

POINT PROCESSES ASSOCIATED WITH STATIONARY STABLE PROCESSES

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ABSTRACT. Point processes induced by stationary symmetric α -stable (SaS) processes can have diverse behavior. We distinguish two cases, depending on whether the stationary SaS process is governed by a dissipative or conservative flow. In the case of dissipative flows, the process is a mixed moving average and the family of scaled point processes converge to a Poisson cluster process. In the case of a conservative flow, we give two examples showing how diverse the behavior can be.

1. INTRODUCTION

Let $\mathbf{X} = (X_n, n = 0, 1, 2, \dots)$ be a stationary symmetric α -stable (SaS) process, $0 < \alpha < 2$. Recall that α -stability means that the linear combinations $\sum_{n=1}^k c_n X_n$ have, for all choice of k and real numbers c_1, \dots, c_k , a symmetric α -stable distribution $S_\alpha(\sigma, 0, 0)$ whose characteristic function is given by $\varphi(\theta) = \exp\{-\sigma^\alpha |\theta|^\alpha\}$, $\theta \in \mathbb{R}$. Here we follow the notation of Samorodnitsky and Taqqu (1994). The scaling constant σ depends, obviously, on k and the choice of c_1, \dots, c_k .

For a sequence of positive constants $b_n \uparrow \infty$ we define

$$(1.1) \quad N_n = \sum_{k=0}^{n-1} \delta_{b_n^{-1} X_k}, \quad n = 1, 2, \dots,$$

which we consider as a sequence of point processes on $[-\infty, \infty] \setminus \{0\}$. Here δ_x is the point mass at x . Weak convergence in the space \mathcal{M} of Radon measures on $[-\infty, \infty] \setminus \{0\}$ of sequences of point processes of the type (1.1) is handled by extreme value techniques. This methodology is attractive because weak convergence of point processes and a clever use of the continuous mapping theorem allows one to obtain a number of limit theorems for various functionals of the stationary process. See for example Resnick (1987).

For a random variable X with a $S_\alpha(\sigma, 0, 0)$ law,

$$(1.2) \quad P(|X| > \lambda) \sim C_\alpha \sigma^\alpha \lambda^{-\alpha} \quad \text{as } \lambda \rightarrow \infty,$$

where C_α is a finite positive constant depending only on α (see Samorodnitsky and Taqqu (1994)). For an iid sequence \mathbf{X} satisfying (1.2), it is well known that an acceptable choice of the scaling sequence (b_n) is

$$(1.3) \quad b_n = n^{1/\alpha}$$

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and in this case the sequence (N_n) converges weakly in the space \mathcal{M} (with the vague topology) to a very particular Poisson random measure, whose intensity blows up near the origin (which is one reason for excluding the origin from the state space). See, once again, Resnick (1987).

It is natural that attention has been focused on removing the assumption of independence in the original process \mathbf{X} . The general sense of the obtained results was that if \mathbf{X} is a stationary process with sufficiently weak dependence, then the sequence (N_n) still converges weakly, and with the same sequence of normalizing constants (1.3); however the limiting random measure is, typically, a cluster Poisson process. See Mori (1977), Davis and Resnick (1985) and Davis and Hsing (1995). These results typically allow the marginal distribution of the stationary process to have balanced regularly varying tails, and no assumption of stability is made.

Our goal in this paper is to understand what may happen when the dependence in the process \mathbf{X} is no longer weak or local. In fact, we would like to see what happens also under long range dependence. This is why we have chosen to concentrate specifically on stationary symmetric α -stable processes. Their structure is rich, and sufficiently well understood to be able us see what happens to the point processes (1.1) when the strength and the length of the memory changes. We will see two important phenomena: the choice of the normalizing constants (1.3) is inappropriate, in general (that means, the normalizing constants are affected not only by how heavy the tails of the marginal distributions are, but also by the length of memory); furthermore, clustering of the extreme observations may so strong that one may need to normalize the sequence (N_n) itself to achieve weak convergence.

We believe that the methods of this paper are extendable to point processes based on certain stationary infinitely divisible processes with regularly varying tails, to many non-symmetric processes and perhaps to a general study of extremal behavior of stationary, regularly varying processes.

In the next section we collect background information and set up the framework of our study. In Section 3 we study point processes corresponding to dissipative maps; these turn out to be processes based on mixed moving averages. Section 4 considers the more intricate case where the stationary stable process is associated with a conservative map.

2. BACKGROUND

Every stationary SaS process \mathbf{X} has an integral representation as a stochastic integral of the type

$$(2.1) \quad X_n = \int_E f_n(x) M(dx), \quad n = 0, 1, 2, \dots,$$

where M is a symmetric α -stable random measure on a measurable space (E, \mathcal{E}) with a σ -finite control measure m , while the functions $f_n \in L^\alpha(m, \mathcal{E})$, $n \geq 0$ are given by

$$(2.2) \quad f_n(x) = a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x), \quad x \in E.$$

Here $f \in L^\alpha(m, \mathcal{E})$ and $\phi : E \rightarrow E$ is a measurable non-singular map (meaning, in this paper, a one-to-one map with both ϕ and ϕ^{-1} measurable, mapping the control measure m into an equivalent measure, but the reader is warned that different authors assign this notion slightly

different meanings). Finally

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x), \quad x \in E,$$

for $n = 0, 1, 2, \dots$, with $u : E \rightarrow \{-1, 1\}$ a measurable function. We refer the reader to Samorodnitsky and Taqqu (1994) for information on α -stable random measures and stochastic integrals with respect to these measures, and to Rosiński (1995) for derivation of the representation (2.1) with the choice (2.2) of the functions (f_n) .

Let $E = C \cup D$ be the Hopf decomposition of the map ϕ into its conservative and dissipative parts. Since ϕ is invertible, C and D are ϕ -invariant measurable sets such that ϕ is conservative on C , while D is the union of translates of a single wandering set. We refer the reader to Krengel (1985)) and Aaronson (1997) for various ergodic theoretical notions we are using. The corresponding decomposition of the process \mathbf{X}

$$(2.3) \quad X_n = \int_C f_n(x) M(dx) + \int_D f_n(x) M(dx) := X_n^C + X_n^D, \quad n = 0, 1, 2, \dots,$$

is a unique (in law) decomposition of a stationary S α S process into a sum $\mathbf{X}^C + \mathbf{X}^D$ of two independent stationary S α S processes, one of which corresponds to a conservative map (empty dissipative part in the Hopf decomposition), and the other corresponds to a dissipative map (empty conservative part in the Hopf decomposition). See Rosiński (1995). Alternative terminology refers to \mathbf{X}^C and \mathbf{X}^D as generated by a conservative flow and a dissipative flow, accordingly.

Stationary S α S processes corresponding to dissipative maps often have “shorter memory” than those corresponding to conservative maps; a clear dichotomy was established in Samorodnitsky (2002). Specifically, consider the sequence of partial maxima of the process \mathbf{X} defined for $n = 1, 2, \dots$ by $W_n = \max(|X_0|, |X_1|, \dots, |X_{n-1}|)$. Then

$$n^{-1/\alpha} W_n \Rightarrow \begin{cases} s_X Z_\alpha & \text{if } \mathbf{X} \text{ corresponds to a dissipative map} \\ 0 & \text{if } \mathbf{X} \text{ corresponds to a conservative map} \end{cases}$$

weakly as $n \rightarrow \infty$. Here Z_α is the standard Fréchet extreme value random variable with distribution function $\exp\{-x^{-\alpha}\}$, $x > 0$, $\alpha > 0$, and s_X is a strictly positive constant depending on the process \mathbf{X} . For stationary S α S processes corresponding to dissipative flows, $n^{1/\alpha}$ is the right normalization for the partial maxima, but for processes corresponding to conservative maps the partial maxima grow at the rate strictly slower than $n^{1/\alpha}$. This clearly implies that if one chooses the normalizing sequence (b_n) in the definition of the point processes (1.1) according to (1.3), and the underlying stationary S α S process corresponds to a conservative map, then the sequence (N_n) converges to the null measure weakly in the space \mathcal{M} , meaning that a normalization according to (1.3) is inappropriate in this case.

The surprising thing is that, for stationary S α S processes corresponding to conservative maps, even if one uses in (1.1) the normalization that makes the the partial maxima of the process converge weakly to an almost surely positive limit, the sequence of point processes (N_n) may not converge weakly in the space \mathcal{M} .

A useful representation of stationary S α S processes corresponding to dissipative maps, also due to Rosiński (1995), is the mixed moving average representation

$$(2.4) \quad X_n = \int_W \int_{\mathbb{R}} f(v, x - n) M(dv, dx), \quad n = 0, 1, 2, \dots,$$

where M is a symmetric α -stable random measure on a product measurable space $(W \times \mathbb{R}, \mathcal{W} \times \mathcal{B})$ with control measure $m = \nu \times \text{Leb}$, with ν a σ -finite measure on (W, \mathcal{W}) , and $f \in L^\alpha(m, \mathcal{W} \times \mathcal{B})$.

Finally we review a series representation of SaS processes, that can be traced back to LePage et al. (1981). Let \mathbf{X} be a SaS process given as a stochastic integral (2.1), and m_n a probability measure on (E, \mathcal{E}) concentrated on a set supporting f_0, \dots, f_{n-1} , $n = 1, 2, \dots$ and equivalent to m on this set. Let $\rho_n = dm_n/dm$. Then the following representation in law holds:

$$(2.5) \quad X_k = C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} \left(\rho_n \left(Y_j^{(n)} \right) \right)^{-1/\alpha} f_k \left(Y_j^{(n)} \right), \quad k = 0, 1, \dots, n-1.$$

Here

$$(2.6) \quad C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha) \cos(\pi\alpha/2)} & \text{if } \alpha \neq 1 \\ 2/\pi & \text{if } \alpha = 1 \end{cases},$$

while (ε_j) , (Γ_j) and $(Y_j^{(n)})$ are three independent sequences of random variables, such that (ε_j) are iid Rademacher random variables ($P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$), (Γ_j) is the sequence of arrival times of a unit rate Poisson process on $(0, \infty)$, and $(Y_j^{(n)})$ are iid E -valued random variables with a common law m_n . See Samorodnitsky and Taqqu (1994).

3. STATIONARY SaS PROCESSES CORRESPONDING TO DISSIPATIVE MAPS AND WEAK CONVERGENCE OF CORRESPONDING POINT PROCESSES

Let \mathbf{X} be a stationary SaS process corresponding to a dissipative map. This process does not have to have only weak or local dependence. However, our discussion in the previous section indicates that the right normalization in the definition of the point processes (1.1) is via (1.3). The following theorem indicates that this is indeed the case, and that the limiting random measure is a cluster Poisson random measure. As discussed above, we may assume without loss of generality that the original stable process is given in the form (2.4). We work with a representation of $\{X_n\}$ determined as follows.

Let

$$N = \sum_l \delta_{(j_l, v_l, u_l)} = PRM(\nu_\alpha \times \nu \times \text{Leb}),$$

be Poisson random measure on $([-\infty, \infty] \setminus \{0\}) \times W \times R$ with mean measure $\nu_\alpha \times \nu \times \text{Leb}$. Here ν_α is the symmetric measure on $[-\infty, \infty] \setminus \{0\}$ given by

$$\nu_\alpha(x, \infty] = \nu_\alpha[-\infty, -x] = x^{-\alpha}, \quad x > 0.$$

Then

$$(3.1) \quad X_n := \sum_l j_l f(v_l, u_l - n), \quad n = 0, \pm 1, \pm 2, \dots$$

This is distributionally the same as (2.4) except we have dropped the factor $C_\alpha^{1/\alpha}$.

Theorem 3.1. *Let \mathbf{X} be the mixed moving average (3.1), and $N_n = \sum_{k=0}^{n-1} \delta_{n-1/\alpha X_k}$, $n = 1, 2, \dots$. Then $N_n \Rightarrow N_*(\cdot)$ as $n \rightarrow \infty$, weakly in the space \mathcal{M} , where N_* is a cluster Poisson random*

measure with a representation

$$(3.2) \quad N_* = \sum_{l=1}^{\infty} \sum_{i=-\infty}^{\infty} \delta_{j_l f(v_l, U_l - i)},$$

where j_l, v_l are described before (3.1) and $\{U_l\}$ are iid $U(0,1)$ random variables independent of points of N . Furthermore, N_* is Radon on $[-\infty, \infty] \setminus \{0\}$ with Laplace functional ($g \geq 0$ continuous with compact support)

$$(3.3) \quad Ee^{-N_*(g)} = \exp\left\{-\iiint_{([-\infty, \infty] \setminus \{0\}) \times W \times [0,1]} \left(1 - e^{-\sum_{i=-\infty}^{\infty} g(xf(v, u-i))}\right) \nu_{\alpha}(dx) \nu(dv) du\right\}.$$

Proof. To compute the Laplace functional of N_* , let

$$\psi(x, v, u) = \sum_{i=-\infty}^{\infty} g(xf(v, u-i))$$

and then

$$Ee^{-N_*(g)} = E \exp\left\{-\sum_l \sum_i g(j_l f(v_l, U_l - i))\right\} = E \exp\left\{-\sum_l \psi(j_l, v_l, U_l)\right\}$$

which, because $\sum_l \delta_{(j_l, v_l, U_l)}$ is PRM with mean measure $\nu_{\alpha} \times \nu \times \text{Leb}_{|[0,1]}$, is equal to

$$= \exp\left\{-\iiint_{([-\infty, \infty] \setminus \{0\}) \times W \times [0,1]} \left(1 - e^{-\psi(x, v, u)}\right) \nu_{\alpha}(dx) \nu(dv) du\right\}$$

and the result follows.

To show N_* is Radon, it is enough to show with $h(x) = 1_{(\delta, \infty] \cup [-\infty, -\delta]}$ and $\delta > 0$,

$$EN_*(h) < \infty.$$

However, this is easy since

$$EN_*(h) = E \sum_l \sum_i h(j_l f(v_l, u_l - i)) = \sum_l E \psi(j_l, v_l, u_l)$$

where we define ψ in terms of h as was done in the previous paragraph for g , and this yields

$$\begin{aligned} &= \iiint \psi(x, v, u) \nu_{\alpha}(dx) \nu(dv) du = \iiint \sum_i h(xf(v, u-i)) \nu_{\alpha}(dx) \nu(dv) du \\ &= \sum_i \int_{u \in [0,1]} \int_{v \in W} \left[\int_{|x| > \delta / |f(v, u-i)|} \nu_{\alpha}(dx) \right] \nu(dv) du \\ &= 2\delta^{-\alpha} \sum_i \iint_{W \times [0,1]} |f(v, u-i)|^{\alpha} \nu(dv) du \\ &= 2\delta^{-\alpha} \iint_{W \times \mathbb{R}} |f(v, u)|^{\alpha} \nu(dv) du < \infty, \end{aligned}$$

the finiteness following by assumption.

A key insight for understanding how the point process based on $\{X_n\}$ becomes a cluster process is that only one Poisson point j_l in the definition of X_n is likely to be large enough as not to be driven to zero by the normalization $n^{-1/\alpha}$ (remember that the origin is excluded from the state

space). See Samorodnitsky (2002) and also Davis and Resnick (1985). Therefore, one expects that N_n has the same weak limit as

$$(3.4) \quad N_n^{(2)} := \sum_{l=1}^{\infty} \sum_{k=0}^{n-1} \delta_{n^{-1/\alpha} j_l f(v_l, u_l - k)}$$

as $n \rightarrow \infty$. We will, first of all, establish convergence of $N_n^{(2)}$, and then show that N_n converges to the same limit.

The Laplace functional of $N_n^{(2)}$ can be computed by the same simple method as used for N_* . Using the scaling property of ν_α we get for $g \geq 0$ continuous with compact support

$$(3.5) \quad Ee^{N_n^{(2)}(g)} = \exp \left\{ - \iiint_{([-\infty, \infty] \setminus \{0\}) \times W \times \mathbb{R}} \left(1 - e^{-\sum_{k=0}^{n-1} g(xf(v, u-k))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \right\}$$

and this must be shown to converge to (3.3). We show this with a series of steps.

STEP 1. Assume (temporarily) that the function f in (2.4) is compactly supported in the second variable, in the sense that for some positive integer K

$$(3.6) \quad f(v, u) = 0, \quad \text{for all } (v, u) \in W \times \mathbb{R} \text{ such that } |u| \geq K.$$

Then the functions $f(\cdot, \cdot - k)$, $k = 0, 1, \dots, n-1$ are supported by $W \times [-K, K+n-1]$.

Step 1a. We begin by examining the integral in (3.5) with u restricted to $[K, n-K]$, assuming $n > 2K+1$, and show

$$(3.7) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_{|x|>0} \int_{v \in W} \int_K^{n-K} \left(1 - e^{-\sum_{k=0}^{n-1} g(xf(v, u-k))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\ = \int_{|x|>0} \int_{v \in W} \int_{[0,1]} \left(1 - e^{-\sum_{k=-K}^K g(xf(v, u-k))} \right) \nu_\alpha(dx) \nu(dv) du. \end{aligned}$$

The triple integral on the left side of (3.7) is

$$\begin{aligned} \int_{|x|>0} \int \sum_{i=K}^{n-K} \int_i^{i+1} &= \int_{|x|>0} \int \sum_{i=K}^{n-K} \int_{u=0}^1 \left(1 - e^{-\sum_{k=0}^{n-1} g(xf(v, u-(k-i)))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\ &= \int_{|x|>0} \int \sum_{i=K}^{n-K} \int_{u=0}^1 \left(1 - e^{-\sum_{l=-i}^{n-i-1} g(xf(v, u-l))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\ &= \int_{|x|>0} \int \sum_{i=K}^{n-K} \int_{u=0}^1 \left(1 - e^{-\sum_{l=-i}^K g(xf(v, u-l))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \end{aligned}$$

because of the compact support of f . Using the compact support assumption once more we see that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{|x|>0} \int \int_{u=0}^1 \left(1 - e^{-\sum_{l=-i}^K g(xf(v, u-l))} \right) \nu_\alpha(dx) \nu(dv) du \\ = \int_{|x|>0} \int \int_{u=0}^1 \left(1 - e^{-\sum_{l=-K}^K g(xf(v, u-l))} \right) \nu_\alpha(dx) \nu(dv) du, \end{aligned}$$

and so the same is true for the averages:

$$\int_{|x|>0} \int \sum_{i=K}^{n-K} \int_{u=0}^1 \left(1 - e^{-\sum_{l=-i}^K g(xf(v, u-l))} \right) \nu_\alpha(dx) \nu(dv) \frac{du}{n}$$

$$\begin{aligned}
 &= \frac{n-2K+1}{n} \frac{1}{n-2K+1} \sum_{i=K}^{n-K} \int_{|x|>0} \int \int_{u=0}^1 \left(1 - e^{-\sum_{l=-i}^K g(xf(v, u-l))}\right) \nu_\alpha(dx) \nu(dv) du \\
 &\quad \rightarrow \int_{|x|>0} \int \int_{u=0}^1 \left(1 - e^{-\sum_{l=-K}^K g(xf(v, u-l))}\right) \nu_\alpha(dx) \nu(dv) du.
 \end{aligned}$$

Step 1b. Now we show that in (3.5), $u \notin [K, n-K]$ leads to a negligible asymptotic contribution to the triple integral:

$$(3.8) \quad \lim_{n \rightarrow \infty} \int_{|x|>0} \int_{v \in W} \int_{u \notin [K, n-K]} \left(1 - e^{-\sum_{k=0}^{n-1} g(xf(v, u-k))}\right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} = 0.$$

We focus on $u < K$ with the explanation for $u > n-K$ being similar. We have, when $u < K$ that $u-k < -K$ (and hence $g(xf(v, u-k)) = 0$) when $k > 2K$. Thus, the left side of (3.8) is the same as

$$\begin{aligned}
 &\int_{|x|>0} \int_{v \in W} \int_{u < K} \left(1 - e^{-\sum_{k=0}^{2K} g(xf(v, u-k))}\right) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\
 &\quad \leq \iiint_{u < K, |x| > \frac{\delta}{\sqrt{2K} |f(v, u-k)|}} 1 \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\
 &\quad \leq \iint_{W \times \mathbb{R}} \delta^{-\alpha} \bigvee_{k=0}^{2K} |f(v, u-k)|^\alpha \nu(dv) \frac{du}{n} \\
 &\quad = \frac{1}{n} \sum_{k=0}^{2K} \iint_{W \times \mathbb{R}} \delta^{-\alpha} |f(v, u)|^\alpha \nu(dv) du \rightarrow 0,
 \end{aligned}$$

as $n \rightarrow \infty$ since $f \in L_\alpha$.

This completes the proof that the Laplace functional of $N_n^{(2)}$ in (3.5) converges to that of N_* in (3.3) in the case where f has compact support in the second variable. We now remove this restriction of f having a compact support.

Step 1c. To remove the assumption of compact support on function f , for a general $f \in L^\alpha(\nu \times \text{Leb})$ define

$$(3.9) \quad f_K(v, x) = f(v, u) \mathbf{1}(|u| \leq K), \quad K \geq 1.$$

Notice that each f_K satisfies (3.6) and that $f_K \rightarrow f$ in $L^\alpha(\nu \times \text{Leb})$ as $K \rightarrow \infty$. Denote

$$(3.10) \quad N_n^{(2, K)} = \sum_{l=1}^{\infty} \sum_{k=0}^{n-1} \delta_{n^{-1/\alpha} j_l f_K(v_l, u_l - k)},$$

for $K, n \geq 1$, and

$$(3.11) \quad N_*^{(K)} = \sum_{l=1}^{\infty} \sum_{i=-\infty}^{\infty} \delta_{j_l f_K(v_l, U_l - i)}, \quad K \geq 1,$$

with the notation of (3.2). We already know that for every $K \geq 1$, $N_n^{(2, K)} \Rightarrow N_*^{(K)}$ weakly in the space \mathcal{M} as $n \rightarrow \infty$. Therefore, to establish $N_n^{(2)} \Rightarrow N_*$, it is enough to prove two things:

$$(3.12) \quad N_*^{(K)} \Rightarrow N_* \text{ weakly in the space } \mathcal{M} \text{ as } K \rightarrow \infty$$

and

$$(3.13) \quad \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left(\left| N_n^{(2,K)}(g) - N_n^{(2)}(g) \right| > \epsilon \right) = 0, \quad \epsilon > 0,$$

for every non-negative continuous function with compact support g on $[-\infty, \infty] \setminus \{0\}$. Assume the support is contained in $[|x| > \delta]$, for $\delta > 0$ and that $g(x) \leq \|g\|$, a finite positive constant.

The claim (3.12) is easy. We have proved that the measure N_* is Radon, and so for every Borel set A bounded away from the origin, N_* has finitely many points in A ; the collection of those points contains, for every $K \geq 1$, the collection of the points of $N_*^{(K)}$. Furthermore, for K large enough, the two collections coincide. Therefore, $N_*^{(K)} \rightarrow N_*$ a.s. in the space \mathcal{M} as $K \rightarrow \infty$.

To check (3.13), notice that

$$N_n^{(2,K)}(g) - N_n^{(2)}(g) = \sum_{l=1}^{\infty} \sum_{k=0}^{n-1} g(n^{-1/\alpha} j_l f(v_l, u_l - k)) \mathbf{1}(|u_l - k| > K)$$

and so

$$\begin{aligned} E \left| N_n^{(2,K)}(g) - N_n^{(2)}(g) \right| &= \sum_{k=0}^{n-1} E \left(\sum_l g(n^{-1/\alpha} j_l f(v_l, u_l - k)) \mathbf{1}(|u_l - k| > K) \right) \\ &= \sum_{k=0}^{n-1} \iiint g(n^{-1/\alpha} x f(v, u - k)) \mathbf{1}(|u - k| > K) \nu_\alpha(dx) \nu(dv) du \\ &= \sum_{k=0}^{n-1} \iiint g(x f(v, u - k)) \mathbf{1}(|u - k| > K) \nu_\alpha(dx) \nu(dv) \frac{du}{n} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \iiint g(x f(v, u)) \mathbf{1}(|u| > K) \nu_\alpha(dx) \nu(dv) du \\ &\leq \|g\| \iiint_{\{(x,v,u): |u| > K, |x| > \frac{\delta}{|f(v,u)|}\}} \nu_\alpha(dx) \nu(dv) du \\ &\leq \|g\| \int_{|u| > K} \delta^{-\alpha} |f(v, u)|^\alpha \nu(dv) du \rightarrow 0, \end{aligned}$$

as $K \rightarrow \infty$, since $f \in L_\alpha$. This proves $N_n^{(2)} \Rightarrow N_*$ without the assumption of compact support.

STEP 2. To complete the proof of the theorem we need to prove (with ρ being the vague metric on \mathcal{M}) that

$$P[\rho(N_n, N_n^{(2)}) > \epsilon] \rightarrow 0, \quad (n \rightarrow \infty),$$

and for this, it suffices to show for $g \in C_K^+([-\infty, \infty] \setminus \{0\})$, the non-negative continuous functions with compact support, that as $n \rightarrow \infty$,

$$(3.14) \quad P[|N_n(g) - N_n^{(2)}(g)| > \epsilon] = P\left[\left| \sum_{k=0}^{n-1} \left[g\left(\frac{X_k}{n^{1/\alpha}}\right) - \sum_l g\left(\frac{j_l}{n^{1/\alpha}} f(v_l, u_l - k)\right) \right] \right| > \epsilon \right] \rightarrow 0.$$

Suppose the support of g is contained in $\{x : |x| > \delta\}$ and let

$$\omega(\theta) = \sup\{|g(x) - g(y)| : |x - y| \leq \theta\}$$

be the modulus of continuity of g . Choose p and an integer m such that

$$(3.15) \quad p > \alpha \quad \text{and} \quad \frac{p}{\alpha} < m + 1.$$

We note the following facts.

(i) For any $\eta > 0$,

$$(3.16) \quad nP\left[\left|\sum_{l=m+1}^{\infty} \epsilon_l \Gamma_l^{-1/\alpha}\right| > n^{1/\alpha} \eta\right] \leq \eta^{-p} n \frac{E\left(\left|\sum_{l=m+1}^{\infty} \epsilon_l \Gamma_l^{-1/\alpha}\right|^p\right)}{n^{p/\alpha}} \rightarrow 0,$$

as $n \rightarrow \infty$, which follows by the method of (Samorodnitsky and Taqqu, 1994, page 27). Here (ϵ_j) and (Γ_j) are independent sequences of random variables, such that (ϵ_j) are iid Rademacher random variables and (Γ_j) is the sequence of arrival times of a unit rate Poisson process on $(0, \infty)$.

(ii) The point process $\sum_l \delta_{j_l f(v_l, u_l)}$ is PRM($\|f\|_{\alpha}^{\nu}$) where

$$\|f\|_{\alpha}^{\nu} = \iint |f(v, u)|^{\alpha} \nu(dv) du < \infty.$$

Therefore

$$(3.17) \quad \sum_l \delta_{j_l f(v_l, u_l)} \stackrel{d}{=} \sum_{l=1}^{\infty} \delta_{\|f\|_{\alpha} \epsilon_l \Gamma_l^{-1/\alpha}}.$$

(iii) For any $\theta > 0$, consider the event

$$[\text{AMO}(\theta)] = [\text{AtMostOne}] = \left(\bigcup_{k=0}^{n-1} \left[\sum_l \delta_{|j_l f(v_l, u_l - k)|} (n^{1/\alpha} \theta, \infty] \geq 2 \right] \right)^c$$

and observe

$$\begin{aligned} P[(\text{AMO}(\theta))^c] &\leq nP\left[\sum_l \delta_{|j_l f(v_l, u_l)|} (n^{1/\alpha} \theta, \infty] \geq 2\right] \\ &\leq n \left(E\left(\sum_l \delta_{|j_l f(v_l, u_l)|} (n^{1/\alpha} \theta, \infty]\right) \right)^2 = (\text{const})n \cdot ((n^{1/\alpha} \theta)^{-\alpha})^2 \rightarrow 0. \end{aligned}$$

For $X_k = \sum_l j_l f(v_l, u_l - k)$, define the random variable Y_k on the set $[\text{AMO}(\frac{\theta}{m+1})]$ to be the summand of largest modulus. For any $\theta < \delta$

$$\begin{aligned} &P\left[\bigvee_{k=0}^{n-1} \left| \frac{X_k}{n^{1/\alpha}} - \frac{Y_k}{n^{1/\alpha}} \right| > \theta, \text{AMO}\left(\frac{\theta}{m+1}\right)\right] \\ &= P\left[\bigvee_{k=0}^{n-1} \left| \frac{X_k}{n^{1/\alpha}} - \frac{Y_k}{n^{1/\alpha}} \right| > \theta, \bigvee_{k=0}^{n-1} \bigvee_l \left| \frac{j_l}{n^{1/\alpha}} f(v_l, u_l - k) \right| \leq \frac{\theta}{m+1}\right] \\ &+ P\left[\bigvee_{k=0}^{n-1} \left| \frac{X_k}{n^{1/\alpha}} - \frac{Y_k}{n^{1/\alpha}} \right| > \theta, \text{AMO}\left(\frac{\theta}{m+1}\right), \bigvee_{k=0}^{n-1} \bigvee_l \left| \frac{j_l}{n^{1/\alpha}} f(v_l, u_l - k) \right| > \frac{\theta}{m+1}\right] =: A + B. \end{aligned}$$

Now

$$A \leq nP\left[\left| \frac{X_0}{n^{1/\alpha}} - \frac{Y_0}{n^{1/\alpha}} \right| > \theta, \bigvee_l \left| \frac{j_l}{n^{1/\alpha}} f(v_l, u_l) \right| \leq \frac{\theta}{m+1}\right]$$

$$\leq nP\left[\left|\sum_{l=m+1}^{\infty} \frac{\epsilon_l \Gamma_l^{-1/\alpha}}{n^{1/\alpha}}\right| > \frac{\theta}{m+1}\right] \rightarrow 0, \quad (n \rightarrow \infty),$$

while B is bounded by

$$\begin{aligned} B &\leq nP\left[\left|\sum_{l=2}^{\infty} \frac{\epsilon_l \Gamma_l^{-1/\alpha}}{n^{1/\alpha}}\right| > \theta, \bigvee_{l=2}^{\infty} \frac{\Gamma_l^{-1/\alpha}}{n^{1/\alpha}} \leq \frac{\theta}{m+1}\right] \\ &\leq nP\left[\left|\sum_{l=m+2}^{\infty} \frac{\epsilon_l \Gamma_l^{-1/\alpha}}{n^{1/\alpha}}\right| > \frac{\theta}{m+1}\right] \rightarrow 0, \quad (n \rightarrow \infty). \end{aligned}$$

Assume now that $\theta < \delta/2$. We have from (3.14)

$$\begin{aligned} (3.18) \quad &P[|N_n(g) - N_n^{(2)}(g)| > \epsilon] = o(1) + P[|N_n(g) - N_n^{(2)}(g)| > \epsilon, \text{AMO}\left(\frac{\theta}{m+1}\right)] \\ &= o(1) + P\left[\sum_{k=0}^{n-1} \left|g\left(\frac{X_k}{n^{1/\alpha}}\right) - g\left(\frac{Y_k}{n^{1/\alpha}}\right)\right| > \epsilon, \text{AMO}\left(\frac{\theta}{m+1}\right), \bigvee_{k=0}^{n-1} \left|\frac{X_k}{n^{1/\alpha}} - \frac{Y_k}{n^{1/\alpha}}\right| \leq \theta\right]. \end{aligned}$$

If $|\frac{Y_k}{n^{1/\alpha}}| \leq \delta/2$, then on $[\bigvee_{k=0}^{n-1} |\frac{X_k}{n^{1/\alpha}} - \frac{Y_k}{n^{1/\alpha}}| \leq \theta]$ we also have $|\frac{X_k}{n^{1/\alpha}}| \leq \delta/2$ and thus $g(\frac{X_k}{n^{1/\alpha}}) = g(\frac{Y_k}{n^{1/\alpha}}) = 0$. Therefore, the probability in (3.18) is bounded by

$$\begin{aligned} o(1) + P[\omega(\theta) \sum_{k=0}^{n-1} \delta_{|Y_k/n^{1/\alpha}|}(\frac{\delta}{2}, \infty) > \epsilon] &\leq o(1) + P[\omega(\theta) \sum_{k=0}^{n-1} \sum_l \delta_{|j_l f(v_l, u_l - k)/n^{1/\alpha}|}(\frac{\delta}{2}, \infty) > \epsilon] \\ &\rightarrow P[\omega(\theta) N_*\{x : |x| > \frac{\delta}{2}\} > \epsilon] \quad (n \rightarrow \infty) \end{aligned}$$

by Step 1 and as $\theta \rightarrow 0$, this expression converges to 0. This suffices. \square

Remark 3.2. Notice that in the case when the kernel f in (2.4) is compactly supported, the claim of Theorem 3.1 can also be obtained from the results of Davis and Hsing (1995).

4. STATIONARY S α S PROCESSES CORRESPONDING TO CONSERVATIVE MAPS AND THE CORRESPONDING POINT PROCESSES

Let \mathbf{X} be a stationary S α S process corresponding to a conservative map. According to our discussion in Section 2, the choice of $b_n = n^{-1/\alpha}$ in (1.1) is inappropriate. This leads to two natural questions: is there a choice of b_n that ensures weak convergence of the sequence of point processes (N_n) and, if yes, what normalizing sequence (b_n) achieves that?

Surprisingly, it turns out that for some stationary S α S processes corresponding to conservative maps, such a normalizing sequence exists, and for some other processes it does not exist; we will see examples of both in this section. This is in contrast with S α S processes corresponding to dissipative maps, whose corresponding point processes have well understood behavior. In fact, the variety of different classes of stationary S α S processes corresponding to conservative maps is so great, that we do not have a full picture of what may happen to the corresponding point processes in all cases. Nonetheless, the examples provided in this section demonstrate that the range of possibilities is wide.

Suppose that a stationary S α S process is given in an integral representation (2.1), and define

$$(4.1) \quad g_n(x) := \max_{i=0,1,\dots,n-1} |f_i(x)|, \quad x \in E, \quad n \geq 0.$$

A plausible guess for an appropriate choice of the normalizing sequence (b_n) in (1.1) is

$$(4.2) \quad b_n = \left(\int_E g_n(x)^\alpha m(dx) \right)^{1/\alpha}, \quad n = 1, 2, \dots$$

Indeed, it follows from the results in Samorodnitsky (2002) that, under very mild assumptions, the partial maximum of the stable process (corresponding to the boundary of the support of the point process) grows at the rate prescribed by (4.2). Furthermore, for representations (2.2) with dissipative maps, b_n given in (4.2) is asymptotically proportional to $n^{-1/\alpha}$.

The following example demonstrates a situation where using the normalizing sequence given by (4.2) ensures that the sequence of point processes (N_n) converges weakly.

Example 4.1. Let $0 < \alpha < \beta < 2$, and Y_0, Y_1, \dots be iid $S_\beta(\sigma, 0, 0)$ random variables. Let A be a positive strictly α/β -stable random variable independent of the sequence (Y_0, Y_1, \dots) , with a Laplace transform $Ee^{-\gamma A} = e^{-\gamma^{\alpha/\beta}}$, $\gamma \geq 0$ and define

$$(4.3) \quad X_i = A^{1/\beta} Y_i, \quad i = 0, 1, 2, \dots$$

Then, marginally, each X_i is a $S_\alpha(d_{\alpha,\beta}\sigma, 0, 0)$ random variable (for some finite positive constant $d_{\alpha,\beta}$), and the stationary S α S process \mathbf{X} defined by (4.3) is called a *sub-stable process*; see Section 3.8 in Samorodnitsky and Taqqu (1994)). Sub-stable processes correspond to conservative maps in the representation (2.1).

Since Y_0, Y_1, \dots are iid, it follows immediately that

$$\sum_{k=0}^{n-1} \delta_{n^{-1/\beta} Y_k} \Rightarrow \sum_{j=1}^{\infty} \delta_{C_\beta^{-1/\beta} \varepsilon_j \Gamma_j^{-1/\beta}}$$

as $n \rightarrow \infty$, weakly in the space \mathcal{M} , where, again, (ε_j) are iid Rademacher random variables, independent of a sequence of Poisson arrivals (Γ_j) , and C_β is, again, given by (2.6). We immediately conclude, for example using Laplace functionals, that

$$(4.4) \quad \sum_{k=0}^{n-1} \delta_{n^{-1/\beta} X_k} \Rightarrow \sum_{j=1}^{\infty} \delta_{C_\beta^{-1/\beta} A^{1/\beta} \varepsilon_j \Gamma_j^{-1/\beta}},$$

weakly in the space \mathcal{M} , if in the right hand side of (4.4) we take A to be independent of the sequences (ε_j) and (Γ_j) .

Note that for the sub-stable process (4.3), the choice of the normalizing sequence prescribed by (4.2) is ((Samorodnitsky and Taqqu, 1994, Proposition 3.8.2))

$$b_n = \left(E \max_{0,1,\dots,n-1} |Y_i|^\alpha \right)^{1/\alpha} \sim c n^{1/\beta}$$

as $n \rightarrow \infty$ for a finite positive constant c , where the asymptotic equivalence follows from, say, Resnick (1987, Section 2.1). Hence, for the sub-stable process (4.3) one can achieve weak convergence of the sequence of point processes (N_n) , and an appropriate choice of normalizing sequence (b_n) is precisely (4.2).

Of course, the same will be true if we replace, in the above construction of a sub-stable process, an iid sequence Y_0, Y_1, \dots , with any symmetric β -stable mixed moving average independent of A , as guaranteed by Theorem 3.1 above.

In marked contrast to Example 4.1, the following example shows that even with the apparently appropriate normalization given by (4.2), the sequence of point processes (N_n) may not converge weakly in the space \mathcal{M} .

Example 4.2. As in Example 5.3 of Samorodnitsky (2002), let $P_i, i \in \mathbb{Z}$ be the laws on $E = \mathbb{Z}^{\mathbb{Z}}$ of an irreducible null-recurrent Markov chain on \mathbb{Z} that corresponds to the different positions of the chain at time zero. Let $\pi = (\pi_i)_{i \in \mathbb{Z}}$ be the unique (σ -finite) invariant measure for this Markov chain satisfying $\pi_0 = 1$. Then

$$(4.5) \quad m(\cdot) = \sum_{i=-\infty}^{\infty} \pi_i P_i(\cdot)$$

is a σ -finite measure on E invariant under the left shift map ϕ ; the latter map is, further, conservative (see Harris and Robbins (1953)).

For $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, x_2, \dots) \in E$ let

$$(4.6) \quad f(\mathbf{x}) = \mathbf{1}_0(x_0).$$

Then $f \in L^\alpha(m)$ and we can define a stationary S α S process \mathbf{X} by the integral representation (2.1), with M a S α S random measure with control measure m , and

$$f_n(\mathbf{x}) = f \circ \phi^n(\mathbf{x}), \quad \mathbf{x} \in E, \quad n = 0, 1, 2, \dots$$

Notice that this is a representation of the form (2.2), with the functions a_n (the cocycle) equal identically to 1.

Let $S_1(\mathbf{x}) = \inf\{n > 0 : x_n = 0\}$ be the first entrance time of zero, and for $n \geq 2$ let $S_n(\mathbf{x}) = \inf\{n > 0 : x_{S_{n-1}(\mathbf{x})+n} = 0\}$ be the n th excursion length outside of zero. By our assumptions, the sequence (S_1, S_2, \dots) is, under the measure P_0 , an iid sequence of a.s. finite random variables with infinite mean. Let F_0 be the distribution of S_1 (under P_0). We assume additionally that

$$(4.7) \quad \overline{F}_0(k) = P_0(S_1 > k) = k^{\beta-1} L(k), \quad k \geq 1$$

for some $1/2 \leq \beta < 1$ and a function L , slowly varying at infinity. It follows from Lemma 3.3 in Resnick et al. (2000) that in this case the sequence (4.2) satisfies

$$(4.8) \quad b_n \sim \left(\frac{1}{\beta} n^\beta L(n) \right)^{1/\alpha} = \left(\sum_{k=1}^n P_0[S_1 \geq k] \right)^{1/\alpha},$$

as $n \rightarrow \infty$.

With the notation and set-up just described, we have the following result.

Proposition 4.3. *Suppose we use the normalizing sequence (4.2) to define a point process (1.1). The random measures $\{\overline{F}_0(n) \sum_{i=1}^{n-1} \delta_{X_i/b_n}, n \geq 1\}$ converge weakly in \mathcal{M} to a limiting random measure $N_*^{(\beta)}(C_\alpha^{1/\alpha} \cdot)$:*

$$(4.9) \quad \overline{F}_0(n) N_n \Rightarrow N_*^{(\beta)} \left(C_\alpha^{1/\alpha} \cdot \right)$$

where

$$(4.10) \quad N_*^{(\beta)} = \sum_{j=1}^{\infty} W_j^{(\beta)} \delta_{\varepsilon_j \Gamma_j^{-1/\alpha}}$$

with $(W_j^{(\beta)})$ an iid sequence independent of the sequences (ε_j) and (Γ_j) , such that

$$(4.11) \quad W_1^{(\beta)} \stackrel{d}{=} \left(\frac{B^{(\beta)}}{c_\beta S^{(1-\beta)}} \right)^{1-\beta}.$$

Here $B^{(\beta)}$ and $S^{(1-\beta)}$ are two independent random variables; $B^{(\beta)}$ has the Beta(1, β) distribution, and $S^{(1-\beta)}$ has the $S_{1-\beta}(1, 1, 0)$ distribution. Finally,

$$c_\beta = \left(2\Gamma(1 + \beta) \sin \frac{\pi\beta}{2} \right)^{1/\beta}.$$

The Laplace functional of $N_*^{(\beta)}$ is ($g \in C_K^+([-\infty, \infty] \setminus \{0\})$)

$$(4.12) \quad Ee^{-N_*^{(\beta)}(g)} = \exp \left\{ \iint_{[-\infty, \infty] \setminus \{0\} \times \mathbb{R}} (1 - e^{-wg(x)}) \nu_\alpha(dx) F_\beta(dw) \right\}$$

where F_β is the distribution of $W_1^{(\beta)}$.

Remark: Observe that (4.9) implies that the sequence of point processes (N_n) does not converge weakly in the space \mathcal{M} ; in fact, it is not even tight (see Lemma 3.20 in Resnick (1987)). Furthermore, the sequence of point processes (N_n) will not converge weakly to a non-trivial limit for any choice of normalizing constants in (1.1). If we select b_n to grow faster than prescribed by (4.2), we will obtain the zero measure in the limit, and if b_n grows at a slower rate than that prescribed by (4.2), then we will have accumulation of mass at infinity. The choice of the normalizing sequence according to (4.2) places the points at the right places, but the points cluster so much, that the cluster sizes themselves have to be normalized in order to obtain convergence. Finally notice that the limit in (4.9) is a random measure but not a point process.

Proof. To prove (4.9), we pursue a strategy similar to that used in the proof of Theorem 3.1. Let $\sum_l \delta_{(\mathbf{t}_l, j_l)}$ be PRM($m \times \nu_\alpha$) and we represent the process $\{X_n\}$ as

$$X_n = \sum_l f_n(\mathbf{t}_l) j_l = \sum_l j_l 1_{[t_l(n)=0]}(\mathbf{t}_l), \quad n = 0, \pm 1, \pm 2, \dots$$

Neglecting the factor C_α , we claim the random measures

$$(4.13) \quad \bar{F}_0(n) N_n = \bar{F}_0(n) \sum_{k=0}^{n-1} \delta_{X_k/b_n} \text{ and } \bar{F}_0(n) N_n^{(2)} := \bar{F}_0(n) \sum_l \sum_{k=0}^{n-1} \delta_{j_l f_k(\mathbf{t}_l)/b_n}$$

have the same limits. Assuming this to be so, compute the Laplace functional of $\bar{F}_0(n) N_n^{(2)}$ as in Theorem 3.1 to get for $g \in C_K^+([-\infty, \infty] \setminus \{0\})$

$$\begin{aligned} Ee^{-\bar{F}_0(n) N_n^{(2)}(g)} &= Ee^{-\sum_l \left(\sum_{k=0}^{n-1} \bar{F}_0(n) g(j_l f_k(\mathbf{t}_l)/b_n) \right)} \\ &= \exp \left\{ - \iint \left(1 - e^{-\sum_{i=0}^{n-1} \bar{F}_0(n) g(b_n^{-1} x 1_{[t(i)=0]}(\mathbf{t}))} \right) \nu_\alpha(dx) m(d\mathbf{t}) \right\} \end{aligned}$$

$$= \exp \left\{ - \iint \left(1 - e^{-\bar{F}_0(n)K_n(\mathbf{t})g(x)} \right) \frac{\nu_\alpha(dx)}{b_n^\alpha} m(d\mathbf{t}) \right\}$$

(where $K_n(\mathbf{t}) = \sum_{i=0}^{n-1} 1_{[\mathbf{t}(i)=0]}(\mathbf{t})$ is the number of visits to state 0 in $0, \dots, n-1$)

$$(4.14) \quad = \exp \left\{ - \int \left[\int \left(1 - e^{-\bar{F}_0(n)K_n(\mathbf{t})g(x)} \right) m_n(d\mathbf{t}) \right] \nu_\alpha(dx) \right\}.$$

Here

$$m_n(d\mathbf{t}) = b_n^{-\alpha} 1_{[S_1(\mathbf{t}) \leq n]}(\mathbf{t}) m(d\mathbf{t})$$

is a probability measure. (See Resnick et al. (2000, Lemma 3.3, page 329).)

Under P_0 , we have the functional limit theorem

$$\xi_n(s) = \beta_n^{-1} \sum_{i=1}^n S_i \Rightarrow c_\beta X_{1-\beta}(s),$$

in $D[0, \infty)$, where $\beta_n = \left(\frac{1}{1-\bar{F}_0}\right)^\leftarrow(n)$, and $X_{1-\beta}(\cdot)$ is a stable subordinator with index $1-\beta$ such that $X_{1-\beta}(1) = S^{(1-\beta)}$. By inversion, we also get with respect to P_0 that

$$(4.15) \quad \bar{F}_0(n)K_{[ns]} \Rightarrow X_{1-\beta}^\leftarrow(s)$$

in $D[0, \infty)$.

This helps us compute the limit distribution with respect to m_n of $\bar{F}_0(n)K_n(\mathbf{t})$ as follows: For $\lambda > 0$,

$$\begin{aligned} m_n[\bar{F}_0(n)K_n > \lambda] &= b_n^{-\alpha} m[\bar{F}_0(n)K_n > \lambda, S_1 \leq n] = b_n^{-\alpha} m[\bar{F}_0(n)K_n > \lambda] \\ &= \sum_{i=-\infty}^{\infty} \pi_i b_n^{-\alpha} P_i[\bar{F}_0(n)K_n > \lambda] \end{aligned}$$

and using a renewal argument, this is

$$\begin{aligned} &= \sum_{i=-\infty}^{\infty} \sum_{j=1}^n \pi_i b_n^{-\alpha} P_i[S_1 = j] P_0[\bar{F}_0(n)K_{n-j} > \lambda] \\ &= \sum_{j=1}^n P_0[\bar{F}_0(n)K_{n-j} > \lambda] \sum_{i=-\infty}^{\infty} b_n^{-\alpha} \pi_i P_i[S_1 = j]. \end{aligned}$$

Now

$$\sum_{i=-\infty}^{\infty} b_n^{-\alpha} \pi_i P_i[S_1 = j] = b_n^{-\alpha} m[S_1 = j] = b_n^{-\alpha} P_0[S_1 \geq j] =: p_n(j)$$

where we used Resnick et al. (2000, Lemma 3.3). Thus

$$(4.16) \quad \begin{aligned} m_n[\bar{F}_0(n)K_n > \lambda] &= \sum_{j=1}^n p_n(j) P_0[\bar{F}_0(n)K_{n-j} > \lambda] \\ &= E \left(P_0[F_0(n)K_{n(1-\frac{T_n}{n})} > \lambda] \right) \end{aligned}$$

where T_n is a random variable independent of K_n with mass function $\{p_n(j), j = 1, \dots, n\}$. Note for $0 < \theta < 1$,

$$P\left[\frac{T_n}{n} \leq \theta\right] = \sum_{j \leq n\theta} p_n(j) = \frac{\sum_{j \leq n\theta} P_0[S_1 \geq j]}{\sum_{j \leq n} P_0[S_1 \geq j]} \rightarrow P[T_\infty \leq \theta] = \theta^\beta,$$

by Karamata's theorem. Thus, from (4.15) and (4.16),

$$\begin{aligned} m_n[\bar{F}_0(n)K_n > \lambda] &\rightarrow EP_0[X_{1-\beta}^-(1 - T_\infty) > \lambda] = P_0[1 - T_\infty > X_{1-\beta}(\lambda)] \\ &= P_0[1 - T_\infty > \lambda^{1/(1-\beta)} X_{1-\beta}(1)] \\ &= P_0\left[\left(\frac{1 - T_\infty}{S^{(1-\beta)}}\right)^{1-\beta} > \lambda\right] = P_0[W_1^{(\beta)} > \lambda]. \end{aligned}$$

It follows from (4.14), that the Laplace functional of $\bar{F}_0(n)N_n^{(2)}$ converges to (4.12) as desired.

Finally, we must show (4.9). Dropping the factor C_β , it is enough to show

$$(4.17) \quad P[\rho(\bar{F}_0(n)N_n, \bar{F}_0(n)N_n^{(2)}) > \epsilon] \rightarrow 0, \quad (n \rightarrow \infty),$$

where recall $\rho(\cdot, \cdot)$ is the vague metric. Showing (4.17), amounts to showing for any $g \in C_K^+([-\infty, \infty] \setminus \{0\})$ (with compact support in, say, $\{x : |x| > \delta\}$) that

$$(4.18) \quad P[\bar{F}_0(n)|N_n(g) - N_n^{(2)}(g)| > \epsilon] \leq P[\bar{F}_0(n) \sum_{k=0}^{n-1} |g(\sum_l \frac{j_l}{b_n} f_k(\mathbf{t}_l)) - \sum_l g(\frac{j_l}{b_n} f_k(\mathbf{t}_l))| > \epsilon] \rightarrow 0.$$

This demonstration is similar to the one given in Theorem 3.1 and we only outline the steps. This time we define, for any $\theta > 0$ $AMO(\theta)$ as

$$(AMO(\theta))^c = \bigcup_{k=0}^{n-1} \left[\sum_l \delta_{|j_l f_k(\mathbf{t}_l)/b_n|}(\theta, \infty] \geq 2 \right].$$

So

$$\begin{aligned} P((AMO(\theta))^c) &\leq nP\left[\sum_l \delta_{|j_l f_0(\mathbf{t}_l)/b_n|}(\theta, \infty] \geq 2\right] \leq n \left(E\left(\sum_l \delta_{|j_l f_0(\mathbf{t}_l)/b_n|}(\theta, \infty]\right) \right)^2 \\ &= n \left(\iint_{|x| > \theta, f_k(\mathbf{t})=1} \frac{\nu_\alpha(dx)}{b_n^\alpha} m(dt) \right)^2 = \frac{n}{b_n^{2\alpha}} \theta^{-\alpha} m[f_0 = 1] \rightarrow 0, \end{aligned}$$

if $\frac{1}{2} < \beta < 1$, since $b_n \in RV_{\beta/\alpha}$. The fact that this probability converges to zero in the case $\beta = 1/2$ as well can be shown as in Example 5.3 of Samorodnitsky (2002).

Now pick m large. In this case, we need to pick an integer $p > \alpha/\beta$ and $m + 1 > p/\alpha$. For $X_k = \sum_l j_l f_k(\mathbf{t}_l)$, define Y_k on $AMO(\theta/(m + 1))$ to be the summand in the definition of X_k of largest modulus. So

$$\begin{aligned} P[\bar{F}_0(n) \sum_{k=0}^{n-1} |g(\sum_l \frac{j_l}{b_n} f_k(\mathbf{t}_l)) - \sum_l g(\frac{j_l}{b_n} f_k(\mathbf{t}_l))| > \epsilon] \\ = P[\bar{F}_0(n) \sum_{k=0}^{n-1} |g(\sum_l \frac{j_l}{b_n} f_k(\mathbf{t}_l)) - \sum_l g(\frac{j_l}{b_n} f_k(\mathbf{t}_l))| > \epsilon, AMO(\frac{\theta}{m + 1})] + o(1) \end{aligned}$$

and as in the proof of Theorem 3.1, this is

$$\leq P\left[\bar{F}_0(n)\omega(\theta) \sum_{k=0}^{n-1} \sum_l \delta_{|j_l f_k(\mathbf{t}_l)/b_n|}(\frac{\delta}{2}, \infty] > \epsilon, AMO(\frac{\theta}{m+1}), \right. \\ \left. \bigvee_{k=0}^{n-1} \left| \frac{X_k}{b_n} - \frac{Y_k}{b_n} \right| \leq \frac{\theta}{m+1} \right] + o(1) + o(1)$$

which goes to zero as first $n \rightarrow \infty$ and then $\theta \rightarrow 0$. \square

Remark. For the point process $N_n^{(2)}$ we have

$$N_n^{(2)} = \sum_{k=0}^{n-1} \sum_l \delta_{j_l f_k(\mathbf{t}_l)/b_n} = \sum_l K_n(\mathbf{t}_l) \delta_{j_l/b_n}$$

so the cluster sizes are represented by $K_n(\mathbf{t}_l)$.

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