

Liquidity Risk and Risk Measure Computation

Robert A. Jarrow*, and Philip Protter †

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Abstract

This paper shows how to apply the recent liquidity risk model of Çetin, Jarrow and Protter [3] to compute a simple and robust adjustment to standard risk measures (e.g. value-at-risk, coherent, or convex) for liquidity risk.

1 Introduction

Much has been written on how to properly incorporate both market and credit risk into various measures of financial risk such as value-at-risk (*var*) or coherent risk measures, see Jorion [8] and references therein. In contrast, much less has been written on how to include liquidity risk in such risk measures.¹ The purpose of this paper is to provide a simple and robust method for including liquidity risk into *var* and other risk measure computations. This is done using a new approach for modeling liquidity risk as presented in Çetin, Jarrow and Protter [3].

This approach hypothesizes the existence of a stochastic supply curve for a security's price as a function of transaction size. Specifically, a second argument incorporates the size (number of shares) and direction (buy versus sell) of a transaction to determine the price at which the trade is executed. For a given supply curve, traders act as price takers. The more liquid an asset,

*Johnson Graduate School of Management, Cornell University, Ithaca, NY, 14853.

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¹One exception is Jarrow and Subramanian [7].

the more horizontal its supply curve. In the context of continuous trading, they characterize necessary and sufficient conditions on the supply curve's evolution such that there are no arbitrage opportunities in the economy. Furthermore, given an arbitrage free evolution for the supply curve, conditions for an approximately complete market are also provided. In the most general setting with unrestricted predictable trading strategies, they obtain three primary conclusions with respect to the pricing of derivatives. First, all liquidity costs are avoidable when (approximately) replicating a derivative's payoff using continuous trading strategies of finite variation. Second, as a consequence of the previous conclusion, the derivative's price is the price obtained if the bid-ask spread and other illiquidities are ignored. Third, there are no implied bid-ask spreads or illiquidities for a derivative's price. It is important to emphasize that these conclusions follow from continuous trading of infinitesimal quantities. In particular, prices for "zero" shares or marginal transactions are required to be well defined and attainable.

This paper shows how to apply the insights from Çetin, Jarrow and Protter [3], in a simple and robust manner, to modify current risk measures (like var or coherent) to account for liquidity risk. The adjustments are formulated with implementation in mind, so the procedures are straightforward in their construction. More complex adjustments are possible, but these await subsequent research.

An outline of this paper is as follows. Section 2 describes the model of Çetin, Jarrow and Protter [3]. Section 3 presents the linear supply curve formulation and estimation procedure used in Çetin, Jarrow, Protter, Warachka [4] and Blais and Protter [2], but modified for "crisis" market situations. Section 4 presents the liquidity adjusted risk measures, and section 5 concludes.

2 The Model

This section reviews the framework of Çetin, Jarrow and Protter [3] for usage in our subsequent analysis. We are given a filtered probability space $(\Omega, F, (F_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfying the usual conditions where T is a fixed time, and \mathbb{P} represents the statistical or empirical probability measure. We consider a market for a stock that pays no dividends. Also traded is a money market account that accumulates value at the spot rate of interest denoted r .

Let $S(t, x)$ represent the stock price, *per share*, at time $t \in [0, T]$ that

a trader pays/receives for order flow x normalized by the value of a money market account. A positive order ($x > 0$) represents a buy, a negative order ($x < 0$) represents a sale, and the zeroth order ($x = 0$) corresponds to the *marginal trade*.

2.1 Trading Strategies and Liquidity Costs

We assume throughout that the interest rate is 0. This hypothesis can be removed by the usual procedure (for example see [9]). A *trading strategy* is a triplet $((X_t, Y_t : t \in [0, T]), \tau)$ where X_t represents the trader's aggregate stock holding at time t (units of the stock), Y_t represents the trader's aggregate money market account position at time t (units of the money market account), and τ represents the liquidation time of the stock position, subject to the following restrictions²:

1. $X_{0-} \equiv Y_{0-} \equiv 0$,
2. $X_T = 0$, and
3. $X = H1_{[0, \tau]}$ for some process $H(t, \omega)$ where $\tau \leq T$ is a stopping time.³

These restrictions ensure that the trading strategy is liquidated prior to time T which ensures that round trip liquidity costs are incurred. The stopping time τ allows the portfolio to be liquidated before time T .

A *self-financing trading strategy (s.f.t.s)* generates no cash flows for all times $t \in (0, T)$ after the initial purchase. More formally, a self-financing trading strategy is a trading strategy $((X_t, Y_t : t \in [0, T]), \tau)$ where:

1. X_t is càdlàg with finite quadratic variation ($[X, X]_T < \infty$),⁴
2. $Y_0 = -X_0S(0, X_0)$, and
3. for $0 < t \leq T$,

$$Y_t = Y_0 + X_0S(0, X_0) + \int_0^t X_{u-}dS(u, 0) - X_tS(t, 0) - L_t \quad (1)$$

² X_t and Y_t are predictable and optional processes, respectively.

³Here, $H(t, \omega)$ is a predictable process and τ is a predictable ($F_t : 0 \leq t \leq T$) stopping time.

⁴see [10] for the definition of all undefined notation.

where L_T is the liquidity cost, defined as

$$L_t = \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c \geq 0 \quad (2)$$

with $L_{0-} = 0$.

The expression $[X, X]_t^c$ denotes the quadratic variation of the continuous part of X at time t (see Protter [10]).

Observe that the liquidity cost consists of two components. The first is due to discontinuous changes in the share holdings while the second results from continuous changes. For a continuous trading strategy, the first term in expression (2) always equals $L_0 = X_0[S(0, X_0) - S(0, 0)]$ after time zero. Note that this expression is the total dollars paid for the shares $X_0 S(0, X_0)$ less the total dollars that would have been paid if there were no quantity impact on the price $X_0 S(0, 0)$. If the trading strategy is also of finite variation, then the second term in equation (2) is zero because $[X, X]_t^c = 0$. Thus, if one employs a trading strategy that is both continuous and of finite variation, then the entire liquidity cost of the s.f.t.s. is due to forming the initial position, and manifested in L_0 .

2.2 Fundamental Theorems of Finance with Illiquidity

As is standard in the literature, an *arbitrage opportunity* is any s.f.t.s. (X, Y, τ) such that $\mathbb{P}\{Y_T \geq 0\} = 1$ and $\mathbb{P}\{Y_T > 0\} > 0$.

A modified first fundamental theorem of asset pricing is available in our economy with illiquidity. For $\beta \geq 0$, define Θ_β as

$$\Theta_\beta \equiv \left\{ \text{s.f.t.s } (X, Y, \tau) \mid \int_0^t X_{u-} dS(u, 0) \geq -\beta \text{ for all } t \text{ almost surely} \right\}.$$

The set Θ_β represents trading strategies whose values are bounded below, excluding doubling strategies from consideration. The modified first fundamental theorem of finance with illiquidity states that if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $S(\cdot, 0)$ is a \mathbb{Q} -local martingale, then there is no arbitrage for $(X, Y, \tau) \in \Theta_\beta$ for any β .

For pricing derivatives, we assume the existence of such a \mathbb{Q} -local martingale for $S(\cdot, 0)$. Next, a market is defined to be *approximately complete* if

given any contingent claim C , defined as a \mathbb{Q} square integrable random variable, there exists a sequence of self financing trading strategies (X^n, Y^n, τ^n) such that $Y_T^n \rightarrow C$ as $n \rightarrow \infty$ in $L^2(d\mathbb{Q})$.⁵ This definition is similar to the standard definition of a complete market. The difference is that the cash flow C of any contingent claim is only approximately attained.

Given this definition, a modified second fundamental theorem of asset pricing also holds in this setting. The modified second theorem states that the existence of a unique probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $S(\cdot, 0) = s$ is a \mathbb{Q} -local martingale implies the market is approximately complete.

It is perhaps surprising that in an approximately complete market, a continuous and finite variation trading strategy may always be constructed to approximate any contingent claim. Indeed, Çetin, Jarrow and Protter [3] demonstrate that given any contingent claim C , there exists a predictable process X such that $C = c + \int_0^T X_u ds_u$ and a sequence of s.f.t.s. (X^n, Y^n, τ^n) , where X^n is continuous and of finite variation with the properties $X_0^n = 0$, $X_T^n = 0$, and $Y_0^n = c$ for all n such that $Y_T^n \rightarrow C$ in $L^2(d\mathbb{Q})$. The liquidity cost of this sequence of s.f.t.s. is equal to zero (since the first trade is of zero magnitude and the s.f.t.s. is continuous and of finite variation), and the contingent claim's cash flows at time T are approximated by

$$Y_T^n = c + \int_0^T X_u^n dS(u, 0). \quad (3)$$

This implies that the unique arbitrage free value for the contingent claim is its classical value

$$E^{\mathbb{Q}}(C). \quad (4)$$

This completes the summary of the three conclusions from Çetin, Jarrow and Protter [3].

The implication of these three conclusions for risk management and pricing is that in “normal” market situations - those when the trading strategies postulated by the model can be utilized - the classical pricing formulas and risk measures apply, even in an economy with liquidity risk. The reason is that by trading (nearly)⁶ continuously and in small quantities, one can avoid

⁵The space $L^2(d\mathbb{Q})$ is the set of F_T -measurable random variables that are square integrable using the probability measure \mathbb{Q} .

⁶Of course, in practice one can only approximate continuous trading strategies, but the argument still applies as long as there is no market structures that limit the ability of the approximation to become better and better. For an example of such an approximation

accumulating significant liquidity costs. But, in “crisis” market situations, like market crashes or when a company experiences huge trading losses - *when market conditions make continuous and small quantity trading impossible* - then liquidity costs become a significant consideration. Risk management procedures and risk measures need to anticipate and to quantify the impact of liquidity costs in these ”crisis” conditions. This is the task to which we now turn.

3 A Linear Supply Curve

The first step in quantifying liquidity risk for risk management applications is to obtain a procedure for estimating the supply curve $S(t, x)$. This is the purpose of this section. As a useful first approximation, we consider a linear supply curve formulation where the slope depends on the state of the economy (crisis or normal). Recent studies by Çetin, Jarrow, Protter, Warachka [4] and Blais and Protter [2] support the linear supply curve formulation, with randomly changing slope coefficients for illiquid stocks. By analogy, in a “crisis” situation, all stocks become illiquid due to market hysteria and the flight to quality.

3.1 The Supply Curve

These insights motivate the following form for the supply curve

$$S(t, x) = S(t, 0)[1 + \alpha_c 1_c x + \alpha_n (1 - 1_c)x] \quad (5)$$

where $\alpha_c \geq \alpha_n \geq 0$ are constants and 1_c is an indicator function for a crisis market.

The first term in the supply curve (the right side of 5) is the classical asset value’s process $S(t, 0)$. This would be the asset process assumed in the classical option pricing approach, for example, in the Black Scholes model this would be a geometric Brownian motion. The second term captures the quantity impact on the asset’s price. It is assumed to be linear, with a different slope coefficient in a normal versus a crisis scenario. The slope

prohibiting market structure, Çetin, Jarrow, Protter, Warachka [4] consider an economy where there is a fixed time unit $\Delta > 0$, after which no additional trades can take place. This discreteness would destroy convergence of any approximating sequence of trading strategies to its continuous time limit.

coefficient for a crisis situation is represented by α_c . It exceeds the slope coefficient for normal times α_n , indicating a larger quantity impact on the price. The indicator variable identifying a crisis market situation is the random variable 1_c .

3.2 Time Series Estimation

To apply the above technology, we need to estimate the slope coefficients of the supply curve. To facilitate this estimation, we assume that the asset values evolve according to a diffusion process

$$dS^i(t, 0) = \mu^i(t)S^i(t, 0)dt + \sigma^i(t)S^i(t, 0)dW^i(t) \quad (6)$$

where $\mu^i(t), \sigma^i(t)$ are suitably bounded so that expression (6) is well-defined, $dW^i(t)dW^j(t) = \eta_{ij}dt$. For $i = 0$, the money market account, $\sigma^i(t, S^i(t, 0)) \equiv 0$, $\mu^0(t) = r$, and $S^0(0, 0) = 1$.

In terms of the available data, let us suppose that we observe at discrete time points the set $(t_j, S^i(t_j, x_j), x_j^i)_{j=1}^m$ for all i . This information includes for each asset both the price and quantity traded (a purchase or sale).

Consider a single asset (dropping the superscript).

$$\log S(t_1, x_{t_1}) = \log S(t_1, 0) + \log[1 + \alpha_c 1_c x_{t_1} + \alpha_n(1 - 1_c)x_{t_1}] \quad (7)$$

For our liquidity cost adjustment (as seen below), we need only estimate the slope coefficient when the market is in a crisis situation. Hence, we can partition the data into two sets - normal and crisis - and only use the observations from the crisis set for the estimation^{7,8}. We assume that this is the case. Then, taking the difference of the stock price process at two consecutive crisis dates gives

$$\begin{aligned} \log \frac{S(t_2, x_{t_2})}{S(t_1, x_{t_1})} &= \log \frac{S(t_2, 0)}{S(t_1, 0)} + \log \frac{1 + \alpha_c x_{t_2}}{1 + \alpha_c x_{t_1}} \\ &= \left[\int_{t_1}^{t_2} \mu(t) - \left(\frac{1}{2}\right)\sigma(t)^2 \right] dt + \int_{t_1}^{t_2} \sigma(t) dW(t) + \log \frac{1 + \alpha_c x_{t_2}}{1 + \alpha_c x_{t_1}} \end{aligned} \quad (8)$$

⁷Of course, we are intentionally being vague about our definition of a market crisis. One could use, for example, a weak definition of a crisis market as just a declining market. It depends on the quantity of data available for the asset in a true "crisis" like a stock price crash.

⁸Of course, the same procedure applies to normal times, utilizing the "normal" market time series data instead.

So that

$$\log \frac{S(t_2, x_{t_2})}{S(t_1, x_{t_1})} \simeq \left[\int_{t_1}^{t_2} \mu(t) - \left(\frac{1}{2}\right)\sigma(t)^2 \right] dt + \int_{t_1}^{t_2} \sigma(t) dW(t) + \alpha_c [x_{t_2} - x_{t_1}] \quad (9)$$

Assuming μ, σ are constants, we can run a time series regression for each asset to get α_c where $\int_{t_1}^{t_2} \sigma(t) dW(t)$ is the regression error.⁹ These estimated coefficients are the necessary inputs to quantify liquidity risk in the risk measures introduced below.

4 Risk Measures with Liquidity Risk

This section shows how to apply the insights from Çetin, Jarrow and Protter [3] to the computation of risk measures that include liquidity risk. We take a simple and robust approach to this computation. As discussed in section 2, we see that trading continuously and in infinitesimal amounts, one can avoid all liquidity costs. In computing risk measures for risk management, one needs to be conservative. Thus, the worst case scenario is a crisis situation, where one has to liquidate assets immediately. The idea is that if the market is declining quickly, then one does not have the luxury to sell assets slowly (continuously) in “small” quantities until the entire position is liquidated. In this situation, liquidity costs become relevant and their consideration essential.

Conceptually, to incorporate liquidity costs in a crisis situation, one would need to specify the trading strategies feasible in such markets, and then determine the optimal trading strategy for liquidation (as for example in Jarrow and Subramanian [7]). This formulation would require the solution of a complex dynamic stochastic control problem. The derived solution would be model specific, and unless the model is specified correctly, it could provide misleading recommendations. Instead, we pursue a more robust and simple approach, leaving the optimal solution to subsequent research. We make the conservative (and perhaps suboptimal) assumption that immediate liquidation is required in a crisis situation. This will generate liquidity costs under the worst case scenario, hence, those appropriate for risk management considerations.

⁹If the error term is heteroskedastic, then standard techniques to handle heteroskedasticity can be applied.

4.1 Portfolio Value Determination

The key input to a risk measure is the value of a firm's portfolio (trading strategy) at some future time T . Depending on the application, this future horizon could be a day, a week, or a year. We now discuss how to compute this value. Using the liquidity risk structure from Çetin, Jarrow and Protter [3], we can easily compute the value of a portfolio in the worst case scenario of immediate liquidation. From expression (1), the value of the position at time T including liquidity costs, denoted V_T^L , is:

$$V_T^L \equiv Y_T + X_T S(T, 0) = Y_0 + X_0 S(0, X_0) + \int_0^T X_{u-} dS(u, 0) - L_T \quad (10)$$

where

$$L_T = \sum_{0 \leq u \leq T} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^T \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c. \quad (11)$$

We suppose that prior to time T markets are "normal," so that we can avoid all liquidity costs until then. The value of our position at time $T-$, an instant before time T , is therefore

$$V_T \equiv Y_{T-} + X_T S(T, 0) = Y_0 + X_0 S(0, X_0) + \int_0^T X_{u-} dS(u, 0). \quad (12)$$

We emphasize that this value equals the time T value of the position in the classical model without liquidity costs.

Due to the crisis at time T , we assume that we need to liquidate $\theta \in [0, 1]$ percent of holdings, so that the liquidity costs are:

$$L_T = -\theta X_T [S(T, -\theta X_T) - S(T, 0)]. \quad (13)$$

Note, that if $X_T > 0$, then liquidation implies that shares are sold, and $L_T > 0$; ¹⁰ if $X_T < 0$, then liquidation implies that shares are purchased, and $L_T > 0$. This liquidity cost represents the total dollars generated $-\theta X_T S(T, -\theta X_T)$ due to liquidation at T , less the total dollars generated if there were no quantity impact on the price $-\theta X_T S(T, 0)$. Then, from expressions (10) and (12):

$$V_T^L = V_T - L_T = V_T + \theta X_T [S(T, -\theta X_T) - S(T, 0)] \leq V_T. \quad (14)$$

¹⁰Note that in this case $[S(T, -\theta X_T) - S(T, 0)] < 0$.

Hence, the time T value of the position with liquidity costs is the classical value less the time T liquidation costs. Liquidity costs from immediate liquidation shifts the entire distribution of the terminal value V_T to the left (with probability one). We now provide explicit computations of the portfolio value given liquidity costs under the linear supply curve formulation.

4.1.1 A Single Asset Portfolio

Consider a single asset position with zero holdings in the money market account. We apply the supply curve in expression (5) to determine the asset's time T value in the worst case scenario as in expression (14). Recall that we have a position of size X_T and we need to liquidate θ percent of holdings. Then, the value of the portfolio without liquidity risk is $V_T = X_T S(T, 0)$ and the liquidity costs are $L_T = -\theta X_T [S(T, -\theta X_T) - S(T, 0)]$. Note, that if $X_T > 0$, then this implies shares are sold; if $X_T < 0$, then this implies shares are purchased. Simple substitution yields

$L_T = -\theta X_T [S(T, 0)[1 - \alpha_c \theta X_T] - S(T, 0)] = \theta^2 X_T^2 S(T, 0) \alpha_c > 0$. Combined,

$$\begin{aligned} V_T^L &= X_T S(T, 0) - L_T & (15) \\ &= X_T S(T, 0) [1 - \alpha_c \theta^2 X_T] \\ &= V_T [1 - \alpha_c \theta^2 X_T] \leq V_T \end{aligned}$$

The time T value including liquidity costs is equal to $[1 - \alpha_c \theta^2 X_T]$ times the classical time T value. This adjustment shifts the portfolio's distribution to the left, i.e. it reduces the portfolio's value for all possible states of the economy (with probability one). Indeed, if $V_T > 0$, then $X_T > 0$ and $[1 - \alpha_c \theta^2 X_T] < 1$, implying that less dollars are received when selling shares. If $V_T < 0$, then $X_T < 0$ and $[1 - \alpha_c \theta^2 X_T] > 1$ implying more dollars are paid when buying back shares (covering short positions). Note that the decline in value is greater when the slope of the supply curve α_c is larger or when the percent of the position that is liquidated θ is larger.

4.1.2 Multiple Asset Portfolios

The liquidity cost for multiple asset portfolios is a straightforward modification of expression (15). For a portfolio consisting on N assets, let the

different assets be indexed by $i = 0, 1, \dots, N$ where $i = 0$ corresponds to the money market account with $\alpha_c^0 \equiv 0$. We suppose that the money market account has no quantity impact. Other assumptions are possible.

Given this notation, the i th asset has a position of size X^i . In a crisis, we need to liquidate θ^i percent of its holdings. Then, repeating the algebra from the single asset portfolio case, the time T value of the multi-asset portfolio including liquidity costs is:

$$\begin{aligned} V_T^L &= \sum_{i \geq 1} X_T^i S^i(T, 0) [1 - \alpha_c^i (\theta^i)^2 X_T^i] + X_T^0 S^0(T, 0) \\ &\leq V_T = \sum_{i \geq 1} X_T^i S^i(T, 0) + X_T^0 S^0(T, 0). \end{aligned} \quad (16)$$

This is a simple adjustment. It indicates that one needs to multiply the final value of each asset by its liquidity discount $[1 - \alpha_c^i (\theta^i)^2 X_T^i]$. This value $[1 - \alpha_c^i (\theta^i)^2 X_T^i] < 1$ if $X_T^i > 0$ and shares are sold at liquidation, and $[1 - \alpha_c^i (\theta^i)^2 X_T^i] > 1$ if $X_T^i < 0$ and shares are purchased at liquidation. Liquidity costs shifts (with probability one) the value of the portfolio at liquidation to the left.

4.2 Risk Measures

As mentioned earlier, the key input to a risk management risk measure is the value of a firm's portfolio at some prespecified future date T . Depending on the application, this time horizon could be a day, a week, or a year. This section briefly discusses how to incorporate a liquidity risk adjustment to standard risk measures like *var* (value-at-risk) or coherent risk measures (see Jorion [8], Artzner, Delbaen, Eber, Heath [1], Jarrow [6]).

First, let us recall the definition of a risk measure. Let G be the space of random variables at time T , representing the set of possible portfolio values. Included in this set of random variables is a holding of just the money market account, with time T value denoted $S^0 \in G$. A *risk measure* is defined as a mapping $\rho : G \rightarrow R^+$. Different risk measures have different properties. The liquidity cost adjustment is straightforward. The risk measure for a firm with no liquidity risk is given by

$$\text{no liquidity risk: } \rho(V_T)$$

where V_T is defined as in expression (16). In contrast, the risk measure for a portfolio with liquidity risk is given by

$$\textbf{liquidity risk: } \rho(V_T^L)$$

where V_T^L is also defined as in expression (16). We illustrate our liquidity cost adjustment with common examples: VAR, coherent, and convex risk measures. We treat average-value-at-risk separately.

4.2.1 Value-at-risk

The *var* risk measure is defined as follows. For a given risk V , and a prespecified risk level $0 \leq \lambda \leq 1$,

$$\text{var}_\lambda(V) = \inf\{\alpha \mid \mathbb{P}(V < -\alpha) \leq \lambda\}.$$

Given liquidity risk, we would replace V_T with V_T^L in the computation of *var*. If we were using a Monte Carlo simulation, one would adjust each individual asset's payoff $S^i(T, 0)$ from the classical model by the quantity $[1 - \alpha_c^i(\theta^i)^2 X_T^i]$ from expression (16). This is a minimal change to the simulation.

4.2.2 Coherent Risk Measures

A coherent risk measure is defined to be any risk measure ρ that satisfies the following four axioms:

Definition 1 *A coherent risk measure is a risk measure that satisfies the following four axioms:*

- *(Translation Invariance)* For all $V \in G$ and all $\alpha \in R$, $\rho(V + \alpha S^0) = \rho(V) - \alpha$.
- *(Subadditivity)* For all $V_1, V_2 \in G$, $\rho(V_1 + V_2) \leq \rho(V_1) + \rho(V_2)$.
- *(Positive Homogeneity)* For all $V \in G$ and all $\lambda \geq 0$, $\rho(\lambda V) = \lambda \rho(V)$.
- *(Inverse Monotonicity)* For all $V_1, V_2 \in G$ with $V_1 \leq V_2$, $\rho(V_2) \leq \rho(V_1)$.

The inverse monotonicity is assumed because we think of a risk measure as the amount of capital needed *to be added* before one accepts the risk.

We apply the coherent risk measure ρ to the liquidity adjusted value V_T^L computed in expression (16). For single asset portfolios, the positive homogeneity axiom is sufficient to generate the liquidity cost adjustment since $[1 - \alpha_c \theta^2 X_T] > 0$:

$$\rho(V_T^L) = \rho([1 - \alpha_c \theta^2 X_T] V_T) = [1 - \alpha_c \theta^2 X_T] \rho(V_T) \quad (17)$$

where V_T is the portfolio's value excluding liquidity costs. As indicated, the risk measure without liquidity risk is premultiplied by the quantity $[1 - \alpha_c \theta^2 X_T]$.

For a multiple asset portfolio, there is no simple adjustment.¹¹ One just replaces V_T with V_T^L in the computation of ρ . Note that by monotonicity,

$$V_T^L \leq V_T \text{ implies } \rho(V_T) \leq \rho(V_T^L),$$

that is, the risk measure including liquidity risk is greater than that without.

4.2.3 Convex Risk Measures

Recently, another set of risk measures has been introduced to the literature, called convex risk measures. In this section, we apply our liquidity risk adjustment to this new set of convex risk measures. Our formulation follows that of [5] and [11].

Definition 2 *A convex risk measure is a risk measure that satisfies the following four axioms, where L^∞ denotes the set of bounded random variables.*

- *(Translation Invariance)* For all $V \in L^\infty$ and all $\alpha \in R$, $\rho(V + \alpha S^0) = \rho(V) - \alpha$.
- *(Inverse Monotonicity)* For all $V_1, V_2 \in L^\infty$ with $V_1 \leq V_2$, $\rho(V_2) \leq \rho(V_1)$.
- *(Distribution Invariance)* If the distributions of $V_1, V_2 \in L^\infty$ are equal under P , then $\rho(V_1) = \rho(V_2)$.
- *(Convexity)* If V_1, V_2 are two elements of L^∞ and $0 \leq \alpha \leq 1$, then

$$\rho(\alpha V_1 + (1 - \alpha) V_2) \leq \alpha \rho(V_1) + (1 - \alpha) \rho(V_2).$$

¹¹ By subadditivity, we have that $\rho(V_T^L) = \rho(\sum_{i \geq 1} X_T^i S^i(T, 0) [1 - \alpha_c^i (\theta^i)^2 X_T^i] + X_T^0 S^0(T, 0)) \leq \sum_{i \geq 1} [1 - \alpha_c^i (\theta^i)^2 X_T^i] \rho(X_T^i S^i(T, 0)) + \rho(X_T^0 S^0(T, 0))$.

Convex risk measures include as a special case coherent risk measures. Indeed, note that if a risk measure has both positive homogeneity and convexity, then it is a coherent risk measure. The replacement of coherent risk measures with convex risk measures (the omission of the positive homogeneity axiom) is important because it captures alternative economic scenarios not captured by the positive homogeneity axiom. For example, in the case of market illiquidity, scale expansion of a portfolio position may not lead to a linear growth in risk, e.g., when large orders must be processed in quantity (as might happen during market crashes or bubbles). Due to the convexity axiom, we once again have the relationship

$$\rho(V_T) \leq \rho(V_T^L).$$

4.2.4 Average Value-at-risk

One of the objections to *var* as a risk measure is that it only considers the probability of a loss, and disregards the size of the loss. It is well known that this has the consequence of discouraging diversification as a way of controlling risk. Furthermore, *var* is not a coherent risk measure nor a convex risk measure. And, since it is equal to the infimum of all dominating convex risk measures, there is no closest convex risk measure to *var*, thereby making it impossible to approximate its behavior via a convex risk measure. There is, however, a risk measure which is defined in terms of *var* that avoids these objections (it is a coherent risk measure), namely *Average value-at-risk*.

Definition 3 We define **Average value-at-risk**, or **Avar**, for a given risk V at the level λ with $0 \leq \lambda \leq 1$, as

$$Avar_\lambda(V) = \frac{1}{\lambda} \int_0^\lambda var_\gamma(V) d\gamma.$$

Average Value at Risk is sometimes alternatively called *Conditional Value at Risk* or *Expected Shortfall*. Note that if V has a continuous distribution, then

$$Avar_\lambda(V) = E\{-V \mid -V \geq var_\lambda(V)\},$$

and in general, for $0 < \lambda < 1$,

$$Avar_\lambda(V) = \max_{\mathbb{Q} \in \mathcal{Q}_\lambda} E^{\mathbb{Q}}(-V), \quad V \in L^\infty$$

where \mathcal{Q}_λ is the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ bounded by $\frac{1}{\lambda}$. As previously stated, one can show that AVaR is a coherent risk measure, and hence the calculations are relatively simple (in the one dimensional case), and similar to those in equation (17).

5 Conclusion

This paper shows how to apply the insights from Çetin, Jarrow and Protter [3], in a simple and robust manner, to modify current risk measures (like VAR or coherent) to account for liquidity risk. The adjustments are formulated with implementation in mind, so the procedures are straightforward in their construction and estimation.

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