation combined with the subdifferential regularity result from [4] implies that

$$\partial a(\lambda^n)=\left\{\sum_{k=0}^{n}\mu_k\lambda^{n-k}\,\middle|\,\mu_0=0,\ \mu_1=-\frac{1}{n},\ \text{and } \mathrm{Re}\mu_2\leqslant 0\right\}.$$

It is shown in [4] that the representation of $\partial a(p)$ for a general polynomial p can be derived from these facts.

Finally, we observe that the results of Levantovskii [8] are easily recovered from our formula for the tangent cone to the epigraph of the abscissa mapping a. The key to this is the following general result relating the tangent cone to lower level sets of a function to the tangent cone of its epigraph. The result is a simple consequence of [11, Proposition 10.3].

PROPOSITION 3.3. *Let* $\mathbf{E}$ *be a finite dimensional Euclidean space and suppose* $C=\{x\,|\,f(x)\leqslant f(\bar{x})\}$ *where* $f:\mathbf{E}\to\mathbb{R}\cup\{\infty\}$ *is lower semicontinuous and* $\bar{x}\in\mathbf{E}$ *is such that* $f(\bar{x})<\infty$. *Then*

$$T_C(\bar{x})\subset\left\{w\,\middle|\,(w,0)\in T_{epi(f)}(\bar{x},f(\bar{x}))\right\}\tag{9}$$

with equality holding whenever f *is subdifferentially regular at* $\bar{x}$ *and* $0\notin\partial f(\bar{x})$.

Proof. By [11, Theorem 8.2] the *subderivative* of f at $\bar{x}$, denoted $df(\bar{x})$, is the function whose epigraph is the set $T_{epi(f)}(\bar{x},f(\bar{x}))$. Hence

$$\{w\,|\,df(\bar{x})(w)\leqslant 0\}=\left\{w\,\middle|\,(w,0)\in T_{epi(f)}(\bar{x},f(\bar{x}))\right\}.$$

Therefore this result is equivalent to [11, Proposition 10.3]. □

Let $\mathcal{P}^n_-$ denote the set of polynomials for which $a(p)\leqslant 0$. The set $\mathcal{P}^n_-$ is called the set of stable polynomials. In [8] Levantovskii derives a formula for the tangent cone to $\mathcal{P}^n_-$ at any point in $\mathcal{P}^n_-$. Our original proof of Theorem 3.2 is based on Levantovskii's derivation. However, the new proof presented here is based on the Gauss-Lucas Theorem. We now show how Theorem 3.2 and Proposition 3.3 combine to provide an alternative path to Levantovskii's formulas. First observe that by Theorem 2.1 a is everywhere subdifferentially regular on $\mathcal{M}^n$. Hence, since $\mathcal{P}^n_- = \{p \,|\, a(p) \leqslant 0\}$, Theorem 3.2 and Proposition 3.3 immediately yield

$$T_{\mathcal{P}^n_-}(\lambda^n) = T_{epi(a)}(\lambda^n)$$
$$= \left\{ b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} \,|\, \mathrm{Re}\, b_1 \geqslant 0, \mathrm{Re}\, b_2 \geqslant 0, \mathrm{Im}\, b_2 = 0 \right\}.$$

Formulas for the tangent cone to $\mathcal{P}^n_-$ at other stable polynomials are just as easily obtained from the more general formula for the tangent cone to epi(a) given in [4].

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