

# An Introduction to Financial Asset Pricing

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## Abstract

We present an introduction to mathematical Finance Theory, covering the basic issues as well as some selected special topics.

## 1 Introduction

Stock markets date back to at least 1531, when one was started in Antwerp, Belgium.<sup>1</sup> Today there are over 150 stock exchanges (see [54]). The mathematical modeling of such markets however, came hundreds of years after Antwerp, and it was embroiled in controversy at its beginnings. The first attempt known to the authors to model the stock market using probability is due to L. Bachelier in Paris about 1900. Bachelier's model was his thesis, and it met with disfavor in the Paris mathematics community, mostly because the topic was not thought worthy of study. Nevertheless we now realize that Bachelier essentially modeled Brownian motion five years before the 1905 paper of Einstein (albeit twenty years after T. N. Thiele of Copenhagen [26]) and of course decades before Kolmogorov gave mathematical legitimacy to the subject of probability theory. Poincaré was hostile to Bachelier's thesis, remarking that his thesis topic was “somewhat remote from those our candidates are in the habit of treating” and Bachelier ended up spending his

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<sup>1</sup>For a more serious history than this thumbnail sketch, we refer the reader to the recent article [34].

career in Besançon, far from the French capital. His work was then ignored and forgotten for some time.

Following work by A. Cowles (1930's), M Kendall and M. F. M. Osborne (1950's), it was the renowned statistician L. J. Savage who re-discovered Bachelier's work in the 1950's, and he alerted Paul Samuelson (see [5, pp. 22-23]). Samuelson further developed Bachelier's model to include stock prices that evolved according to a geometric Brownian motion, and thus (for example) always remained positive. This built on the earlier observations of Cowles and others that it was the increments of the logarithms of the prices that behaved independently.

The development of financial asset pricing theory over the 35 years since Samuelson's 1965 article [50] has been intertwined with the development of the theory of stochastic integration. A key breakthrough occurred in the early 1970's when Black, Scholes, and Merton ([6],[45]) proposed a method to price European options via an explicit formula. In doing this they made use of the Itô stochastic calculus and the Markov property of diffusions in key ways. The work of Black, Merton, and Scholes brought order to a rather chaotic situation, where the previous pricing of options had been done by intuition about ill defined market forces. Shortly after the work of Black, Merton, and Scholes, the theory of stochastic integration for semimartingales (and not just Itô processes) was developed in the 1970's and 1980's, mostly in France, due in large part to P. A. Meyer of Strasbourg and his collaborators. These advances in the theory of stochastic integration were combined with the work of Black, Scholes and Merton to further advance the theory, by Harrison and Kreps [27] and Harrison and Pliska [28] in seminal articles published in 1979 and 1980. In particular they established a connection between complete markets and martingale representation. Much has happened in the intervening two decades, and the subject has attracted the interest and curiosity of a large number of researchers and of course practitioners. The interweaving of finance and stochastic integration continues today. This article has the hope of introducing researchers to the subject at more or less its current state, for the special topics addressed here. We take an abstract approach, attempting to introduce simplifying hypotheses as needed, and we signal when we do so. In this way it is hoped that the reader can see the underlying structure of the theory.

The subject is much larger than the topics of this article, and there are several books that treat the subject in some detail (e.g., [19],[38],[46],[52]), including the new lovely book by Shreve [53]. Indeed, the reader is some-

times referred to books such as [19] to find more details for certain topics. Otherwise references are provided for the relevant papers.

## 2 Introduction to Derivatives and Arbitrage

Let  $S = (S_t)_{0 \leq t \leq T}$  represent the (nonnegative) price process of a risky asset (e.g., the price of a stock, a commodity such as “pork bellies,” a currency exchange rate, etc.). The present is often thought of as time  $t = 0$ . One is interested in the unknown price at some future time  $T$ , and thus  $S_T$  constitutes a “risk.” For example, if an American company contracts at time  $t = 0$  to deliver machine parts to Germany at time  $T$ , then the unknown price of Euros at time  $T$  (in dollars) constitutes a risk for that company. In order to reduce this risk, one may use “derivatives”: one can purchase — at time  $t = 0$  — the right to buy Euros at time  $T$  at a price that is fixed at time 0, and which is called the “strike price.” If the price of Euros is higher at time  $T$ , then one exercises this right to buy the Euros, and the risk is removed. This is one example of a derivative, called a *call option*.

A *derivative* is any financial security whose value is derived from the price of another asset, financial security, or commodity. For example, the call option just described is a derivative because its value is derived from the value of the underlying Euro. In fact, almost all traded financial securities can be viewed as derivatives.<sup>2</sup> Returning to the *call option* with strike price  $K$ , its payoff at time  $T$  can be represented mathematically as

$$C = (S_T - K)^+$$

where  $x^+ = \max(x, 0)$ . Analogously, the payoff to a *put option* with strike price  $K$  at time  $T$  is

$$P = (K - S_T)^+$$

and this corresponds to the right to *sell* the security at price  $K$  at time  $T$ . These are two simple examples of derivatives, called a *European call option* and *European put option*, respectively. They are clearly related, and we have

$$S_T - K = (S_T - K)^+ - (K - S_T)^+.$$

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<sup>2</sup>A fun exercise is to try to think of a financial security whose value does not depend on the price of some other asset or commodity. An example is a precious metal itself, like gold, trading as a commodity. But, gold stocks are a derivative as well as gold futures!

This simple equality leads to a relationship between the price of a call option and the price of a put option known as *put – call parity*. We return to this in Section 3.7.

We can also use these two simple options as building blocks for more complicated derivatives. For example, if

$$V = \max(K, S_T)$$

then

$$V = S_T + (K - S_T)^+ = K + (S_T - K)^+.$$

More generally, if  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is convex then we can use the well known representation

$$f(x) = f(0) + f'_+(0)x + \int_0^\infty (x - y)^+ \mu(dy) \quad (1)$$

where  $f'_+(x)$  is the right continuous version of the (mathematical) derivative of  $f$ , and  $\mu$  is a positive measure on  $\mathbb{R}$  with  $\mu = f''$ , where the mathematical derivative is in the generalized function sense. In this case if

$$V = f(S_T)$$

is our financial derivative, then  $V$  is effectively a portfolio consisting of a continuum of European call options, using (1) (see [7]):

$$V = f(0) + f'_+(0)S_T + \int_0^\infty (S_T - K)^+ \mu(dK).$$

For the derivatives discussed so far, the derivative's time  $T$  value is a random variable of the form  $V = f(S_T)$ , that is, a function of the value of  $S$  at one fixed and prescribed time  $T$ . One can also consider derivatives of the form

$$\begin{aligned} V &= F(S)_T \\ &= F(S_t; 0 \leq t \leq T) \end{aligned}$$

which are functionals of the paths of  $S$ . For example if  $S$  has càdlàg paths (càdlàg is a French acronym for “right continuous with left limits”) then  $F: D \rightarrow \mathbb{R}_+$ , where  $D$  is the space of functions  $f: [0, T] \rightarrow \mathbb{R}_+$  which are right continuous with left limits.

If the derivative's value depends on a decision of its holder at only the expiration time  $T$ , then they are considered to be of the *European type*, although their analysis for pricing and hedging is more difficult than for simple European call and put options. The decision in the case of a call or put option is whether to exercise the right to buy or sell, respectively.<sup>3</sup> Hence, such decisions are often referred to as *exercise* decisions.

An *American type derivative* is one in which the holder has a decision to make with respect to the security at any time before or at the expiration time. For example, an *American call option* allows the holder to buy the security at a striking price  $K$  not only at time  $T$  (as is the case for a European call option), but at any time between times  $t = 0$  and time  $T$ . (It is this type of option that is listed, for example, in the "Listed Options Quotations" in the Wall Street Journal.) Deciding when to exercise such an option is complicated. A strategy for exercising an American call option can be represented mathematically by a *stopping rule*  $\tau$ . (That is, if  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is the underlying filtration of  $S$  then  $\{\tau \leq t\} \in \mathcal{F}_t$  for each  $t$ ,  $0 \leq t \leq T$ .) For a given  $\tau$ , the American call's payoff at time  $\tau(\omega)$  is

$$C(\omega) = (S_{\tau(\omega)}(\omega) - K)^+.$$

We now turn to the *pricing of derivatives*. Let  $C$  be a random variable in  $\mathcal{F}_T$  representing the time  $T$  payoff to a derivative. Let  $V_t$  be its *value* (or price) at time  $t$ . What then is  $V_0$ ? From a traditional point of view based on an analysis of fair (gambling) games, classical probability tells us that<sup>4</sup>

$$V_0 = E\{C\}. \tag{2}$$

One should pay the expected payoff of participating in the gamble. But, one should also discount for the time value of money (the interest forgone or earned) and assuming a fixed spot interest rate  $r$ , one would have

$$V_0 = E\left\{\frac{C}{(1+r)^T}\right\} \tag{3}$$

instead of (2). Surprisingly, this value is not correct, because it ignores the impact of risk aversion on the part of the purchaser. For simplicity, we will

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<sup>3</sup>This decision is explicitly represented by the maximum operator in the payoff of the call and put options.

<sup>4</sup>This assumes, implicitly, that there are no intermediate cash flows from holding the derivative security.

take  $r = 0$  and then show why the obvious price given in (2) does not work (!).

Let's consider a simple binary example. At time  $t = 0$ , 1 Euro = \$1.15. We assume at time  $t = T$  that the Euro will be worth either \$0.75 or \$1.45. Let the probability that it goes up to \$1.45 be  $p$  and the probability that it goes down be  $1 - p$ .

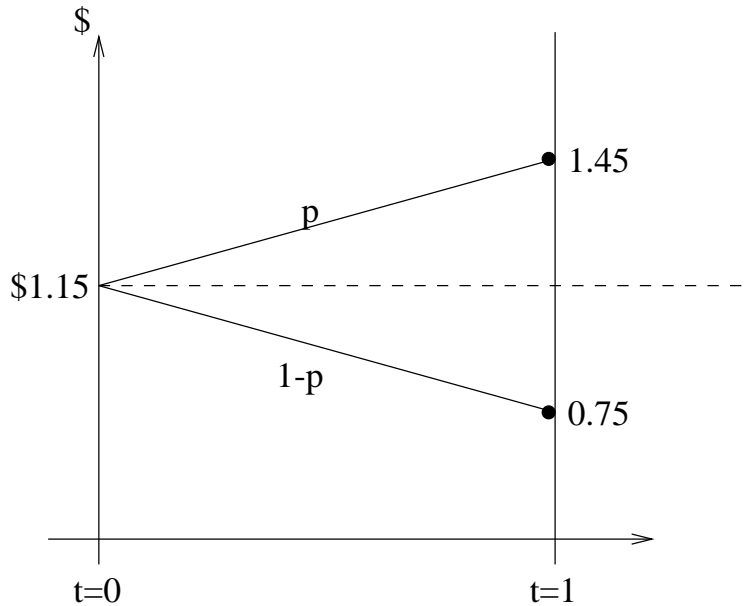


Figure 1: Binary Schematic

Consider a European call option with exercise price  $K = \$1.15$ . That is,  $C = (S_T - \$1.15)^+$ , where  $S = (S_t)_{0 \leq t \leq T}$  is the price of one Euro in U.S. dollars. The classical rules for calculating probabilities dating back to Huygens and Bernoulli give a fair price of  $C$  as

$$E\{C\} = (1.45 - 1.15)p = (0.30)p.$$

For example if  $p = 1/2$  we get  $V_0 = 0.15$ .

The *Black-Scholes* method<sup>5</sup> for calculating the option's price, however, is quite different. We first replace  $p$  with a new probability  $p^*$  that (in the

<sup>5</sup>The "Black-Scholes method" dates back to the fundamental and seminal articles [6] and [45] of 1973, where partial differential equations were used; the ideas implicit in that

absence of interest rates) makes the security price  $S = (S_t)_{t=0,T}$  a martingale. Since this is a two-step process, **we need only to choose  $p^*$  so that  $S$  has a constant expectation under  $P^*$ , the probability measure implied by the choice of  $p^*$ .** Since  $S_0 = 1.15$ , we need

$$E^*\{S_T\} = 1.45p^* + (1 - p^*)0.75 = 1.15 \quad (4)$$

where  $E^*$  denotes mathematical expectation with respect to the probability measure  $P^*$  given by  $P^*(\text{Euro} = \$1.45 \text{ at time } T) = p^*$ , and  $P^*(\text{Euro} = \$0.75 \text{ at time } T) = 1 - p^*$ . Solving for  $p^*$  gives

$$p^* = 4/7.$$

We get now

$$V_0 = E^*\{C\} = (0.30)p^* = \frac{6}{35} \simeq 0.17.$$

The change from  $p$  to  $p^*$  seems arbitrary. But, there is an *economics* argument to justify it. This is where the economics concept of *no arbitrage opportunities* changes the usual intuition dating back to the 16th and 17th centuries.

Suppose, for example, at time  $t = 0$  you sell the call option, giving the buyer of the option the right to purchase 1 Euro at time  $T$  for \$1.15. He then gives you the price  $v(C)$  of the option. Again we assume  $r = 0$ , so there is no cost to borrow money. You can then follow a safety strategy to prepare for the option you sold, as follows (calculations are to two decimal places):

Action at time $t = 0$	Result
Sell the option at price $v(C)$	$+v(C)$
Borrow \$ $\frac{9}{28}$	$+0.32$
Buy $\frac{3}{7}$ euros at \$ 1.15	$-0.49$

**The balance at time  $t = 0$  is  $v(C) - 0.17$ .**

**At time T there are two possibilities:**

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(and subsequent) articles are now referred to as the Black-Scholes methods. More correctly, it should be called the Black-Merton-Scholes method. M. S. Scholes and R. Merton received the Nobel prize in economics for [6],[45], and related work (F. Black died and was not able to share in the prize.)

What happens to the euro	Result
<b>The euro has risen:</b>	
Option is exercised	-0.30
Sell $\frac{3}{7}$ euros at 1.45	+0.62
Pay back loan	-0.32
End balance:	0
<b>The euro has fallen:</b>	
Option is worthless	0
Sell $\frac{3}{7}$ euros at 0.75	+0.32
Pay back loan	-0.32
End balance:	0

Since the balance at time  $T$  is zero in both cases, the balance at time 0 should also be 0; therefore we must have  $v(C) = 0.17$ . Indeed any price other than  $v(C) = 0.17$  would allow either the option seller or buyer to make a sure profit *without any risk*. Such a sure profit with no risk is called an *arbitrage opportunity* in economics, and it is a standard assumption that such opportunities do not exist. (Of course if they were to exist, market forces would, in theory, quickly eliminate them.)

Thus we see that — at least in the case of this simple example — that the “no arbitrage price” of the derivative  $C$  is not  $E\{C\}$ , but rather it must be  $E^*\{C\}$ . We emphasize that this is contrary to our standard intuition based on fair games, since  $P$  is the probability measure governing the true laws of chance of the security, while  $P^*$  is an artificial construct.

**Remark 1 (Heuristic Explanation)** *We offer two comments here. The first is that the change of probability measures from  $P$  to  $P^*$  is done with the goal of keeping the expectation constant. (See equation (4).) It is this property of constant expectation of the price process which excludes the possibility of arbitrage opportunities, when the price of the derivative is chosen to be the expectation under  $P^*$ . Since one can have many different types of processes with constant expectation, one can ask: what is the connection to martingales? The answer is that a necessary and sufficient condition for a process  $M = (M_t)_{t \geq 0}$  to be a uniformly martingale is that  $E(M_\tau) = E(M_0)$  for every stopping time  $\tau$ . The key here is that it is required for every stopping time, and not just for fixed times. In words, the price process must have constant expectation at all random times (stopping times) under a measure  $P^*$  in order for the expectation of the contingent claim under  $P^*$  to be an arbitrage free price of the claim.*

The second comment refers to Figure 1 (Binary Schematic). Intuition tells us that as  $p \nearrow 1$ , that the price of a call or put option must change, since as it becomes almost certain that the price will go up, the call might be worth less (or more) to the purchaser. And yet our no arbitrage argument tells us that it cannot, and that  $p^*$  is fixed for all  $p$ ,  $0 < p < 1$ . How can this be? An economics explanation is that if one lets  $p$  increase to 1, one is implicitly perverting the economy. In essence, this perversion of the economy a fortiori reflects changes in participants' levels of risk aversion. If the price can change to only two prices, and it is near certain to go up, how can we keep the current price fixed at \$ 1.15? Certainly this change in perceived probabilities should affect the current price too. In order to increase  $p$  towards 1 and simultaneously keep the current price fixed at \$ 1.15, we are forced to assume that people's behavior has changed, and either they are very averse to even a small potential loss (the price going down to \$ 0.75), or they now value much less the near certain potential price increase to \$ 1.45.

This simple binary example can do more than illustrate the idea of using the lack of arbitrage to determine a price. We can also use it to approximate some continuous time models for the evolution of an asset's price. We let the time interval become small ( $\Delta t$ ), and we let the binomial model already described become a recombinant tree, which moves up or down to a neighboring node at each time "tick"  $\Delta t$ . For an actual time "tick" of interest of length say  $\delta$ , we can have the price go to  $2^n$  possible values for a given  $n$ , by choosing  $\Delta t$  small enough in relation to  $n$  and  $\delta$ . Thus for example if the continuous time process follows Geometric Brownian motion:

$$dS_t = \sigma S_t dB_t + \mu S_t dt$$

(as is often assumed in practice); and if the security price process  $S$  has value  $S_t = s$ , then it will move up or down at the next tick  $\Delta t$  to

$$s \exp(\mu\Delta t + \sigma\sqrt{\Delta t}) \quad \text{if up;} \quad s \exp(\mu\Delta t - \sigma\sqrt{\Delta t}) \quad \text{if down;}$$

with  $p$  being the probability of going up or down (here take  $p = \frac{1}{2}$ ). Thus for a time  $t$ , if  $n = \frac{t}{\Delta t}$ , we get

$$S_t^n = S_0 \exp \left( \mu t + \sigma \sqrt{t} \left( \frac{2X_n - n}{\sqrt{n}} \right) \right),$$

where  $X_n$  counts the number of jumps up. By the Central Limit Theorem  $S_t^n$  converges, as  $n$  tends to infinity, to a log normal process  $S = (S_t)_{t \geq 0}$ ; that is,  $\log S_t$  has a normal distribution with mean  $\log(S_0 + \mu t)$  and variance  $\sigma^2 t$ .

Next we use the absence of arbitrage to change  $p$  from  $\frac{1}{2}$  to  $p^*$ . We find  $p^*$  by requiring that  $E^*\{S_t\} = E^*\{S_0\}$ , and we get  $p^*$  approximately equal to

$$p^* = \frac{1}{2} \left( 1 - \sqrt{\Delta t} \left( \frac{\mu + \frac{1}{2}\sigma^2}{\sigma} \right) \right).$$

Thus under  $P^*$ ,  $X_n$  is still Binomial, but now it has mean  $np^*$  and variance  $np^*(1 - p^*)$ . Therefore  $\left( \frac{2X_n - n}{\sqrt{n}} \right)$  has mean  $-\sqrt{t}(\mu + \frac{1}{2}\sigma^2)/\sigma$  and a variance which converges to 1 asymptotically. The Central Limit Theorem now implies that  $S_t$  converges as  $n$  tends to infinity to a log normal distribution:  $\log S_t$  has mean  $\log S_0 - \frac{1}{2}\sigma^2 t$  and variance  $\sigma^2 t$ . Thus

$$S_t = S_0 \exp\left(\sigma\sqrt{t}Z - \frac{1}{2}\sigma^2 t\right)$$

where  $Z$  is  $N(0, 1)$  under  $P^*$ . This is known as the “binomial approximation.” The binomial approximation can be further used to derive the Black-Scholes equations, by taking limits, leading to simple formulas in the continuous case. (We present these formulas in Section 3.10). A simple derivation can be found in Cox, Ross and Rubinstein in 1979 [12] or in Chapter 12B of [19], pp. 294–299.

## 3 The Core of the Theory

### 3.1 Basic Definitions

Throughout this section we will assume that we are given an underlying probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . We further assume  $\mathcal{F}_s \subset \mathcal{F}_t$  if  $s < t$ ;  $\mathcal{F}_0$  contains all the  $P$ -null sets of  $\mathcal{F}$ ; and also that  $\bigcap_{s > t} \mathcal{F}_s \equiv \mathcal{F}_{t+} = \mathcal{F}_t$  by hypothesis. This last property is called the *right continuity of the filtration*. These hypotheses, taken together, are known as the *usual hypotheses*. (When the usual hypotheses hold, one knows that every martingale has a version which is càdlàg, one of the most important consequences of these hypotheses.)

## 3.2 The Price Process

We let  $S = (S_t)_{t \geq 0}$  be a semimartingale<sup>6</sup> which will be the *price process* of a risky security. For simplicity, after the initial purchase or sale, we assume that the security has no cash flows associated with it (for example, if the security is a common stock, we assume that there are no dividends paid). This assumption is easily relaxed, but its relaxation unnecessarily complicates the notation and explanation, so we leave it to outside references.

## 3.3 Spot Interest Rates

Let  $r$  be a fixed spot rate of interest. If one invests 1 dollar at rate  $r$  for one year, at the end of the year one has  $1 + r$  dollars. If interest is paid at  $n$  evenly spaced times during the year and compounded, then at the end of the year one has  $\left(1 + \frac{r}{n}\right)^n$ . This leads to the notion of an *interest rate compounded continuously*:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$$

or, for a fraction  $t$  of the year, one has \$  $e^{rt}$  after  $t$  units of time for a spot interest rate  $r$  compounded continuously.

We define

$$R(t) = e^{rt};$$

then  $R$  satisfies the ODE (ODE abbreviates Ordinary Differential Equation)

$$dR(t) = rR(t)dt; \quad R(0) = 1. \tag{5}$$

Using the ODE (5) as a basis for interest rates, one can treat a variable interest rate  $r(t)$  as follows: ( $r(t)$  can be random: that is  $r(t) = r(t, \omega)$ ):<sup>7</sup>

$$dR(t) = r(t)R(t)dt; \quad R(0) = 1 \tag{6}$$

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<sup>6</sup>One *definition of a semimartingale* is a process  $S$  that has a decomposition  $S = M + A$ , with  $M$  a local martingale and  $A$  an adapted process with càdlàg paths of finite variation on compacts. See [48] for all information regarding semimartingales.

<sup>7</sup>An example is to take  $r(t)$  to be a diffusion; one can then make appropriate hypotheses on the diffusion to model the behavior of the spot interest rate.

and solving yields  $R(t) = \exp\left(\int_0^t r(s)ds\right)$ . We think of the interest rate process  $R(t)$  as the *time  $t$  value of a money market account*.

### 3.4 Trading Strategies and Portfolios

We will assume as given a risky asset with price process  $S$  and a money market account with price process  $R$ . Let  $(a_t)_{t \geq 0}$  and  $(b_t)_{t \geq 0}$  be our *time  $t$  holdings* in the security and the bond, respectively.

We call our holdings of  $S$  and  $R$  our *portfolio*. Note that for the model to make sense, we must have both the risky asset and the money market account present. When we receive money through the sale of risky assets, we place the cash in the money market account; and when we purchase risky assets, we use the cash from the money market account to pay for the expenditure. The money market account is allowed to have a negative balance.

**Definition 1** *The value at time  $t$ <sup>8</sup> of a portfolio  $(a, b)$  is:*

$$V_t(a, b) = a_t S_t + b_t R_t.$$

Now we have our first problem. Later we will want to change probabilities so that  $V = (V_t(a, b))_{t \geq 0}$  is a martingale. One usually takes the right continuous versions of a martingale, so we want the right side of (4) to be at least càdlàg. Typically this is not a real problem. Even if the process  $a$  has no regularity, one can always choose  $b$  in such a way that  $V_t(a, b)$  is càdlàg.

Let us next define two sigma algebras on the product space  $\mathbb{R}_+ \times \Omega$ . We recall that we are given an underlying probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  with  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ , satisfying the “usual hypotheses.”

**Definition 2** *Let  $\mathbb{L}$  denote the space of left continuous processes whose paths have right limits (càglàd), and which are adapted: that is,  $H_t \in \mathcal{F}_t$ , for  $t \geq 0$ . The predictable  $\sigma$ -algebra  $\mathcal{P}$  on  $\mathbb{R}_+ \times \Omega$  is*

$$\mathcal{P} = \sigma\{H : H \in \mathbb{L}\}.$$

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<sup>8</sup>This concept of value is a commonly used approximation. If one were to liquidate one’s risky assets at time  $t$  all at once to realize this “value”, one would find less money in the savings account, due to liquidity and transaction costs. For simplicity, we are assuming there are no liquidity and transaction costs. Such an assumption is not necessary, however, and we recommend the interested reader to [35] in this volume.

That is  $\mathcal{P}$  is the smallest  $\sigma$ -algebra that makes all of  $\mathbb{L}$  measurable.

**Definition 3** The optional  $\sigma$ -algebra  $\mathcal{O}$  on  $\mathbb{R}_+ \times \Omega$  is

$$\mathcal{O} = \sigma\{H : H \text{ is càdlàg and adapted}\}.$$

In general we have  $\mathcal{P} \subset \mathcal{O}$ . In the case where  $B = (B_t)_{t \geq 0}$  is a standard Wiener process (or “Brownian motion”), and  $\mathcal{F}_t^0 = \sigma(B_s; s \leq t)$  and  $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$  where  $\mathcal{N}$  are the  $P$ -null sets of  $\mathcal{F}$ , then we have  $\mathcal{O} = \mathcal{P}$ . In general  $\mathcal{O}$  and  $\mathcal{P}$  are not equal. Indeed if they are equal, then every stopping time is predictable: that is, there are no totally inaccessible stopping times.<sup>9</sup> Since the jump times of (reasonable) Markov processes are totally inaccessible, any model which contains a Markov process with jumps (such as a Poisson Process) will have  $\mathcal{P} \subset \mathcal{O}$ , where the inclusion is strict.

**Remark on Filtration Issues:** The predictable  $\sigma$ -algebra  $\mathcal{P}$  is important because it is the natural  $\sigma$ -field for which stochastic integrals are defined. In the special case of Brownian motion one can use the optional  $\sigma$ -algebra (since they are the same). There is a third  $\sigma$ -algebra which is often used, known as the progressively measurable sets, and denoted  $\pi$ . One has, in general, that  $\mathcal{P} \subset \mathcal{O} \subset \pi$ ; however in practice one gains very little by assuming a process is  $\pi$ -measurable instead of optional, if – as is the case here – one assumes that the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is right-continuous (that is  $\mathcal{F}_{t+} = \mathcal{F}_t$ , all  $t \geq 0$ ). The reason is that the primary use of  $\pi$  is to show that adapted, right-continuous processes are  $\pi$ -measurable and in particular that  $S_T \in \mathcal{F}_T$  for  $T$  a stopping time and  $S$  progressive; but such processes are already optional if  $(\mathcal{F}_t)_{t \geq 0}$  is right continuous. Thus there are essentially no “naturally occurring” examples of progressively measurable processes that are not

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<sup>9</sup>A *totally inaccessible stopping time* is a stopping time that comes with no advance warning: it is a complete surprise. A stopping time  $T$  is *totally inaccessible* if whenever there exists a sequence of non-decreasing stopping times  $(S_n)_{n \geq 1}$  with  $\Lambda = \bigcap_{n=1}^{\infty} \{S_n < T\}$ , then

$$P(\{w : \lim_{n \rightarrow \infty} S_n = T\} \cap \Lambda) = 0.$$

A stopping time  $T$  is *predictable* if there exists a non-decreasing sequence of stopping times  $(S_n)_{n \geq 1}$  as above with

$$P(\{w : \lim_{n \rightarrow \infty} S_n = T\} \cap \Lambda) = 1.$$

Note that the probabilities above need not be only 0 or 1; thus there are in general stopping times which are neither predictable nor totally inaccessible.

already optional. An example of such a process, however, is the indicator function  $1_G(t)$ , where  $G$  is described as follows: let  $\mathbb{Z} = \{(t, \omega) : B_t(\omega) = 0\}$ . ( $B$  is standard Brownian motion.) Then  $\mathbb{Z}$  is a perfect (and closed) set on  $\mathbb{R}_+$  for almost all  $\omega$ . For fixed  $\omega$ , the complement is an open set and hence a countable union of open intervals.  $G(\omega)$  denotes the left end-points of these open intervals. One can then show (using the Markov property of  $B$  and P. A. Meyer's section theorems) that  $G$  is progressively measurable but not optional. In this case note that  $1_G(t)$  is zero except for countably many  $t$  for each  $\omega$ , hence  $\int 1_G(s)dB_s \equiv 0$ . Finally we note that if  $a = (a_s)_{s \geq 0}$  is progressively measurable, then  $\int_0^t a_s dB_s = \int_0^t \dot{a}_s dB_s$ , where  $\dot{a}$  is the predictable projection of  $a$ .<sup>10</sup>

Let us now recall a few details of stochastic integration. First, let  $S$  and  $X$  be any two càdlàg semimartingales. The integration by parts formula can be used to define the quadratic co-variation of  $X$  and  $S$ :

$$[X, S]_t = X_t Y_t - \int_0^t X_{s-} dS_s - \int_0^t S_{s-} dX_s.$$

However if a càdlàg, adapted process  $H$  is not a semimartingale, one can still give the quadratic co-variation a meaning, by using a limit in probability as the definition. This limit always exists if both  $H$  and  $S$  are semimartingales:

$$[H, S]_t = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi^n[0, t]} (H_{t_{i+1}} - H_{t_i})(S_{t_{i+1}} - S_{t_i})$$

where  $\pi^n[0, t]$  be a sequence of finite partitions of  $[0, t]$  with  $\lim_{n \rightarrow \infty} \text{mesh}(\pi^n) = 0$ .

Henceforth let  $S$  be a (càdlàg) semimartingale, and let  $H$  be càdlàg and adapted, or alternatively  $H \in \mathbb{L}$ . Let  $H_- = (H_{s-})_{s \geq 0}$  denote the left-continuous version of  $H$ . (If  $H \in \mathbb{L}$ , then of course  $H = H_-$ .) We have:

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<sup>10</sup>Let  $H$  be a bounded, measurable process. ( $H$  need not be adapted.) The *predictable projection* of  $H$  is the unique predictable process  $\dot{H}$  such that

$$\dot{H}_T = E\{H | \mathcal{F}_{T-}\} \quad \text{a.s. on } \{T < \infty\}$$

for all predictable stopping times  $T$ . Here  $\mathcal{F}_{T-} = \sigma\{A \cap \{t < T\}; A \in \mathcal{F}_t\} \vee \mathcal{F}_0$ . For a proof of the existence and uniqueness of  $\dot{H}$  see [48, p. 119].

**Theorem 1**  $H$  càdlàg, adapted or  $H \in \mathbb{L}$ . Then

$$\lim_{n \rightarrow \infty} \sum_{t_i \in \pi^n[0,t]} H_{t_i}(S_{t_{i+1}} - S_{t_i}) = \int_0^t H_{s-} dS_s,$$

with convergence uniform in  $s$  on  $[0, t]$  in probability.

We remark that it is crucial that we sample  $H$  at the left endpoint of the interval  $[t_i, t_{i+1}]$ . Were we to sample at, say, the right endpoint or the midpoint, then the sums would not converge in general (they converge for example if the quadratic covariation process  $[H, S]$  exists); in cases where they do converge, the limit is in general different. Thus while the above theorem gives a pleasing “limit as Riemann sums” interpretation to a stochastic integral, it is not at all a perfect analogy.

The basic idea of the preceding theorem can be extended to bounded predictable processes in a method analogous to the definition of the Lebesgue integral for real-valued functions. Note that

$$\sum_{t_i \in \pi^n[0,t]} H_{t_i}(S_{t_{i+1}} - S_{t_i}) = \int_{0+}^t H_s^n dS_s,$$

where  $H_t^n = \sum H_{t_i} 1_{(t_i, t_{i+1}]}$  which is in  $\mathbb{L}$ ; thus these “simple” processes are the building blocks, and since  $\sigma(\mathbb{L}) = \mathcal{P}$ , it is unreasonable to expect to go beyond  $\mathcal{P}$  when defining the stochastic integral.

There is, of course, a maximal space of integrable processes where the stochastic integral is well defined and still gives rise to a semimartingale as the integrated process; without describing it (see any book on stochastic integration such as [48]), we define:

**Definition 4** For a semimartingale  $S$  we let  $L(S)$  denote the space of predictable processes  $a$ , where  $a$  is integrable with respect to  $S$ .

We would like to fix the underlying semimartingale (or vector of semimartingales)  $S$ . The process  $S$  represents the price process of our risky asset. A way to do that is to introduce the notion of a *model*. We present two versions. The first is the more complete, as it specifies the probability space and the underlying filtration. However it is also cumbersome, and thus we will abbreviate it with the second:

**Definition 5** A sextuple  $(\Omega, \mathcal{F}, \mathbb{F}, S, L(S), P)$ , where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ , is called an asset pricing model; or more simply, the triple  $(S, L(S), P)$  is called a model, where the probability space and  $\sigma$ -algebras are implicit: that is,  $(\Omega, \mathcal{F}, \mathbb{F})$  is implicit.

We are now ready for a key definition.

A trading strategy in the risky asset is a predictable process  $a = (a_t)_{t \geq 0}$  with  $a \in L(S)$ ; its economic interpretation is that at time  $t$  one holds an amount  $a_t$  of the asset. We also remark that it is reasonable that  $a$  be predictable:  $a$  is the trader's holdings at time  $t$ , and this is based on information obtained at times strictly before  $t$ , but not  $t$  itself. Often one has in concrete situations that  $a$  is continuous or at least càdlàg or càglàd (left continuous with right limits). (Indeed, it is difficult to imagine a practical trading strategy with pathological path irregularities.) In the case  $a$  is adapted and càglàd, then

$$\int_0^t a_s dS_s = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi^n[0, t]} a_{t_i} \Delta_i S$$

where  $\pi^n[0, t]$  is a sequence of partitions of  $[0, t]$  with mesh tending to 0 as  $n \rightarrow \infty$ ;  $\Delta_i S = S_{t_{i+1}} - S_{t_i}$ ; and with convergence in u.c.p. (uniform in time on compacts and converging in probability). Thus inspired by (1) we let

$$G_t = \int_{0+}^t a_s dS_s$$

and  $G$  is called the (*financial*) gain process generated by  $a$ . A trading strategy in the money market account,  $b = (b_t)_{t \geq 0}$ , is defined in an analogous fashion except that we only require that  $b$  is optional and  $b \in L(R)$ . We will call the pair  $(a, b)$ , as defined above, a trading strategy.

**Definition 6** A trading strategy  $(a, b)$  is called self-financing if

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s \quad (7)$$

for all  $t \geq 0$ .

Note that the equality (7) above implies that  $a_t S_t + b_t R_t$  is càdlàg.

To justify this definition heuristically, let us assume the spot interest rate is constant and zero: that is,  $r = 0$  which implies that  $R_t = 1$  for all  $t \geq 0$ ,

a.s. We can do this by the principle of numéraire invariance; see Section (3.6), later in this article. We then have

$$a_t S_t + b_t R_t = a_t S_t + b_t.$$

Assume for the moment that  $a$  and  $b$  are semimartingales, and as such let us denote them  $X$  and  $Y$  respectively.<sup>11</sup> If at time  $t$  we change our position in the risky asset, to be self-financing we must change also the amount in our money market account; thus we need to have the equality:

$$(X_{t+dt} - X_t)S_{t+dt} = -(Y_{t+dt} - Y_t),$$

which is algebraically equivalent to

$$(S_{t+dt} - S_t)(X_{t+dt} - X_t) + (X_{t+dt})S_t = -(Y_{t+dt} - Y_t),$$

which implies in continuous time:

$$S_{t-}dX_t + d[S, X]_t = -dY_t.$$

Using integration by parts, we get:

$$X_t S_t - X_{t-} dS_t = -dY_t,$$

and integrating yields the desired equality

$$X_t S_t + Y_t = \int_0^t X_{s-} dS_s + X_0 S_0 + Y_0. \quad (8)$$

Finally we drop the assumption that  $X$  and  $Y$  are semimartingales, and replacing  $X_-$  with  $a$  and  $Y$  with  $b$  respectively, the equation (8) becomes

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 + \int_0^t a_s dS_s + (b_t - b_0),$$

as we have in equation (7).

The next concept is of fundamental importance. An *arbitrage opportunity* is the chance to make a profit *without risk*. The standard way of modeling this mathematically is as follows:

**Definition 7** *A model is arbitrage free if there does not exist a self-financing trading strategy  $(a, b)$  such that  $V_0(a, b) = 0$ ,  $V_T(a, b) \geq 0$ , and  $P(V_T(a, b) > 0) > 0$ .*

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<sup>11</sup>Since  $X$  is assumed to be a semimartingale, it is right continuous, and thus is not in general predictable; hence when it is the integrand of a stochastic integral we need to replace  $X_s$  with  $X_{s-}$ , which of course denotes the left continuous version of  $X$ .

### 3.5 The Fundamental Theorem of Asset Pricing

In Section 2 we saw that with the “No Arbitrage” assumption, at least in the case of a very simple example, a reasonable price of a derivative was obtained by taking expectations and changing from the “true” underlying probability measure,  $P$ , to an equivalent one,  $P^*$ . More formally, under the assumption that  $r = 0$ , or equivalently that  $R_t = 1$  for all  $t$ , the price of a derivative  $C$  was not  $E\{C\}$  as one might expect, but rather  $E^*\{C\}$ . (If the process  $R_t$  is not constant and equal to one, then we consider the expectation of the discounted claim  $E^*\{C/R_T\}$ .)

The idea underlying the equivalent change of measure was to find a probability  $P^*$  that gave the price process  $S$  a constant expectation. This simple insight readily generalizes to more complex stochastic processes. In continuous time, a sufficient condition for the price process  $S = (S_t)_{t \geq 0}$  to have constant expectation is that it be a martingale. That is, if  $S$  is a martingale then the function  $t \rightarrow E\{S_t\}$  is constant. Actually this property is not far from characterizing martingales. A classic theorem from martingale theory is the following (cf, eg, [48]):

**Theorem 2** *Let  $S = (S_t)_{t \geq 0}$  be càdlàg and suppose  $E\{S_\tau\} = E\{S_0\}$  for any bounded stopping time  $\tau$  (and of course  $E\{|S_\tau|\} < \infty$ ). Then  $S$  is a martingale.*

That is, if we require constant expectation at stopping times (instead of only at fixed times), then  $S$  is a martingale.

Based on this simple pricing example and the preceding theorem, one is lead naturally to the following conjecture.

**Conjecture:** Let  $S$  be a price process on a given space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ . Then there are no arbitrage opportunities if and only if there exists a probability  $P^*$ , equivalent to  $P$ , such that  $S$  is a martingale under  $P^*$ .

The origins of the preceding conjecture can be traced back to Harrison and Kreps [27] in 1979 for the case where  $\mathcal{F}_T$  is finite, and later to Dalang, Morton and Willinger [13] in 1990 for the case where  $\mathcal{F}_T$  is infinite, but time is discrete. Before stating a more rigorous theorem (our version is due to Delbaen and Schachermeyer [14]; see also [15]), let us examine a needed hypothesis.

We need to avoid problems that arise from the classical doubling strategy in gambling. Here a player bets \$1 at a fair bet. If he wins, he stops. If

he loses he next bets \$2. Whenever he wins, he stops, and his profit is \$1. If he continues to lose, he continues to play, each time doubling his bet. This strategy leads to a certain gain of \$1 without risk. However, the player needs to be able to tolerate arbitrarily large losses before he gains his certain profit. Of course, no one has such infinite resources to play such a game. Mathematically one can eliminate this type of problem by requiring trading strategies to give martingales that are bounded below by a constant. Thus the player's resources, while they can be huge, are nevertheless finite and bounded by a non-random constant. This leads to the next definition.

**Definition 8** *Let  $\alpha > 0$ , and let  $S$  be a semimartingale. A predictable trading strategy  $\theta$  is  $\alpha$ -admissible if  $\theta_0 = 0$ ,  $\int_0^t \theta_s dS_s \geq -\alpha$ , all  $t \geq 0$ .  $\theta$  is called admissible if there exists  $\alpha > 0$  such that  $\theta$  is  $\alpha$ -admissible.*

Before we make more definitions, let us recall the basic approach. Suppose  $\theta$  is an admissible, self-financing trading strategy with  $\theta_0 S_0 = 0$  and  $\theta_T S_T \geq 0$ . In the next section we will see that without loss of generality we can neglect the bond or “numéraire” process by a “change of numéraire,” so that the self-financing condition reduces to

$$\theta_T S_T = \theta_0 S_0 + \int_0^T \theta_s dS_s.$$

Then if  $P^*$  exists such that  $\int \theta_s dS_s$  is a martingale, we have

$$E^* \{ \theta_T S_T \} = 0 + E^* \left\{ \int_0^T \theta_s dS_s \right\}.$$

In general, if  $S$  is continuous then  $\int_0^t \theta_s dS_s$  is only a *local* martingale.<sup>12</sup> If  $S$  is merely assumed to be a càdlàg semimartingale, then  $\int_0^t \theta_s dS_s$  need only be a  $\sigma$  martingale<sup>13</sup>. However if for some reason we do know that it is a

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<sup>12</sup>A process  $M$  is a *local martingale* if there exists a sequence of stopping times  $(T_n)_{n \geq 1}$  increasing to  $\infty$  a.s. such that  $(M_{t \wedge T_n})_{t \geq 0}$  is a martingale for each  $n \geq 1$ .

<sup>13</sup>A process  $X$  is a  $\sigma$  martingale if there exists an  $\mathbb{R}^d$  valued martingale  $M$  and a predictable  $\mathbb{R}_+$  valued  $M$ -integrable process  $H$  such that  $X$  is the stochastic integral of  $H$  with respect to  $M$ . See [48, pp. 237–239] for more about  $\sigma$  martingales.

true martingale then  $E^*\{\int_0^T \theta_s dS_s\} = 0$ , whence  $E^*\{\theta_T S_T\} = 0$ , and since  $\theta_T S_T \geq 0$  we deduce  $\theta_T S_T = 0$ ,  $P^*$  a.s., and since  $P^*$  is equivalent to  $P$ , we have  $\theta_T S_T = 0$  a. s. ( $dP$ ) as well. This implies no arbitrage exists. The technical part of this argument is to show  $\int_0^t \theta_s dS_s$  is a  $P^*$  true martingale, and not just a local martingale (see the proof of the Fundamental Theorem that follows). The converse is typically harder: that is, that no arbitrage implies  $P^*$  exists. The converse is proved using a version of the Hahn-Banach theorem.

Following Delbaen and Schachermayer, we make a sequence of definitions:

$$K_0 = \left\{ \int_0^\infty \theta_s dS_s \mid \theta \text{ is admissible and } \lim_{t \rightarrow \infty} \int_0^t \theta_s dS_s \text{ exists a.s.} \right\} C_0 = \{\text{all functions dominated}$$

**Definition 9** *A semimartingale price process  $S$  satisfies*

(i) *the No Arbitrage condition if  $\mathbb{C} \cap L_+^\infty = \{0\}$  (this corresponds to no chance of making a profit without risk);*

(ii) *the No Free Lunch with Vanishing Risk condition (NFLVR) if  $\overline{\mathbb{C}} \cap L_+^\infty = \{0\}$ , where  $\overline{\mathbb{C}}$  is the closure of  $\mathbb{C}$  in  $L^\infty$ .*

**Definition 10** *A probability measure  $P^*$  is called an equivalent martingale measure, or alternatively a risk neutral probability, if  $P^*$  is equivalent to  $P$ , and if under  $P^*$  the price process  $S$  is a  $\sigma$  martingale.*

Clearly condition (ii) implies condition (i). Condition (i) is slightly too restrictive to imply the existence of an equivalent martingale measure  $P^*$ . (One can construct a trading strategy of  $H_t(\omega) = 1_{\{[0,1] \setminus \mathbb{Q} \times \Omega\}}(t, \omega)$ , which means one sells before each rational time and buys back immediately after it; combining  $H$  with a specially constructed càdlàg semimartingale shows that (i) does not imply the existence of  $P^*$  - see [14, p. 511].)

Let us examine then condition (ii). If NFLVR is not satisfied then there exists an  $f_0 \in L_+^\infty$ ,  $f_0 \not\equiv 0$ , and also a sequence  $f_n \in \mathbb{C}$  such that  $\lim_{n \rightarrow \infty} f_n = f_0$  a. s, such that for each  $n$ ,  $f_n \geq f_0 - \frac{1}{n}$ . In particular  $f_n \geq -\frac{1}{n}$ . This is almost the same as an arbitrage opportunity, since any element of  $f \in \overline{\mathbb{C}}$  is the limit in the  $L^\infty$  norm of a sequence  $(f_n)_{n \geq 1}$  in  $\mathbb{C}$ . This means that if  $f \geq 0$  then the sequence of possible losses  $(f_n^-)_{n \geq 1}$  tends to zero uniformly as  $n \rightarrow \infty$ , which means that the risk vanishes in the limit.

**Theorem 3 (Fundamental Theorem; Bounded Case)** *Let  $S$  be a bounded semimartingale. There exists an equivalent martingale measure  $P^*$  for  $S$  if and only if  $S$  satisfies NFLVR.*

**Proof:** Let us assume we have NFLVR. Since  $S$  satisfies the no arbitrage property we have  $\mathbb{C} \cap L_+^\infty = \{0\}$ . However one can use the property NFLVR to show  $\mathbb{C}$  is weak\* closed in  $L^\infty$  (that is, it is closed in  $\sigma(L^1, L^\infty)$ ), and hence there will exist a probability  $P^*$  equivalent to  $P$  with  $E^*\{f\} \leq 0$ , all  $f$  in  $\mathbb{C}$ . (This is the Kreps-Yan separation theorem - essentially the Hahn-Banach theorem; see, e.g., [55]). For each  $s < t$ ,  $B \in \mathcal{F}_s$ ,  $\alpha \in \mathbb{R}$ , we deduce  $\alpha(S_t - S_s)1_B \in \mathbb{C}$ , since  $S$  is bounded. Therefore  $E^*\{(S_t - S_s)1_B\} = 0$ , and  $S$  is a martingale under  $P^*$ .

For the converse, note that NFLVR remains unchanged with an equivalent probability, so without loss of generality we may assume  $S$  is a Martingale under  $P$  itself. If  $\theta$  is admissible, then  $\left(\int_0^t \theta_s dS_s\right)_{t \geq 0}$  is a local martingale, hence it is a supermartingale. Since  $E\{\theta_0 S_0\} = 0$ , we have as well  $E\left\{\int_0^\infty \theta_s dS_s\right\} \leq E\{\theta_s S_0\} = 0$ . This implies that for any  $f \in \mathbb{C}$ , we have  $E\{f\} \leq 0$ . Therefore it is true as well for  $f \in \overline{\mathbb{C}}$ , the closure of  $\mathbb{C}$  in  $L^\infty$ . Thus we conclude  $\overline{\mathbb{C}} \cap L_+^\infty = \{0\}$ .  $\square$

**Theorem 4 (Corollary)** *Let  $S$  be a locally bounded semimartingale. There is an equivalent probability measure  $P^*$  under which  $S$  is a local martingale if and only if  $S$  satisfies NFLVR.*

The measure  $P^*$  in the corollary is known as a *local martingale measure*. We refer to [14, p. 479] for the proof of the corollary. Examples show that in general  $P^*$  can make  $S$  only a local martingale, not a martingale. We also note that any semimartingale with continuous paths is locally bounded. However in the continuous case there is a considerable simplification: the No Arbitrage property alone, properly interpreted, implies the existence of an equivalent local martingale measure  $P^*$  (see [16]). Indeed using the Girsanov theorem this implies that under the No Arbitrage assumption the semimartingale must have the form

$$S_t = M_t + \int_0^t H_s d[M, M]_s$$

where  $M$  is a local martingale under  $P$ , and with restrictions on the predictable process  $H$ . Indeed, if one has  $\int_0^\epsilon H_s^2 d[M, M]_s = \infty$  for some  $\epsilon > 0$ , then  $S$  admits “immediate arbitrage,” a fascinating concept introduced by Delbaen and Schachermayer (see [16]).

For the general case, we have the impressive theorem of Delbaen and Schachermayer (see [16] for a proof), as follows:

**Theorem 5 (Fundamental Theorem; General Case)** *Let  $S$  be a semi-martingale. There exists an equivalent probability measure  $P^*$  such that  $S$  is a sigma martingale under  $P^*$  if and only if  $S$  satisfies NFLVR.*<sup>14</sup>

**Caveat:** In the remainder of the paper we will abuse language by referring to the equivalent probability measure  $P^*$  which makes  $S$  into a sigma martingale, as an **equivalent martingale measure**. For clarity let us repeat: *if  $P^*$  is an equivalent martingale measure, then we can a priori conclude no more than that  $S$  is a sigma martingale (or local martingale, if  $S$  has continuous paths).*

### 3.6 Numéraire Invariance

Our portfolio as described in Section 3.4 consists of

$$V_t(a, b) = a_t S_t + b_t R_t$$

where  $(a, b)$  are trading strategies,  $S$  is the risky security price, and  $R_t = \exp(\int_0^t r_s ds)$  is the price of a money market account. The process  $R$  is often called a *numéraire*. One can then deflate future monetary values by multiplying by  $\frac{1}{R_t} = \exp\left(-\int_0^t r_s ds\right)$ . Let us write  $Y_t = \frac{1}{R_t}$  and we shall refer to the process  $Y_t$  as a *deflator*. By multiplying  $S$  and  $R$  by  $Y = \frac{1}{R}$ , we can effectively reduce the situation to the case where the price of the money market account is constant and equal to one. The next theorem allows us to do just that.

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<sup>14</sup>See [48], Section 9 of Chapter IV (pp. 237ff), for a treatment of sigma martingales; alternatively, see Section 6e of Chapter III (pp. 214ff) of [32].

**Theorem 6 (Numéraire Invariance)** *Let  $(a, b)$  be a trading strategy for  $(S, R)$ . Let  $Y = \frac{1}{R}$ . Then  $(a, b)$  is self-financing for  $(S, R)$  if and only if  $(a, b)$  is self-financing for  $(YS, 1)$ .*

**Proof:** Let  $Z = \int_0^t a_s dS_s + \int_0^t b_s dR_s$ . Then using integration by parts we have (since  $Y$  is continuous and of finite variation)

$$\begin{aligned} d(Y_t Z_t) &= Y_t dZ_t + Z_t dY_t \\ &= Y_t a_t dS_t + Y_t b_t dR_t + \left( \int_0^t a_s dS_s + \int_0^t b_s dR_s \right) dY_t \\ &= a_t (Y_t dS_t + S_t dY_t) + b_t (Y_t dR_t + R_t dY_t) \\ &= a_t d(YS)_t + b_t d(YR)_t \end{aligned}$$

and since  $YR = \frac{1}{R} R = 1$ , this is

$$= a_t d(YS)_t$$

since  $dYR = 0$  because  $YR$  is constant. Therefore

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s$$

if and only if

$$a_t \frac{1}{R_t} S_t + b_t = a_0 S_0 + b_0 + \int_0^t a_s d\left(\frac{1}{R} S\right)_s.$$

The Numéraire Invariance Theorem allows us to assume  $R \equiv 1$  without loss of generality. Note that one can easily check that there is no arbitrage for  $(a, b)$  with  $(S, R)$  if and only if there is no arbitrage for  $(a, b)$  with  $(\frac{1}{R} S, 1)$ . By renormalizing, we no longer write  $(\frac{1}{R} S, 1)$ , but simply  $S$ .

The preceding theorem is the standard version, but in many applications (for example those arising in the modeling of stochastic interest rates), one wants to assume that the numéraire is a strictly positive semimartingale (instead of only a continuous finite variation process as in the previous theorem). We consider here the general case, where the numéraire is a (not necessarily continuous) semimartingale. For examples of how such a change of numéraire theorem can be used (albeit for the case where the deflator is assumed continuous), see for example [23]. A reference to the literature for a result such as the following theorem is [29, p. 223].

**Theorem 7 (Numéraire Invariance; General Case)** *Let  $S, R$  be semimartingales, and assume  $R$  is strictly positive. Then the deflator  $Y = \frac{1}{R}$  is a semimartingale and  $(a, b)$  is self-financing for  $(S, R)$  if and only if  $(a, b)$  is self-financing for  $(\frac{S}{R}, 1)$ .*

**Proof:** Since  $f(x) = \frac{1}{x}$  is  $\mathcal{C}^2$  on  $(0, \infty)$ , we have that  $Y$  is a (strictly positive) semimartingale by Itô's formula. By the self-financing hypothesis we have

$$\begin{aligned} V_t(a, b) &= a_t S_t + b_t R_t \\ &= a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s. \end{aligned}$$

Let us assume  $S_0 = 0$ , and  $R_0 = 1$ . The integration by parts formula for semimartingales gives

$$d(S_t Y_t) = d\left(\frac{S_t}{R_t}\right) = S_{t-} d\left(\frac{1}{R_t}\right) + \frac{1}{R_{t-}} dS_t + d\left[S, \frac{1}{R}\right]_t$$

and

$$d\left(\frac{V_t}{R_t}\right) = V_{t-} d\left(\frac{1}{R_t}\right) + \frac{1}{R_{t-}} dV_t + d\left[V, \frac{1}{R}\right]_t.$$

We can next use the self-financing assumption to write:

$$\begin{aligned} d\left(\frac{V_t}{R_t}\right) &= a_t S_{t-} d\left(\frac{1}{R_t}\right) + b_t R_{t-} d\left(\frac{1}{R_t}\right) + \frac{1}{R_{t-}} a_t dS_t + \frac{1}{R_{t-}} b_t dR_t \\ &\quad + a_t d\left[S, \frac{1}{R}\right]_t + b_t d\left[R, \frac{1}{R}\right]_t \\ &= a_t \left( S_{t-} d\left(\frac{1}{R}\right) + \frac{1}{R_{t-}} dS + d\left[S, \frac{1}{R}\right] \right) \\ &\quad + b_t \left( R_{t-} d\left(\frac{1}{R}\right) + \frac{1}{R_{t-}} dR + d\left[R, \frac{1}{R}\right] \right) \\ &= a_t d\left(S \frac{1}{R}\right) + b_t d\left(R \frac{1}{R}\right). \end{aligned}$$

Of course  $R_t \frac{1}{R_t} = 1$ , and  $d(1) = 0$ ; hence

$$d\left(\frac{V_t}{R_t}\right) = a_t d\left(S_t \frac{1}{R_t}\right).$$

In conclusion we have

$$V_t = a_t S_t + b_t R_t = b_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s,$$

and

$$a_t \left( \frac{S_t}{R_t} \right) + b_t = \frac{V_t}{R_t} = b_0 + \int_0^t a_s d \left( \frac{S_s}{R_s} \right).$$

□

### 3.7 Redundant Derivatives

Let us assume given a security price process  $S$ , and by the results in Section 3.6 we take  $R_t \equiv 1$ . Let  $\mathcal{F}_t^0 = \sigma(S_r; r \leq t)$  and let  $\mathcal{F}_t^\sim = \mathcal{F}_t^0 \vee \mathcal{N}$  where  $\mathcal{N}$  are the null sets of  $\mathcal{F}$  and  $\mathcal{F} = \bigvee_t \mathcal{F}_t^0$ , under  $P$ , defined on  $(\Omega, \mathcal{F}, P)$ . Finally we take  $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u^\sim$ . A *derivative* on  $S$  is then a random variable  $C \in \mathcal{F}_T$ , for some fixed time  $T$ . Note that we pay a small price here for the simplification of taking  $R_t \equiv 1$ , since if  $R_t$  were to be a non-constant stochastic process, it might well change the minimal filtration we are taking, because then the processes of interest would be  $(R_t, S_t)$ , in place of just  $S_t/R_t$ .

One goal of Finance Theory is to show there exists a self financing trading strategy  $(a, b)$  that one can use either to obtain  $C$  at time  $T$ , or to come as close as possible – in an appropriate sense – to obtaining  $C$ . This is the issue we discuss in this section.

**Definition 11** *Let  $S$  be the price process of a risky security and let  $R$  be the price process of a money market account (numéraire), which we setting equal to the constant process 1.<sup>15</sup> A derivative  $C \in \mathcal{F}_T$  is said to be redundant if there exists an admissible self-financing trading strategy  $(a, b)$  such that*

$$C = a_0 S_0 + b_0 R_0 + \int_0^T a_s dS_s + \int_0^T b_s dR_s.$$

*Let us normalize  $S$  by writing  $M = \frac{1}{R} S$ ; then  $C$  will still be redundant under  $M$  and hence we have (taking  $R_t = 1$ , all  $t$ ):*

$$C = a_0 M_0 + b_0 + \int_0^T a_s dM_s.$$

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<sup>15</sup>Although  $R$  is taken to be constant and equal to 1, we include it initially in the definition to illustrate the role played by being able to take it a constant process.

Next note that if  $P^*$  is any equivalent martingale measure making  $M$  a martingale, and if  $C$  has finite expectation under  $P^*$ , we then have

$$E^*\{C\} = E^*\{a_0M_0 + b_0\} + E^*\left\{\int_0^T a_s dM_s\right\}$$

provided all expectations exist,

$$= E^*\{a_0M_0 + b_0\} + 0.$$

**Theorem 8** *Let  $C$  be a redundant derivative such that there exists an equivalent martingale measure  $P^*$  with  $C \in \mathcal{L}^*(M)$ . (See the second definition following for a definition of  $\mathcal{L}^*(M)$ ). Then there exists a unique no arbitrage price of  $C$  and it is  $E^*\{C\}$ .*

**Proof:** First we note that the quantity  $E^*\{C\}$  is the same for every equivalent martingale measure. Indeed if  $Q_1$  and  $Q_2$  are both equivalent martingale measures, then

$$E_{Q_i}\{C\} = E_{Q_i}\{a_0M_0 + b_0\} + E_{Q_i}\left\{\int_0^T a_s dM_s\right\}.$$

But  $E_{Q_i}\left\{\int_0^T a_s dM_s\right\} = 0$ , and  $E_{Q_i}\{a_0M_0 + b_0\} = a_0M_0 + b_0$ , since we assume  $a_0$ ,  $M_0$ , and  $b_0$  are known at time 0 and thus without loss of generality are taken to be constants.

Next suppose one offers a price  $v > E^*\{C\} = a_0M_0 + b_0$ . Then one follows the strategy  $a = (a_s)_{s \geq 0}$  and (we are ignoring transaction costs) at time  $T$  one has  $C$  to present to the purchaser of the option. One thus has a sure profit (that is, risk free) of  $v - (a_0M_0 + b_0) > 0$ . This is an arbitrage opportunity. On the other hand if one can buy the claim  $C$  at a price  $v < a_0M_0 + b_0$ , analogously at time  $T$  one will have achieved a risk-free profit of  $(a_0M_0 + b_0) - v$ .  $\square$

**Definition 12** *If  $C$  is a derivative, and there exists an admissible self-financing trading strategy  $(a, b)$  such that*

$$C = a_0M_0 + b_0 + \int_0^T a_s dM_s;$$

*then the strategy  $a$  is said to replicate the derivative  $C$ .*

**Theorem 9 (Corollary)** *If  $C$  is a redundant derivative, then one can replicate  $C$  in a self-financing manner with initial capital equal to  $E^*\{C\}$ , where  $P^*$  is any equivalent martingale measure for the normalized price process  $M$ .*

At this point we return to the issue of *put-call parity* mentioned in the introduction (Section 2). Recall that we had the trivial relation

$$M_T - K = (M_T - K)^+ - (K - M_T)^+,$$

which, by taking expectations under  $P^*$ , shows that the price of a call at time 0 equals the price of a put plus the stock price minus  $K$ . More generally at time  $t$ ,  $E^*\{(M_T - K)^+ | \mathcal{F}_t\}$  equals the value of a put at time  $t$  plus the stock price at time  $t$  minus  $K$ , by the  $P^*$  martingale property of  $M$ .

It is tempting to consider markets where all derivatives are redundant. Unfortunately, this is too large a space of random variables; we wish to restrict ourselves to derivatives that have good integrability properties as well.

Let us fix an equivalent martingale measure  $P^*$ , so that  $M$  is a martingale (or even a local martingale) under  $P^*$ . We consider all self-financing trading strategies  $(a, b)$  such that the process  $\left(\int_0^t a_s^2 d[M, M]_s\right)^{1/2}$  is locally integrable: that means that there exists a sequence of stopping times  $(T_n)_{n \geq 1}$  which can be taken  $T_n \leq T_{n+1}$ , a. s. , such that  $\lim_{n \rightarrow \infty} T_n \geq T$  a. s. and

$$E^* \left\{ \left( \int_0^{T_n} a_s^2 d[M, M]_s \right)^{1/2} \right\} < \infty, \text{ each } T_n. \text{ Let } \mathcal{L}^*(M) \text{ denote the class}$$

of such strategies, under  $P^*$ . We remark that we are cheating a little here: we are letting our definition of a complete market (which follows) depend on the measure  $P^*$ , and it would be preferable to define it in terms of the objective probability  $P$ . How to go about doing this is a non trivial issue. In the happy case where the price process is already a local martingale under the objective probability measure, this issue of course disappears.

**Definition 13** *A market model  $(M, \mathcal{L}^*(M), P^*)$  is complete if every derivative  $C \in L^1(\mathcal{F}_T, dP^*)$  is redundant for  $\mathcal{L}^*(M)$ . That is, for any  $C \in L^1(\mathcal{F}_T, dP^*)$ , there exists an admissible self-financing trading strategy  $(a, b)$  with  $a \in \mathcal{L}^*(M)$  such that*

$$C = a_0 M_0 + b_0 + \int_0^T a_s dM_s,$$

and such that  $(\int_0^t a_s dM_s)_{t \geq 0}$  is uniformly integrable. In essence, then, a complete market is one for which every derivative is redundant.

We point out that the above definition is one of many possible definitions of a complete market. For example, one could limit attention to nonnegative random payoffs and/or payoffs that are in  $L^2(\mathcal{F}_T, dP^*)$ .

We note that in probability theory a martingale  $M$  is said to have the *predictable representation property* if for any  $C \in L^2(\mathcal{F}_T)$  one has

$$C = E\{C\} + \int_0^T a_s dM_s$$

for some predictable  $a \in \mathcal{L}(M)$ . This is, of course, essentially the property of market completeness. Martingales with predictable representation are well studied and this theory can usefully be applied to Finance. For example, suppose we have a model  $(S, R)$  where by a change of numéraire we take  $R = 1$ . Suppose further there is an equivalent martingale measure  $P^*$  such that  $S$  is a Brownian motion under  $P^*$ . Then the model is complete for all claims  $C$  in  $L^1(\mathcal{F}_T, P^*)$  such that  $C \geq -\alpha$ , for some  $\alpha \geq 0$ . ( $\alpha$  can depend on  $C$ .) To see this, we use martingale representation (see, e.g., [48]) to find a predictable process  $a$  such that for  $0 \leq t \leq T$ :

$$E^*\{C|\mathcal{F}_t\} = E^*\{C\} + \int_0^t a_s dS_s.$$

Let

$$V_t(a, b) = a_0 S_0 + b_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s;$$

we need to find  $b$  such that  $(a, b)$  is an admissible, self-financing trading strategy. Since  $R_t = 1$ , we have  $dR_t = 0$ , hence we need

$$a_t S_t + b_t R_t = b_0 + \int_0^t a_s dS_s,$$

and taking  $b_0 = E^*\{C\}$ , we have

$$b_t = b_0 + \int_0^t a_s dS_s - a_t S_t$$

provides such a strategy. It is admissible since  $\int_0^t a_s dS_s \geq -\alpha$  for some  $\alpha$  which depends on  $C$ .

Unfortunately, having the predictable representation property is rather delicate, and few martingales possess this property. Examples include Brownian motion, the Compensated Poisson process (but *not* mixtures of the two nor even the difference of two Poisson processes)(although see [36] for sufficient conditions when one can mix the two and have completeness), and the Azéma martingales. (One can consult [48] for background, and [18] for more on the Azéma martingales.) One can mimic a complete market in the case (for example) of two independent noises, each of which is complete alone. Several authors have done this with Brownian noise together with compensated Poisson noise, by proposing hedging strategies for each noise separately. A recent example of this is [40] (where the Poisson intensity can depend on the Brownian motion) in the context of default risk models. A more traditional example is [37].

Most models are therefore *not* complete, and most practitioners believe the a financial world being modeled is at best only approximately complete. We will return again to the notion of an incomplete market later on in this section. First, we need to characterize complete markets. In this regard, we have the following result:

**Theorem 10** *Suppose there is an equivalent martingale measure  $P^*$  such that  $M$  is a local martingale. Then  $P^*$  is the unique equivalent martingale measure only if the market is complete.*

This theorem is a trivial consequence of Dellacherie’s approach to Martingale Representation: if there is a unique probability making a process  $M$  a local martingale, then  $M$  must have the martingale representation property. The theory has been completely resolved in the work of Jacod and Yor. (See for example [48, Chapter IV, Section 4], for a pedagogic approach to the theory.)

To give an example of what can happen, let  $\mathcal{M}^2$  be the set of equivalent probabilities making  $M$  an  $L^2$ -martingale. Then  $M$  has the predictable representation property (and hence market completeness) for every extremal element of the convex set  $\mathcal{M}^2$ . If  $\mathcal{M}^2 = \{P^*\}$ , only one element, then of course  $P^*$  is extremal. (See [48, Theorem 40, p. 186]). Indeed  $P^*$  is in fact unique in the proto-typical example of Brownian motion; since many diffusions can be constructed as pathwise functionals of Brownian motion they inherit the completeness of the Brownian model. But there are examples where one has complete markets without the uniqueness of the equivalent

martingale measure (see [1] in this regard, as well as [33]). Nevertheless the situation is simpler when we assume our models have continuous paths.

The next theorem is a version of what is known as **The Second Fundamental Theorem of Asset Pricing**. We state and prove it for the case of  $L^2$  derivatives only. We note that this theorem has a long and illustrious history, going back to the fundamental paper of Harrison and Kreps [27, p. 392] for the discrete case, and to Harrison and Pliska [28, p. 241] for the continuous case, although in [28] the theorem below is stated only for the “only if” direction.

**Theorem 11** *Let  $M$  have continuous paths. There is a unique  $P^*$  such that  $M$  is an  $L^2$   $P^*$ -martingale if and only if the market is complete.*

**Proof:** The theorem follows easily from Theorems 38, 39, and 340 of [48, pp. 185–186]; we will assume those results and prove the theorem. Theorem 39 shows that if  $P^*$  is unique then the market model is complete. If  $P^*$  is not unique but the model is nevertheless complete, then by Theorem 37  $P^*$  is nevertheless extremal in the space of probability measures making  $M$  an  $L^2$  martingale. Let  $Q$  be another such extremal probability, and let  $L_\infty = \frac{dQ}{dP^*}$  and  $L_t = E_P\{L_\infty|\mathcal{F}_t\}$ , with  $L_0 = 1$ . Let  $T_n = \inf\{t > 0 : |L_t| \geq n\}$ .  $L$  will be continuous by Theorem 40 of [48, p. 186], hence  $L_t^n = L_{t \wedge T_n}$  is bounded. We then have, for bounded  $C \in \mathcal{F}_s$ :

$$E_Q\{M_{t \wedge T_n} C\} = E^*\{M_{t \wedge T_n} L_t^n C\}$$

$$E_Q\{M_{s \wedge T_n} C\} = E^*\{M_{s \wedge T_n} L_s^n C\}.$$

The two left sides of the above equalities are equal and this implies that  $ML^n$  is a martingale, and thus  $L^n$  is a bounded  $P^*$ -martingale orthogonal to  $M$ . It is hence constant by Theorem 39 of [48, p. 185]. We conclude  $L_\infty \equiv 1$  and thus  $Q = P^*$ .  $\square$

Note that if  $C$  is a redundant derivative, then the no arbitrage price of  $C$  is  $E^*\{C\}$ , for any equivalent martingale measure  $P^*$ . (If  $C$  is redundant then we have seen the quantity  $E^*\{C\}$  is the same under every  $P^*$ .) However, if a market model is not complete, then

- there will arise non-redundant claims, and
- there will be more than one equivalent martingale measure  $P^*$ .

We now have the conundrum: if  $C$  is non-redundant, what is the no arbitrage price of  $C$ ? We can no longer argue that it is  $E^*\{C\}$ , because there are many such values! The absence of this conundrum is a large part of the appeal of complete markets. One resolution of this conundrum is to use an investor's preferences/tastes to select among the set of possible equivalent martingale measures a unique one, that will make them indifferent between holding the derivative in their portfolio or not. This is an interesting area of current research and for more on this topic see [19] and references cited therein.

Finally, let us note that when  $C$  is redundant there is always a replication strategy  $a$ . However, when  $C$  is non-redundant it cannot be replicated. In the non-redundant case the best we can do is replicate in some approximate sense (for example expected squared error loss), and we call the strategy we follow a *hedging strategy*. See for example [22] and [30] for results about hedging strategies.

### 3.8 The Stochastic Price Process

For simplicity, we will limit our discussion to one dimension. Let  $S = (S_t)_{t \geq 0}$  denote our price process for a risky asset. Let  $s < t$  and suppose  $t - s$  is a small but finite time interval. The randomness in the market price between times  $s$  and  $t$  comes from the cumulative price changes due to the actions of many traders. Let us enumerate the traders' individual price changes over this interval. Let the random variable  $\theta_i$  denote the *change in the price* of the asset due to the different sized purchase or sale by the  $i^{\text{th}}$  trader between the times  $s$  and  $t$ . No activity corresponds to  $\theta_i = 0$ . The total effect of the traders' actions on the price is  $\Theta = \sum_{i=1}^n \theta_i$ .

If  $n$  is large (even  $n = 50$  would suffice in most cases, and typically  $n$  is much larger) and if the  $\theta_i$  are independent with mean  $\mu$  and finite variance  $\sigma^2$ , then by the Central Limit Theorem we have that  $\mathcal{L}(\Theta) = \mathcal{L}(\sum_{i=1}^n \theta_i) \approx N(n\mu, n\sigma^2)$ <sup>16</sup>, where  $\mathcal{L}(Y)$  denotes the law, or distribution, of a random variable  $Y$ . Under these assumptions, and with  $\mu = 0$ , it is reasonable to describe the random forces affecting the asset price as Gaussian. We further remark that, as is well known, one can substantially weaken the hypotheses that the random variables  $(\theta_i)_{i \geq 1}$  are independent, using for example martingale

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<sup>16</sup>As is customary,  $N(\mu, \sigma^2)$  denotes the normal distribution (also known as the Gaussian distribution) with mean  $\mu$  and variance  $\sigma^2$ .

central limit theorems (see, eg, [32] for a definitive treatment, or [31] for an introductory treatment), and one can also weaken the assumption that all variances are identical. One could then use a stochastic differential equation to give a dynamic model of the risky asset price, where we let  $B$  denote a Brownian motion:

$$dS_t = \sigma(t, S_t)dB_t + \mu(t, S_t)dt, \quad (9)$$

since the increments of the Brownian motion are given by  $B_t - B_s \sim N(0, \sigma_0^2(t-s))$ .<sup>17</sup> We usually take  $\sigma_0^2 = 1$ , since otherwise we could simply modify the coefficient function  $\sigma(t, x)$ . The function  $\sigma$  can be thought of as the sensitivity of the price change to “market noise” when  $S_t = x$ . The term given by  $\mu(t, S_t)dt$  is called the “drift,” and it corresponds to changes in the risky asset price which are not due to market noise, but rather due to market fundamentals.

There are many problems with the model given by equation (9), but the most fundamental one is that price process must always take nonnegative values, and there is no *a priori* reason that  $S$  be positive, even with taking  $S_0 > 0$ . Let us address this problem. Henceforth we will consider only *autonomous* coefficients.<sup>18</sup> This means that if the noise process has stationary and independent increments, then the price process will be a time homogeneous strong Markov process.<sup>19</sup> With dependence on time in the coefficients, one loses the time homogeneity, although the solution is still Markov.<sup>20</sup> Suppose instead we let the risky asset price process be  $Y = (Y_t)_{t \geq 0}$  with  $Y_0 = 1$  and  $Y_t > 0$  for all  $t, 0 \leq t \leq T$  for some time horizon  $T$ , a.s. Since  $Y > 0$  always, we can take its logarithm, and define  $X_t = \ln(Y_t)$ , and obviously  $Y_t = e^{X_t}$ .

Let us assume that  $X$  is the unique solution of (9), where appropriate

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<sup>17</sup>By choosing the Brownian motion, which has stationary and independent increments, we are implicitly assuming that the distributions of the traders’ likelihoods to trade in a time interval  $(s, t)$  depends only on the length of the interval  $t - s$  and does not change with time, and are independent for disjoint time intervals. Both of these assumptions have been questioned repeatedly. See for example [11] for the case against the stationarity assumption.

<sup>18</sup>That is, coefficients of the form  $\sigma(x)$ , rather than of the form  $\sigma(t, x)$ .

<sup>19</sup>See [48, p. 36 or p. 299] for example, for a definition of a strong Markov process.

<sup>20</sup>By assuming time homogeneity, however, we are depriving ourselves of a useful possibility to allow for excess kurtosis in our models, by allowing time dependence in the diffusion coefficient; see for example [44]. Kurtosis of a random variable  $X$  with mean  $\mu$  is sometimes defined as  $\gamma = E\{(X - \mu)^4\}/(E\{(X - \mu)^2\})^2$ , and excess kurtosis is simply  $\gamma - 3$ , because the kurtosis of a Gaussian random variable is 3.

hypotheses are made upon the coefficients  $\sigma$  and  $\mu$  to ensure a unique non-exploding solution. We can use Itô's formula to find a dynamic expression for  $Y$ . Indeed,

$$e^{X_t} = e^{X_0} + \int_0^t e^{X_s} dX_s + \frac{1}{2} \int_0^t e^{X_s} d[X, X]_s$$

and substituting  $Y$  for  $e^X$  we get

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \sigma(s, \ln(Y_s)) dB_s + \int_0^t Y_s \mu(s, \ln(Y_s))^2 ds \\ &\quad + \frac{1}{2} \int_0^t Y_s \sigma(s, \ln(Y_s))^2 ds, \end{aligned}$$

and letting  $\hat{\sigma}(t, y) = \sigma(t, \ln(y))$  and  $\hat{\mu}(t, y) = \mu(t, \ln(y))$ , we have:

$$dY_t = Y_t \hat{\sigma}(t, Y_t) dB_t + Y_t \{ \hat{\mu}(t, Y_t) + \frac{1}{2} \hat{\sigma}(t, Y_t)^2 \} dt. \quad (10)$$

Note that we have shown, *inter alia*, that if there exists a unique non-exploding solution to equation (9), then there also exists the same for (10), even though the function  $y \rightarrow y\sigma(t, y)$  need not be globally Lipschitz. We further note that if an equation of the form

$$dY_t = f(Y_t)Y_t dB_t + g(Y_t)Y_t dt; Y_0 > 0 \quad (11)$$

has a unique, strong, non-exploding solution, then  $P(\omega : \exists t \text{ such that } Y_t(\omega) \leq 0) = 0$  (See [48, p. 351]).

The absolute magnitude that  $Y$  changes, that is  $Y_{t+dt} - Y_t$ , is not by itself as meaningful as the relative change. For example if  $Y_{t+dt} - Y_t = \$0.12$ , this can be a large change if  $Y_t = \$1.25$ , or it can be an insignificant change if  $Y_t = \$105.12$ . Therefore we often speak of the **return on the asset**, which is *the change of the price divided by the original value*. Since we now have that  $Y > 0$ , we can rewrite equation (11) as

$$\frac{dY_t}{Y_t} = f(Y_t) dB_t + g(Y_t) dt; Y_0 > 0,$$

and indeed this is often the way the price process is written in the literature.<sup>21</sup>

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<sup>21</sup>The coefficient  $\mu$  is called the **drift** and it reflects the *fundamentals* of the asset: its position in the industry and its expected future earnings or losses. The coefficient  $f$  is called the **volatility** and it represents a the standard deviation of the returns. It is the volatility that creates risk in the investment, and it is the primary object of study.

The simplest form of such a price process is when  $f = \sigma$  and  $g = \mu$  are constants, and then of course

$$Y_t = \exp(\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t) \equiv \mathcal{E}(\sigma B_t + \mu t),$$

where  $\mathcal{E}(Z)$  denotes the *stochastic exponential* of a semimartingale  $Z$ .<sup>22</sup> One reason this simplest form is so popular is that if  $f = \sigma$  a constant, it is easy to estimate this parameter. Indeed, a simple procedure is to sample  $Y$  at  $n + 1$  equal spaced time steps  $\{t_1, t_2, \dots, t_{n+1}\}$  with  $t_i - t_{i-1} = \delta$  in chronological order and let

$$\hat{\mu} = \frac{1}{n\delta} \sum_{i=1}^n \ln\left(\frac{Y_{t_{i+1}}}{Y_{t_i}}\right)$$

and

$$\hat{\sigma}^2 = \frac{1}{(n-1)\delta} \sum_{i=1}^n (\ln\left(\frac{Y_{t_{i+1}}}{Y_{t_i}}\right) - \hat{\mu})^2$$

and then  $\hat{\sigma}^2$  is an unbiased and consistent estimator for  $\sigma^2$ . (If one takes  $\delta$  as a fraction of a year, then the parameters are annualized.) Of course there are other reasons the form is so popular; see Section 3.10.

Let us return to the heuristic derivation of the price process. Recall that  $\theta_i$  denotes the change in the price of the risky asset due to the size of a purchase or sale by the  $i^{\text{th}}$  trader between the times  $s$  and  $t$ , with the total effect of the traders' actions being  $\Theta = \sum_{i=1}^n \theta_i$ . In reality, there are many different rubrics of traders. Some examples are (a) a small trader; (b) a trader for a large mutual fund; (c) a trader for a pension fund; (d) corporate traders; and (e) traders for hedge funds. These traders have different goals and different magnitudes of equity supporting their trades. Let us divide the traders into rubrics, and for rubric  $n$  we enumerate the traders  $(n, 1), (n, 2), \dots, (n, n)$  and we let the traders' impacts on the price between times  $s$  and  $t$  be denoted  $U_{n,1}, U_{n,2}, \dots, U_{n,n}$ . We assume that the random variables  $(U_{n,i})_{1 \leq i \leq n}$  are i.i.d. for every  $n \geq 1$  and independent across all the  $n$ , and moreover for each fixed  $n$  have common law  $l_n$ . Set:

$$\Psi_n = \sum_{i=1}^n U_{n,i}, \text{ and } \theta_n = \frac{\Psi_n - E(\Psi_n)}{V_n},$$

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<sup>22</sup>For a continuous semimartingale  $X$  the stochastic exponential of  $X$ , denoted  $\mathcal{E}(X)$ , is the process  $Y$  given by  $Y_t = \mathcal{E}(X)_t = \exp(X_t - \frac{1}{2}[X, X]_t)$ , and  $Y$  satisfies the exponential type differential equation  $dY_t = Y_t dX_t$ ;  $Y_0 = 1$ . See [48, p. 85] for more about the stochastic exponential, which is also sometimes called the *Doléans-Dade exponential*.

where  $V_n$  is the standard deviation of  $\Psi_n$ . Then

$$\Theta_n = \sum_{i=1}^n \theta_i = \sum_{i=1}^n \sum_{j=1}^i \frac{U_{i,j} - E(U_{i,j})}{V_i}$$

represents the normalized random effect on the market price of the traders' actions between times  $s$  and  $t$ . We have that  $\Theta_n$  converges in distribution to a random variable which is *infinitely divisible*. If we denote this random variable as  $Z_t - Z_s$ , and think of  $Z$  as a noise process with stationary and independent increments, then  $Z$  must be a **Lévy process** (see for example [48, p. 21] for this result, and in general [48] or [4] for more information on Lévy processes in general). Since the only Lévy process with continuous paths is Brownian motion with drift, in order to be different from the classical case, the paths  $t \rightarrow Z_t(\omega)$  must have jumps.

A discontinuous price process requires a different analysis. Let us understand why. We begin as before and let  $Z$  be a Lévy process (with jumps), and then form  $X$  by

$$dX_t = \sigma(X_{t-})dZ_t + \mu(X_{t-})dt \quad (12)$$

and  $Y_t = e^{X_t} > 0$ . Next using Itô's formula we have

$$\begin{aligned} e^{X_t} &= e^{X_0} + \int_0^t e^{X_{s-}} dX_s + \frac{1}{2} \int_0^t e^{X_{s-}} d[X, X]_s^c \\ &\quad + \sum_{s \leq t} (e^{X_s} - e^{X_{s-}} - e^{X_{s-}} \Delta X_s) \end{aligned}$$

and substituting  $Y_t$  for  $e^{X_t}$ , and using that for a Lévy process  $Z$  one *a fortiori* has that  $d[Z, Z]_t^c = \gamma dt$  for some constant  $\gamma \geq 0$ , we have

$$\begin{aligned} Y_t &= Y_0 + \int_0^t Y_{s-} \sigma(\ln(Y_{s-})) dZ_s + \int_0^t Y_{s-} \mu(\ln(Y_{s-})) ds \\ &\quad + \frac{1}{2} \int_0^t Y_{s-} \sigma(\ln(Y_{s-}))^2 \gamma ds + \sum_{s \leq t} (Y_s - Y_{s-} - Y_{s-} \sigma(\ln(Y_{s-})) \Delta Z_s) \\ &= Y_0 + \hat{\sigma}(Y_{s-}) dZ_s + \int_0^t Y_{s-} \{ \hat{\mu}(Y_{s-}) + \frac{\gamma}{2} \hat{\sigma}(Y_{s-})^2 \} ds \\ &\quad + \sum_{s \leq t} (Y_s - Y_{s-} - Y_{s-} \hat{\sigma}(Y_{s-}) \Delta Z_s) \end{aligned}$$

which does not satisfy a stochastic differential equation driven by  $dZ$  and  $dt$ . If we were simply to forget about the series term at the end, as many researchers do, and instead were to consider the following equation as our dynamic model

$$dY_t = Y_{t-}f(Y_{t-})dZ_t + Y_{t-}g(Y_{t-})dt, : Y_0 > 0, \quad (13)$$

then we could no longer ensure that  $Y$  is a positive price process! Indeed, if we consider the simple case where  $f = \sigma$  and  $g = \mu$  are both constants, with  $Y_0 = 1$ , we have

$$dY_t = \sigma Y_{t-}dZ_t + \mu Y_{t-}dt$$

which has a closed form solution

$$Y_t = \exp(\sigma Z_t + \mu t - \frac{1}{2}\sigma^2\gamma t)\prod_{s \leq t} e^{\{-\sigma\Delta Z_s\}}(1 + \sigma\Delta Z_s)$$

and thus as soon as one has a jump  $\Delta Z_s \leq -\frac{1}{\sigma}$ , we have  $Y$  becoming zero or negative. In general, for equations of the form (13), a sufficient condition to have  $Y > 0$  always is that  $|\Delta Z_s| < \frac{1}{\|\sigma\|_{L^\infty}}$ , for all  $s \geq 0$ , a.s. (See [48, p. 352].)

Should one stop here? Perhaps one should consider more general noise processes than Lévy processes? Indeed, one could consider time changes of Lévy processes, since there is empirical evidence of non-stationarity in the increments of the noise process (see for example [11] and more recently [8]); or, and not exclusively, one might think there is some historical dependence in the noise process, violating the assumptions of independence of the increments. The advantage that the assumption of independent increments provides is that the solution  $X$  of the SDE is a strong Markov process (a time homogeneous strong Markov process if the increments are also stationary and the coefficients are autonomous). Therefore, so is  $Y = e^X$ , since the function  $x \rightarrow e^x$  is injective. If however one replaces the Lévy driving process  $Z$  with a strong Markov process, call it  $Q$ , then the solution  $X$  with  $Z$  replaced by  $Q$ , will no longer be a Markov process, although the vector process  $(X, Q)$  will be strong Markov (See [48, Theorem 32 on p. 300 and Theorem 73 on p. 353]).

But why do we care if  $X$ , and hence  $Y$ , is strong Markov? Many researchers claim there is evidence that the price process has short term momentum, which would violate the Markov property. The reason is that it is mathematically convenient to have  $Y$  be Markov, since in this case one has

a hope of calculating (or at least approximating) a hedging strategy for a financial derivative. If however one is willing to forego having a time homogeneous strong Markov price process, then one can consider a price process of the form

$$dY_t = Y_{t-}f(Y_{t-})dZ_t + Y_{t-}g(Y_{t-})dA_t, : Y_0 > 0, \quad (14)$$

where  $Z$  and  $A$  are taken to be semimartingales. There is a danger to this level of generality, since not all such models lead to an absence of arbitrage, as we shall see in Section 3.11.

### 3.9 Determining the Replication Strategy

It is rare that we can actually “explicitly” compute a replication strategy for a derivative security. However, there are simple cases where miracles happen; and when there are no miracles, then we can often approximate hedging strategies using numerical techniques.

Let us consider a standard, and relatively simple derivative security of the form

$$C = f(S_T)$$

where  $S$  is the price of the risky security. The two most important examples (already discussed in Section 2) are

- *The European call option:* Here  $f(x) = (x - K)^+$  for a constant  $K$ , so the contingent claim is  $C = (S_T - K)^+$ .  $K$  is referred to as the *strike price* and  $T$  is the expiration time. In words, the European call option gives the holder the right to *buy* one unit of the security at the price  $K$  at time  $T$ . Thus the (random) value of the option at time  $T$  is  $(S_T - K)^+$ .
- *The European put option:* Here  $f(x) = (K - x)^+$ . This option gives the holder the right to *sell* one unit of the security at time  $T$  at price  $K$ . Hence the (random) value of the option at time  $T$  is  $(K - S_T)^+$ .

To illustrate the ideas involved, let us take  $R_t \equiv 1$  by a change of numéraire, and let us suppose that  $C = f(S_T)$  is a redundant derivative. The *value* of a replicating self-financing trading strategy  $(a, b)$  for the claim, at time  $t$ , is:

$$V_t = E^*\{f(S_T)|\mathcal{F}_t\} = a_0S_0 + b_0 + \int_0^t a_s dS_s.$$

We now make a series of hypotheses in order to obtain a simpler analysis:

**Hypothesis 1.**  $S$  is a Markov process under some equivalent local martingale measure  $P^*$ .

Under hypothesis 1 we have that

$$V_t = E^*\{f(S_T)|\mathcal{F}_t\} = E^*\{f(S_T)|S_t\}.$$

But measure theory tells us that there exists a function  $\varphi(t, \cdot)$ , for each  $t$ , such that

$$E^*\{f(S_T)|S_t\} = \varphi(t, S_t).$$

**Hypothesis 2.**  $\varphi(t, x)$  is  $\mathcal{C}^1$  in  $t$  and  $\mathcal{C}^2$  in  $x$ .

This hypothesis enables us to use Itô's formula:

$$\begin{aligned} V_t &= E^*\{f(S_T)|\mathcal{F}_t\} = \varphi(t, S_t) \\ &= \varphi(0, S_0) + \int_0^t \varphi'_x(s, S_{s-})dS_s \\ &\quad + \int_0^t \varphi'_s(s, S_{s-})ds + \frac{1}{2} \int_0^t \varphi''_{xx}(s, S_{s-})d[S, S]_s^c \\ &\quad + \sum_{0 < s \leq t} \{\varphi(s, S_s) - \varphi(s, S_{s-}) - \varphi'_x(s, S_{s-})\Delta S_s\}. \end{aligned}$$

**Hypothesis 3.**  $S$  has continuous paths.

With hypothesis 3 Itô's formula simplifies:

$$\begin{aligned} V_t = \varphi(t, S_t) &= \varphi(0, S_0) + \int_0^t \varphi'_x(s, S_s)dS_s \\ &\quad + \int_0^t \varphi'_s(s, S_s)ds + \frac{1}{2} \int_0^t \varphi''_{xx}(s, S_s)d[S, S]_s. \end{aligned} \tag{15}$$

Since  $V$  is a  $P^*$  martingale, the right side of (15) must also be a  $P^*$  martingale. This is true if

$$\int_0^t \varphi'_s(s, S_s)ds + \frac{1}{2} \int_0^t \varphi''_{xx}(s, S_s)d[S, S]_s = 0. \tag{16}$$

For equation (16) to hold, it is reasonable to require that  $[S, S]$  have paths which are absolutely continuous almost surely. Indeed, we assume more than that. We assume a specific structure for  $[S, S]$ :

**Hypothesis 4.**  $[S, S]_t = \int_0^t h(s, S_s)^2 ds$  for some jointly measurable function  $h$  mapping  $\mathbb{R}_+ \times \mathbb{R}$  to  $\mathbb{R}$ .

We then get that (16) certainly holds if  $\varphi$  is the solution of the partial differential equation:

$$\frac{1}{2}h(s, x)^2 \frac{\partial^2 \varphi}{\partial x^2}(s, x) + \frac{\partial \varphi}{\partial s}(s, x) = 0$$

with boundary condition  $\varphi(T, x) = f(x)$ . Note that if we combine Hypotheses 1 - 4, we have a continuous Markov process with quadratic variation  $\int_0^t h(s, S_s)^2 ds$ . An obvious candidate for such a process is the solution of a stochastic differential equation

$$dS_s = h(s, S_s)dB_s + k(s; S_r; r \leq s)ds,$$

where  $B$  is a standard Wiener process (Brownian motion) under  $P$ .  $S$  is a continuous Markov process under  $P^*$ , with quadratic variation  $[S, S]_t = \int_0^t h(s, S_s)^2 ds$  as desired.

The quadratic variation is a path property and is unchanged by changing to an equivalent probability measure  $P^*$  (see [48] for example). But what about the Markov property? Why is  $S$  a Markov process under  $P^*$  when  $b$  can be path dependent? Here we digress a bit.

Let us analyze  $P^*$  in more detail. Since  $P^*$  is equivalent to  $P$ , we can let  $Z = \frac{dP^*}{dP}$  and  $Z > 0$  a.s. ( $dP$ ). Let  $Z_t = E\{Z|\mathcal{F}_t\}$ , which is clearly a martingale. By Girsanov's theorem (see, eg, [48]),

$$\int_0^t h(s, S_s)dB_s - \int_0^t \frac{1}{Z_s} d[Z, \int_0^\cdot h(r, S_r)dB_r]_s \quad (17)$$

is a  $P^*$  martingale.

Let us suppose that  $Z_t = 1 + \int_0^t H_s Z_s dB_s$ , which is reasonable since we have martingale representation for  $B$  and  $Z$  is a martingale. We then have that (17) becomes

$$\int_0^t h(s, S_s)dB_s - \int_0^t \frac{1}{Z_s} Z_s H_s h(s, S_s)ds = \int_0^t h(s, S_s)dB_s - \int_0^t H_s h(s, S_s)ds.$$

If we choose  $H_s = \frac{k(s; S_r; r \leq s)}{h(s, S_s)}$ , then we have

$$S_t = \int_0^t h(s, S_s) dB_s + \int_0^t k(s; S_r; r \leq s) ds$$

is a martingale under  $P^*$ . Moreover, we have  $M_t = B_t + \int_0^t \frac{k(s; S_r; r \leq s)}{h(s, S_s)} ds$  is a  $P^*$  martingale. Since  $[M, M]_t = [B, B]_t = t$ , by Lévy's theorem it is a  $P^*$ -Brownian motion (see, e. g. , [48]), and we have

$$dS_t = h(t, S_t) dM_t$$

and thus  $S$  is a Markov process under  $P^*$ .

The last step in this digression is to show that it is possible to construct such a  $P^*$ ! Recall that the *stochastic exponential* of a semimartingale  $X$  is the solution of the “exponential equation”

$$dY_t = Y_t dX_t; \quad Y_0 = 1.$$

The solution is known in closed form and is given by

$$Y_t = \exp\left(X_t - \frac{1}{2}[X, X]_t^c\right) \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

If  $X$  is continuous then

$$Y_t = \exp\left(X_t - \frac{1}{2}[X, X]_t\right),$$

and it is denoted  $Y_t = \mathcal{E}(X)_t$ . Recall that we wanted  $dZ_t = H_t Z_t dB_t$ . Let  $N_t = \int_0^t H_s dB_s$ , and we have  $Z_t = \mathcal{E}(N)_t$ . Then we set  $H_t = \frac{-k(t; S_r, r \leq t)}{h(t, S_t)}$  as planned and let  $dP^* = Z_T dP$ , and we have achieved our goal. Since  $Z_T > 0$  a.s. ( $dP$ ), we have that  $P$  and  $P^*$  are equivalent.

Let us now summarize the foregoing. We assume we have a price process given by

$$dS_t = h(t, S_t) dB_t + k(t; S_r, r \leq t) dt.$$

We form  $P^*$  by  $dP^* = Z_T dP$ , where  $Z_T = \mathcal{E}(N)_T$  and  $N_t = \int_0^t \frac{-k(s; S_r, r \leq s)}{h(s, S_s)} dB_s$ . We let  $\varphi$  be the (unique) solution of the boundary value problem.

$$\frac{1}{2} h(t, x)^2 \frac{\partial^2 \varphi}{\partial x^2}(t, x) + \frac{\partial}{\partial s} \varphi(t, x) = 0 \quad (18)$$

and  $\varphi(T, x) = f(x)$ , where  $\varphi$  is  $\mathcal{C}^2$  in  $x$  and  $\mathcal{C}^1$  in  $t$ . Then

$$V_t = \varphi(t, S_t) = \varphi(0, S_0) + \int_0^t \frac{\partial \varphi}{\partial x}(s, S_s) dS_s.$$

Thus, under these four rather restrictive hypotheses, we have found our replication strategy! It is  $a_s = \frac{\partial \varphi}{\partial x}(s, S_s)$ . We have also found our value process  $V_t = \varphi(t, S_t)$ , provided we can solve the partial differential equation (18). However even if we cannot solve it in closed form, we can always approximate  $\varphi$  numerically.

**Remark 2** *It is a convenient hypothesis to assume that the price process  $S$  of our risky asset follows a stochastic differential equation driven by Brownian motion.*

**Remark 3** *Although our price process is assumed to follow the SDE*

$$dS_t = h(t, S_t)dB_t + k(t; S_r, r \leq t)dt,$$

*we see that the PDE (4) does not involve the “drift” coefficient  $k$  at all! Thus the price and the replication strategy do not involve  $k$  either. The economic explanation of this is two-fold: first, the drift term  $k$  is already reflected in the market price: it is based on the “fundamentals” of the security; and second, what is important is the risk involved as reflected in the term  $h$ .*

**Remark 4** *Hypothesis (2) is not a benign hypothesis. Since  $\varphi$  turns out to be the solution of a partial differential equation (given in (18)), we are asking for regularity of the solution. This is typically true when  $f$  is smooth (which of course the canonical example  $f(x) = (K - x)^+$  is not!). The problem occurs at the boundary, not the interior. Thus for reasonable  $f$  we can handle the boundary terms. Indeed this analysis works for the cases of European calls and puts as we describe in the next Section 3.10.*

### 3.10 The Black Scholes Model

In Section 3.9 we saw the convenience of assuming that  $S$  solves a stochastic differential equation. Let us now assume  $S$  follows a linear SDE (= Stochastic Differential Equation) with constant coefficients:

$$dS_t = \sigma S_t dB_t + \mu S_t dt; \quad S_0 = 1. \tag{19}$$

Let  $X_t = \sigma B_t + \mu t$  and we have

$$dS_t = S_t dX_t; \quad S_0 = 1$$

so that

$$S_t = \mathcal{E}(X)_t = e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t}.$$

The process  $S$  of (19) is known as *geometric Brownian motion* and has been used to study stock prices since at least the 1950's and the work of P. Samuelson.

In this simple case the solution of the PDE (18) of Section 3.9 *can be found explicitly*, and it is given by

$$\varphi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(xe^{\sigma u\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)}) e^{-\frac{u^2}{2}} du.$$

In the case of a *European call option* we have  $f(x) = (x - K)^+$  and

$$\begin{aligned} \varphi(x, t) = & x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{K} + \frac{1}{2}\sigma^2(T-t)\right)\right) \\ & - K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{x}{K} - \frac{1}{2}\sigma^2(T-t)\right)\right). \end{aligned}$$

Here  $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{u^2}{2}} du$ . In the case of this call option we can also compute the replication strategy:

$$a_t = \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S_t}{K} + \frac{1}{2}\sigma^2(T-t)\right)\right). \quad (20)$$

And, we can compute the *price* (here we assume  $S_0 = s$ ):

$$\begin{aligned} V_0 = \varphi(x, 0) = & x\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} + \frac{1}{2}\sigma^2T\right)\right) \\ & - K\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} - \frac{1}{2}\sigma^2T\right)\right). \end{aligned} \quad (21)$$

These formulas, (20) and (21) are the celebrated *Black-Scholes option formulas*, (or as we prefer to call them, *Black-Scholes-Merton option formulas*) with  $R_t \equiv 1$ .

This is a good opportunity to show how things change in the presence of interest rates. Let us now assume a constant interest rate  $r$  so that  $R_t = e^{-rt}$ . Then the formula (21) becomes:

$$V_0 = \varphi(x, 0) = x\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right)\right) - e^{-rT}K\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right)\right).$$

These relatively simple, explicit, and easily computable formulas make working with European call and put options easy. It is perhaps because of this beautiful simplicity that security prices are often assumed to follow geometric Brownian motions, even when there is significant evidence to the contrary. Finally note that – as we observed earlier – the drift coefficient  $\mu$  does not enter into the Black-Scholes formula.

### 3.11 Reasonable Price Processes

This section studies reasonable price processes, which we define to be price processes consistent with no arbitrage. The reason, of course, is that if a price process admits arbitrage, it would be unstable. Traders' actions, taking advantage of the arbitrage opportunities, would change the price process into something else (the mechanism is as discussed in section 3.8).

Here, we consider arbitrary semimartingales as possible price processes, and we study necessary conditions for them to have no arbitrage opportunities. Because of the Delbaen–Schachermayer theory, we know that this is equivalent to finding an equivalent probability measure  $P^*$  such that a semimartingale  $X$  is a  $\sigma$  martingale under  $P^*$ . Note that in Section 3.9 we showed how to construct  $P^*$  by constructing the Radon-Nikodym derivative  $\frac{dP^*}{dP}$ , under the assumption that the price process followed a stochastic differential equation of a reasonable form, driven by a Brownian motion. This is of course in the case of a complete market, where  $P^*$  is unique. In the incomplete case, there are many equivalent local martingale measures, and for these cases we will indicate in Section 3.12 how to explicitly construct at least one of the equivalent probability measures such that  $X$  is a  $\sigma$  martingale.

**Definition 14** *A reasonable price process  $X$  is a nonnegative semimartingale on a filtered probability space satisfying ‘the usual hypotheses’*

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ , such that there exists at least one equivalent probability measure  $P^*$  making  $X$  a  $\sigma$  martingale under  $P^*$ .

### 3.11.1 The Continuous Case

Let  $X_t = X_0 + M_t + A_t$ ,  $t \geq 0$  be a continuous semimartingale on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . We seek necessary conditions (and if possible sufficient) such that there exists an equivalent probability measure  $P^*$  where  $X$  is a  $P^*$   $\sigma$  martingale. Since  $X$  is continuous, and since all continuous sigma martingales are in fact local martingales, we need only concern ourselves with local martingales. We give the appropriate theorem without proof, and instead refer the interested reader to [49] for a detailed proof.<sup>23</sup>

**Theorem 12** *Let  $X_t = X_0 + M_t + A_t$ ,  $0 \leq t \leq T$  be a continuous semimartingale on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  where  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ . Let  $C_t = [X, X]_t = [M, M]_t$ ,  $0 \leq t \leq T$ . There exists an equivalent probability measure  $P^*$  on  $\mathcal{F}_T$  such that  $X$  is a  $P^*$  sigma martingale only if the following two conditions are satisfied:*

1.  $dA \ll dC$  a.s.;
2. If  $J$  is such that  $A_t = \int_0^t J_s dC_s$  for  $0 \leq t \leq T$ , then  $\int_0^T J_s^2 dC_s < \infty$  a.s.;

*If in addition one has the condition below, then we have sufficient conditions for there to exist an equivalent probability measure  $P^*$  on  $\mathcal{F}_T$  such that  $X$  is a  $P^*$  sigma martingale:*

3.  $E\{\mathcal{E}(-J \cdot M)_T\} = 1$ , where  $\mathcal{E}(U)$  denotes the stochastic exponential of a semimartingale  $U$ .

**Remark 5** *Recall that the decomposition of a continuous semimartingale is unique (assuming one takes the local martingale  $M$  to be continuous), so  $M$  and  $A$  are uniquely defined. If the martingale  $M$  is Brownian motion, that is if  $M = B$ , then since  $[B, B]_t = t$  we have as a necessary condition that  $A$  must have paths which are absolutely continuous (with respect to Lebesgue measure) almost surely. This means that a semimartingale such as  $X_t =$*

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<sup>23</sup>In the following, the symbol  $C$  is not the payoff to a derivative security as it has been in previous sections.

$1 + |B_t|$  cannot be a reasonable price process, *even though it is a nonnegative semimartingale, since by Tanaka's formula we have  $X_t = 1 + \beta_t + L_t$  where  $\beta$  is another Brownian motion, and  $L$  is the local time of  $B$  at level 0.*<sup>24</sup> We know that the paths of  $L$  are singular with respect to Lebesgue measure, a.s.

**Remark 6** *The sufficiency is not as useful as it might seem, because of condition (3) of Theorem 12. The first two conditions should be, in principle, possible to verify, but the third condition is in general not. On the other hand there do exist other sufficient conditions that can be used to verify condition (3) of Theorem 12, such as Kazamaki's condition and the more well known condition of Novikov (See, e.g, [48] for an expository treatment of these conditions). However in practice, both of these conditions are typically quite difficult or impossible to verify, and other more ad hoc methods are used when appropriate. Typically one uses ad hoc methods to show the process in question is both positive and everywhere finite. Since these process often arise in practice as solutions of stochastic differential equations, this amounts to verifying that there are no explosions. The interested reader can consult [9] for recent results concerning these ad hoc methods.*

### 3.11.2 The General Case

A key step in the proofs for the continuous case is the use of Girsanov's theorem. A problem in the general case is that the analog of the predictable version of Girsanov's theorem is not applicable to arbitrary semimartingales (one needs some finiteness, or integrability conditions). Therefore, one needs to use a version of Girsanov's theorem due to Jacod and Mémmin, that works for random measures, and this naturally leads us to the framework of semimartingale characteristics. For background on characteristics, we refer the reader to the excellent treatment in [32].

Let  $X$  be an arbitrary semimartingale with characteristics  $(B, C, \nu)$  on our usual filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . The random measure  $\nu$  factors as follows:  $\nu(ds, dx) = dA_s(\omega)K_s(\omega, dx)$  in such a way that we can take  $C_t = \int_0^t c_s dA_s$  and  $B_t = \int_0^t b_s dA_s$ . We have the following theorem which gives necessary conditions for  $X$  to have no arbitrage in the

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<sup>24</sup>It is also trivial to construct an arbitrage strategy for this price process: if we buy and hold one share at time 0 for \$1, then at time  $T$  we have  $X_T$  dollars, and obviously  $X_T \geq 1$  a.s., and  $P(X_T > 1) = 1 > 0$ .

Delbaen-Schachermayer sense of NFLVR. We give the theorem without proof; a proof can be found in [49].

**Theorem 13** *Let  $P^*$  be another probability measure equivalent to  $P$ . Then of course  $X$  is a semimartingale under  $P^*$ , with characteristics  $(B^*, C, \nu^*)$ <sup>25</sup>. We then know (see Theorem 3.17 on page 170 of [32]) that the random measure  $\nu^*$  is absolutely continuous with respect to  $\nu$ , and that there exists a predictable process (predictable in the extended sense)  $Y(s, x)_{s \geq 0, x \in \mathbb{R}}$  such that*

$$\nu^* = Y \cdot \nu. \tag{22}$$

*If  $X$  is a  $P^*$   $\sigma$  martingale, then we must have the following four conditions satisfied:*

1.  $b_t + \beta_t c_t + \int \{x(Y(t, x) - 1_{\{|x| \leq 1\}})K_t(dx) = 0; P(d\omega)dA_s(\omega)$  almost everywhere;
2.  $\int_0^T \beta_s^2 dC_s < \infty$ , a.s.;
3.  $\Delta A_t > 0$  implies that  $\int xY(s, x)K(s, dx) = 0$ ;
4.  $\int |x^2| \wedge |x|Y(t, x)K_t(dx) < \infty$ ,  $P(d\omega)dA_s(\omega)$  almost everywhere.

**Remark 7** *Distinct from the continuous case, we only have necessary conditions for  $P^*$  to exist, and not sufficient conditions. The proof of the sufficiency in the continuous case breaks down here.*

Often we impose the assumption of *quasi left continuity*<sup>26</sup> of the underlying filtration. This is a standard assumption in most of Markov process theory, for example, due to Meyer's theorem (cf [48, p. 105]). A simple example of a quasi left continuous filtration is the natural (completed) filtration of a Lévy process.

**Theorem 14** *Let  $X$  be a semimartingale as in Theorem 13. Suppose in addition that  $\mathbb{F}$  is a quasi-left continuous filtration, and that  $A$  is continuous. If  $X$  is a  $P^*$   $\sigma$  martingale, then we must have the following three conditions satisfied:*

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<sup>25</sup>We write  $C$  instead of  $C^*$  because it is the same process for any equivalent probability measure.

<sup>26</sup>See [48, p. 191] for a definition and discussion of quasi left continuity of a filtration. The primary implication of a filtration being quasi left continuous is that no martingale can jump at a predictable stopping time.

1.  $b_t + \beta_t c_t + \int \{x(Y(t, x) - 1_{\{|x| \leq 1\}})K_t(dx) = 0; P(d\omega)dA_s(\omega)$  almost everywhere;
2.  $\int_0^T \beta_s^2 dC_s < \infty$ , a.s.;
3.  $\int |x^2| \wedge |x| Y(t, x) K_t(dx) < \infty$ ,  $P(d\omega)dA_s(\omega)$  almost everywhere.

**Remark 8** *Since the filtration  $\mathbb{F}$  is quasi-left continuous, all martingales jump at totally inaccessible times, so the assumption that  $A$  be continuous is not a restriction on the martingale terms, but rather a condition eliminating predictable jump times in the drift. Since  $A$  is continuous, obviously we are able to remove the condition on the jumps of  $A$ .*

**Remark 9 (General Remarks)** *Comparing Theorem 12 and Theorem 13 illustrates how market incompleteness corresponding to the price process  $X$  can arise in two different ways. First, Theorem 12 shows that (in the continuous case) the choice of the orthogonal martingale  $M$  is essentially arbitrary, and each such choice leads to a different equivalent probability measure rendering  $X$  a local martingale. Second, Theorem 13 shows that in the general case (the case where jumps are present) incompleteness can still arise for the same reasons as in the continuous case, but also because of the jumps, through the choice of  $Y$ . Indeed, we are free to change  $Y$  appropriately at the cost of changing  $b$ . Only if  $K$  reduces to a point mass is it then possible to have uniqueness of  $P^*$  (and hence market completeness), and then of course only if  $C = 0$ . What this means is that if there are jumps in the price process, our only hope for a complete market is for there to be only one kind of jump, and no continuous martingale component.*

We also wish to remark that through clever constructions, one can indeed have complete markets with jumps in more interesting settings than point processes; see for example [18]. In addition, one can combine (for example) a Brownian motion and a compensated Poisson process via a clever trick to get a complete market which has a continuous martingale and a jump martingale as components of the price process. (See [36].)

### 3.12 The Construction of Martingale Measures

In Section 3.9 we showed how, in the special continuous case, one can construct (in that section the unique) martingale measure (also known as a risk neutral measure). We can use the same idea more generally in incomplete

markets, and we illustrate this technique here. Note that we are not trying for maximum generality, and we will make some rather strong finiteness assumptions in order to keep the presentation simple. Here is the primary result.

**Theorem 15** *Let  $S$  be a price process, and assume it is a special semimartingale<sup>27</sup> with canonical decomposition  $S = M + A$ . Assume that the conditional quadratic variation process  $\langle M, M \rangle$  exists, and that  $dA_t \ll d\langle M, M \rangle_t$  such that if  $dA_t = K_t d\langle M, M \rangle_t$  for some predictable process  $K$ , then  $E(e^{\int_0^T K_s^2 d\langle M, M \rangle_s}) < \infty$ . Assume further that for any stopping time  $\tau$ ,  $0 \leq \tau \leq T$ , we have  $K_\tau \Delta M_\tau > -1$ . Let*

$$Z_t = 1 + \int_0^t Z_{s-} (-K_s) dM_s, : 0 \leq t \leq T$$

and set  $dP^* = Z_T dP$ . Then  $P^*$  is a equivalent martingale measure for  $P$ .

**Proof:** Since we know by hypothesis that  $K_\tau \Delta M_\tau > -1$  for any stopping time  $\tau$  with values in  $[0, T]$ , we have that  $Z > 0$  on  $[0, T]$  almost surely. Thus  $Z$  is a positive supermartingale. The hypothesis  $E(e^{\int_0^T K_s^2 d\langle M, M \rangle_s}) < \infty$  allows us to assume that  $Z$  is a true martingale, by Shimbo's theorem (see [48, p. 142], or [51]).<sup>28</sup> Therefore  $E(Z_T) = 1$  and  $P^*$  is a true probability measure, equivalent to  $P$ . We therefore have, by the Girsanov-Meyer theorem, that the canonical decomposition of  $S$  under  $P^*$  is:

$$S_t = \{S_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M \rangle_s\} + \{A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M \rangle_s\}. \quad (23)$$

We next note that

$$\int_0^t \frac{1}{Z_{s-}} d\langle Z, M \rangle_s = \int_0^t \frac{1}{Z_{s-}} Z_{s-} (-K_s) d\langle M, M \rangle_s = - \int_0^t K_s d\langle M, M \rangle_s \quad (24)$$

and this equals  $-A_t$  by our hypothesis on  $K$  (and hence  $A$ ). This implies that  $P^*$  renders  $S$  into a local martingale, and hence  $P^*$  is a choice for an equivalent martingale measure.

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<sup>27</sup>A semimartingale is called *special* if it has a decomposition where the finite variation term can be taken predictable. See [48, pp. 130ff] for more information on special semimartingales.

<sup>28</sup>If  $S$  is assumed continuous, we have that the condition  $E(e^{\frac{1}{2} \int_0^T K_s^2 d\langle M, M \rangle_s}) < \infty$  is sufficient, by Novikov's criterion.

**Example 1** Suppose we have a price process which satisfies:

$$dS_t = \sigma_1(S_t)dB_t + \sigma_2(S_t)dW_t + \sigma_3(S_{t-})dM_t + \mu(S_t)dt; : S_0 > 0 \quad (25)$$

where  $B$  and  $W$  are independent Brownian motions,  $M_t = N_t - \lambda t$ , a compensated standard Poisson process with arrival intensity  $\lambda$ . We let  $\mathbb{M}$  denote the sum of the three martingales. Moreover we assume that  $\sigma_1, \sigma_2, \sigma_3$  and  $\mu$  all bounded, Lipschitz functions. (We also assume of course that  $N$  is independent from the two Brownian motions.) To find a risk neutral measure  $P^*$ , we need only choose it in such a way as to eliminate the drift under  $P^*$ . We have four (and as we shall see hence an infinite number of) obvious choices:

1. We can choose  $Z$  to be the unique solution of

$$Z_{1,t} = 1 + \int_0^t Z_{1,s}(-\mu(S_s))dB_s; : Z_{1,0} = 1, \quad (26)$$

and take  $dP_1^* = Z_{1,T}dP$ . We then get, using equations (23) and (24), that

$$\int_0^t \frac{1}{Z_{1,s-}} d\langle Z_1, \mathbb{M} \rangle_s = \int_0^t \frac{1}{Z_{1,s-}} Z_{1,s-}(-\mu(S_s))d\langle B, \mathbb{M} \rangle_s \quad (27)$$

$$= - \int_0^t \mu(S_s)d\langle B, B \rangle_s = - \int_0^t \mu(S_s)ds \quad (28)$$

where, due to the independence assumption, the second to last equality uses  $\langle B, W \rangle = \langle B, M \rangle = 0$ , whence  $\langle B, \mathbb{M} \rangle = \langle B, B \rangle$ . Finally, we have  $d\langle B, B \rangle_s = d[B, B]_s = ds$ , since  $B$  is a Brownian motion.

2. Instead, we can choose  $Z_2$  to satisfy the SDE

$$Z_{2,t} = 1 + \int_0^t Z_{2,s}(-\mu(S_s))dW_s; : Z_{2,0} = 1.$$

This gives us a new equivalent martingale measure  $dP_2^* = Z_{2,T}dP$  by the same calculations as above. In particular, we get  $\langle W, \mathbb{M} \rangle = \langle W, W \rangle$  at the last step.

3. For the third example, we set

$$Z_{3,t} = 1 + \int_0^t Z_{3,s}(-\mu(S_s))\frac{1}{\lambda}dM_s; : Z_{3,0} = 1,$$

and  $dP_3^* = Z_{3,T}dP$ . This time we repeat the calculation of Equation (28) to get:

$$\begin{aligned} \int_0^t \frac{1}{Z_{1,s-}} d\langle Z_1, \mathbb{M} \rangle_s &= \int_0^t \frac{1}{Z_{1,s-}} Z_{1,s-} (-\mu(S_s)) \frac{1}{\lambda} d\langle M, \mathbb{M} \rangle_s \\ &= - \int_0^t \mu(S_s) \frac{1}{\lambda} d\langle M, M \rangle_s = - \int_0^t \mu(S_s) \frac{1}{\lambda} \lambda ds, \end{aligned}$$

since  $d\langle M, M \rangle_s = \lambda ds$ , and of course we have used once again the independence of  $B, W$ , and  $M$ , which implies that  $\langle B, M \rangle_t = 0$ .

4. In addition to the three equivalent martingale measures constructed above, we can of course combine them, as follows:

$$\begin{aligned} Z_{4,t} &= + \int_0^t Z_{4,s-} \{ \alpha(-\mu(S_s)) dB_s + \beta(-\mu(S_s)) dW_s + \gamma(-\mu(S_s)) \frac{1}{\lambda} dM_s \}; \\ Z_{4,0} &= 1, \end{aligned}$$

where  $\alpha, \beta$ , and  $\gamma$  are all nonnegative, and  $\alpha + \beta + \gamma = 1$ . Then  $dP_4^* = Z_{4,T}dP$ .

One can imagine many more constructions, by combinations of the first three examples via random (rather than deterministic and linear) combinations.

Finally, note that these constructions, even with random combinations of the first three fundamental examples, need not exhaust the possibilities for equivalent martingale measures. Depending on the underlying filtration and probability measure, there could be martingales orthogonal<sup>29</sup> to  $B, W$ , and  $M$  also living on the space, which could generate orthogonal equivalent martingale measures. In this case, there is little hope to explicitly construct these alternative equivalent martingale measures with the given underlying processes. This point is made clear, but in a more abstract setting, in Section 3.11.

### 3.13 More Complex Derivatives in the Brownian Paradigm: A General View

In Sections 3.9 and 3.10 we studied derivatives of the form  $C = f(S_T)$ , that depend only on the final value of the price process. There we showed that

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<sup>29</sup>See Section 3 of Chapter IV of [48] for a treatment of orthogonal martingales, and in particular Corollary 1 on p. 183 of [48].

the computation of the price and also the hedging strategy can be obtained by solving a partial differential equation, provided the price process  $S$  is assumed to be Markov under  $P^*$ . But, this is a limited perspective. There are many other derivative securities whose payoffs depend on the entire path of the price process, and not only on the final value. In this case, the partial differential equation approach is not applicable and other techniques from the theory of stochastic processes must be applied. This section studies the techniques necessary to handle these more complex derivative securities.

We illustrate these techniques by looking at a *look-back option*, a derivative security whose payoff depends on the maximum value of the asset price  $S$  over the entire path from 0 to  $T$ . Let us return to Geometric Brownian motion:

$$dS_t = \sigma S_t dB_t + \mu S_t dt.$$

Proceeding as in Section 3.9 we change to an equivalent probability measure  $P^*$  such that  $B_t^* = B_t + \frac{\mu}{\sigma} t$  is a standard Brownian motion under  $P^*$ . Now,  $S$  is a martingale satisfying:

$$dS_t = \sigma S_t dB_t^*.$$

Let  $F$  be a functional defined on  $\mathbf{C}[0, T]$ , the continuous functions with domain  $[0, T]$ . Then  $F(u) \in \mathbb{R}$ , where  $u \in \mathbf{C}[0, T]$ . Let us suppose that  $F$  is *Fréchet* differentiable and let  $DF$  denote its Fréchet derivative. Under some technical conditions on  $F$  (see, e.g., [10]), if  $C = F(B^*)$ , then one can show

$$C = E^*\{C\} + \int_0^T {}^p(DF(B^*; (t, T)))dB_t^* \quad (29)$$

where  ${}^p(X)$  denotes the predictable projection of  $X$ . (This is often written “ $E^*\{X|\mathcal{F}_t\}$ ” in the literature. The process  $X = (X_t)_{0 \leq t \leq T}$ ,  $E^*\{X_t|\mathcal{F}_t\}$  is defined for each  $t$  a. s. The null set  $N_t$  depends on  $t$ . Thus  $E^*\{X_t|\mathcal{F}_t\}$  does not uniquely define a process, since if  $N = \bigcup_{0 \leq t \leq T} N_t$ , then  $P(N_t) = 0$  for each  $t$ , but  $P(N)$  need not be zero. The theory of predictable projections avoids this problem.)

Using (29) we then have a formula for the hedging strategy:

$$a_t = \frac{1}{\sigma S_t} {}^p(DF(\cdot, (t, T))).$$

For the look-back option, we have the payoff:  $C(\omega) = \sup_{0 \leq t \leq T} S_t(\omega) = S_T^* = F(B^*)$ . Then, we can let  $\tau(B^*)$  denote the random time where the trajectory of  $S$  attains its maximum on  $[0, T]$ . Such an operation is Fréchet differentiable and

$$DF(B^*, \cdot) = \sigma F(B^*) \delta_{\tau(B^*)},$$

where  $\delta_\alpha$  denotes the Dirac measure at  $\alpha$ .

Let

$$M_{s,t} = \max_{s \leq u \leq t} \left( B_u^* - \frac{1}{2} \sigma u \right)$$

with  $M_t = M_{0,t}$ . Then the Markov property gives

$$\begin{aligned} E^* \{ DF(B^*, (t, T)) | \mathcal{F}_t \} (B^*) &= E^* \{ \sigma F(B^*) 1_{\{M_{t,T} > M_t\}} | \mathcal{F}_t \} (B^*) \\ &= \sigma S_t E^* \{ \exp(\sigma M_{T-t}); M_{T-t} > M_t(B^*) \}. \end{aligned}$$

For a given fixed value of  $B^*$ , this last expectation depends only on the distribution of the maximum of a Brownian motion with constant drift. But this distribution is explicitly known. Thus, we obtain an explicit hedging strategy for this look-back option (see [25]):

$$\begin{aligned} a_t(\omega) &= \left( -\log \frac{M_t}{S_t}(\omega) + \frac{\sigma^2(T-t)}{2} + 2 \right) \Phi \left( \frac{-\log \frac{M_t}{S_t}(\omega) + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} \right) \\ &\quad + \sigma \sqrt{T-t} \varphi \left( \frac{-\log \frac{M_t}{S_t}(\omega) + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} \right) \end{aligned}$$

where  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$  and  $\varphi(x) = \Phi'(x)$ .

The value of this look-back option is then:

$$V_0 = E^* \{ C \} = S_0 \left( \frac{\sigma^2 T}{2} + 2 \right) \Phi \left( \frac{1}{2} \sigma \sqrt{T} \right) + \sigma \sqrt{T} S_0 \varphi \left( \frac{1}{2} \sigma \sqrt{T} \right).$$

Requiring that the claim be of the form  $C = F(B^*)$  where  $F$  is Fréchet differentiable is still restrictive. One can weaken this hypothesis substantially by requiring that  $F$  be only Malliavin differentiable. If we let  $D$  denote now the Malliavin derivative of  $F$ , then equation (29) is still valid. Nevertheless explicit strategies and prices can be computed only in a few very special cases, and usually only when the price process  $S$  is Geometric Brownian motion.

## 4 American Type Derivatives

### 4.1 The General View

We begin with an abstract definition, when there is *a unique equivalent martingale measure*.

**Definition 15** *We are given an adapted process  $U$  and an expiration time  $T$ . An American type derivative is a claim to the payoff  $U_\tau$  at a stopping time  $\tau \leq T$ ; the stopping time  $\tau$  is chosen by the holder of the derivative and is called the exercise policy.*

We let  $V_t =$  the price of the security at time  $t$ . One wants to find  $(V_t)_{0 \leq t \leq T}$  and especially  $V_0$ . Let  $V_t(\tau)$  denote the value of the security at time  $t$  if the holder uses exercise policy  $\tau$ . Let us further assume without loss of generality that  $R_t \equiv 1$ . Then

$$V_t(\tau) = E^*\{U_\tau | \mathcal{F}_t\}$$

where of course  $E^*$  denotes expectation with respect to the equivalent martingale measure  $P^*$ .

Let  $\mathcal{T}(t) = \{\text{all stopping times with values in } [t, T]\}$ .

**Definition 16** *A rational exercise policy is a solution to the optimal stopping problem*

$$V_0^* = \sup_{\tau \in \mathcal{T}(0)} V_0(\tau). \quad (30)$$

We want to establish a price for an American type derivative. That is, how much should one pay for the right to purchase  $U$  in between  $[0, T]$  at a stopping rule of one's choice?

Suppose first that the supremum in (30) is achieved. That is, let us assume there exists a rule  $\tau^*$  such that  $V_0^* = V_0(\tau^*)$  where  $V_0^*$  is defined in (30).

**Theorem 16**  *$V_0^*$  is a lower bound for the no arbitrage price of the American type derivative.*

**Proof:** Suppose it is not. Let  $V_0 < V_0^*$  be another price. Then one should buy the security at  $V_0$  and use the stopping rule  $\tau^*$  to purchase  $U$  at time  $\tau^*$ .

One then spends  $-U_{\tau^*}$ , which gives an initial payoff of  $V_0^* = E^*\{U_{\tau^*}|\mathcal{F}_0\}$ ; one's initial profit is  $V_0^* - V_0 > 0$ . This is an arbitrage opportunity.  $\square$

To prove  $V_0^*$  is also an upper bound for the no arbitrage price (and thus finally equal to the price!) is more difficult.

**Definition 17** *A super-replicating trading strategy  $\theta$  is a self-financing trading strategy  $\theta$  such that  $\theta_t S_t \geq U_t$ , all  $t$ ,  $0 \leq t \leq T$ , where  $S$  is the price of the underlying risky security on which the American type derivative is based. (We are again assuming  $R_t \equiv 1$ .)*

**Theorem 17** *Suppose a super replicating strategy  $\theta$  exists with  $\theta_0 S_0 = V_0^*$ . Then,  $V_0^*$  is an upper bound for the no arbitrage price of the American type derivative  $(U, T)$ .*

**Proof:** If  $V_0 > V_0^*$ , then one can sell the American type derivative and adopt a super-replicating trading strategy  $\theta$  with  $\theta S_0 = V_0^*$ . One then has an initial profit of  $V_0 - V_0^* > 0$ , while we are also able to cover the payment  $U_\tau$  asked by the holder of the security at his exercise time  $\tau$ , since  $\theta_\tau S_\tau \geq U_\tau$ . Thus we have an arbitrage opportunity.  $\square$

The existence of super-replicating trading strategies can be established using *Snell Envelopes*. A stochastic process  $Y$  is said to be of “class  $D$ ” if the collection  $\mathcal{H} = \{Y_\tau : \tau \text{ a stopping time}\}$  is uniformly integrable.

**Theorem 18** *Let  $Y$  be a càdlàg, adapted process,  $Y > 0$  a.s., and of “Class  $D$ ”. Then there exists a positive càdlàg supermartingale  $Z$  such that*

- (i)  $Z \geq Y$ , and for every other positive supermartingale  $Z'$  with  $Z' \geq Y$ , also  $Z' \geq Z$ ;
- (ii)  $Z$  is unique and also belongs to Class  $D$ ;
- (iii) For any stopping time  $\tau$

$$Z_\tau = \text{ess sup}_{\nu \geq \tau} E\{Y_\nu | \mathcal{F}_\tau\}$$

( $\nu$  is also a stopping time).

For a proof consult [17] or [38].  $Z$  is called the *Snell Envelope of  $Y$* .

One then needs to make some regularity hypotheses on the American type derivative  $(U, T)$ . For example, if one assumes  $U$  is a continuous semimartingale and  $E^*\{[U, U]_T\} < \infty$ , it is more than enough. One then uses the existence of Snell envelopes to prove:

**Theorem 19** *Under regularity assumptions (for example  $E^*\{[U, U]_T\} < \infty$  suffices), there exists a super-replicating trading strategy  $\theta$  with  $\theta_t S_t \geq k$  for all  $t$  for some constant  $k$  and such that  $\theta_0 S_0 = V_0^*$ . A rational exercise policy is*

$$\tau^* = \inf\{t > 0 : Z_t = U_t\},$$

where  $Z$  is the Snell Envelope of  $U$  under  $P^*$ .

## 4.2 The American Call Option

Let us here assume that for a price process  $(S_t)_{0 \leq t \leq T}$  and a bond process  $R_t \equiv 1$ , there exists a unique equivalent martingale measure  $P^*$  which means that there is no arbitrage and the market is complete.

**Definition 18** *An American call option with terminal time  $T$  and strike price  $K$  gives the holder the right to buy the security  $S$  at any time  $\tau$  between 0 and  $T$ , at price  $K$ .*

It is of course reasonable to consider the random time  $\tau$  where the option is exercised at a stopping time, and the option's payoff is  $(S_\tau - K)^+$ , corresponding to which rule  $\tau$  that the holder uses.

First, we note that since the holder of the American call option is free to choose the rule  $\tau \equiv T$ , he or she is always in a better position than the holder of a European call option, whose worth is  $(S_T - K)^+$ . Thus, the price of an American call option should be bounded below by the price of the corresponding European call option.

As in Section 4.1 we let

$$V_t(\tau) = E^*\{U_\tau | \mathcal{F}_t\} = E^*\{(S_\tau - K)^+ | \mathcal{F}_t\}$$

denote the value of our American call option at time  $t$  assuming  $\tau$  is the exercise rule. The price is then

$$V_0^* = \sup_{\tau; 0 \leq \tau \leq T} E^*\{(S_\tau - K)^+\}.$$

We note however that  $S = (S_t)_{0 \leq t \leq T}$  is a martingale under  $P^*$ , and since  $f(x) = (x - K)^+$  is a convex function, we have  $(S_t - K)^+$  is a submartingale under  $P^*$ . Hence, from (1) we have that

$$V_0^* = E^*\{(S_T - K)^+\}$$

since  $t \rightarrow E^*\{(S_t - K)^+\}$  is an increasing function, and the sup – even for stopping times – of the expectation of a submartingale is achieved at the terminal time (this can be easily seen as a trivial consequence of the Doob-Meyer decomposition theorem). This leads to the following result (however the analogous result is not true for American put options, or even for American call options if the underlying stocks pay dividends):

**Theorem 20** *In a complete market (with no arbitrage) the price of an American call option with terminal time  $T$  and strike price  $K$  is the same as the price for a European call option with the same terminal time and strike price.*

**Theorem 21 (Corollary)** *If the price process  $S_t$  follows the SDE*

$$dS_t = \sigma S_t dB_t + \mu S_t dt;$$

*then the price of an American call option with strike price  $K$  and terminal time  $T$  is the same as that of the corresponding European call option and is given by the formula (21) of Black, Scholes, and Merton.*

Although the prices of the European and American call options are the same, we have said nothing about the replication strategies. But, the above theorem essentially states that the American call option is never exercised early, and hence, is identical to the European call option. Thus, their replication strategies will be identical as well.

### 4.3 Backwards Stochastic Differential Equations and the American Put Option

Let  $\xi$  be in  $L^2$  and suppose  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz in space. Then a simple backwards ordinary differential equation ( $\omega$  by  $\omega$ ) is

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(s, Y_s(\omega)) ds.$$

However if  $\xi \in L^2(\mathcal{F}_T, dP)$  and one requires that a solution  $Y = (Y_t)_{0 \leq t \leq T}$  be *adapted* (that is,  $Y_t \in \mathcal{F}_t$ ), then the equation is more complex. For example, if  $Y_t \in \mathcal{F}_t$  for every  $t$ ,  $0 \leq t \leq T$ , then one has

$$Y_t = E\left\{\xi + \int_t^T f(s, Y_s) ds \mid \mathcal{F}_t\right\}. \quad (31)$$

An equation such as (31) is called a *Backwards Stochastic Differential Equation*.

Next, we write

$$Y_t = E\left\{\xi + \int_0^T f(s, Y_s) ds \middle| \mathcal{F}_t\right\} - \int_0^t f(s, Y_s) ds = M_t - \int_0^t f(s, Y_s) ds$$

where  $M$  is the martingale  $E\left\{\xi + \int_0^T f(s, Y_s) ds \middle| \mathcal{F}_t\right\}$ . We then have

$$\begin{aligned} Y_T - Y_t &= M_T - M_t - \left( \int_0^T f(s, Y_s) ds - \int_0^t f(s, Y_s) ds \right) \xi - Y_t \\ &= M_T - M_t - \int_t^T f(s, Y_s) ds \end{aligned}$$

or, the equivalent equation:

$$Y_t = \xi + \int_t^T f(s, Y_s) ds - (M_T - M_t). \quad (32)$$

Next, let us suppose that we are solving (31) on the canonical space for Brownian motion. Then, we have that the martingale representation property holds, and hence there exists a predictable  $Z \in \mathcal{L}(B)$  such that

$$M_t = M_0 + \int_0^t Z_s dB_s$$

where  $B$  is Brownian motion. We have that (32) becomes:

$$Y_t = \xi + \int_t^T f(s, Y_s) ds - \int_t^T Z_s dB_s. \quad (33)$$

Thus, to find an adapted  $Y$  that solves (31) is equivalent to find a pair  $(Y, Z)$  with  $Y$  adapted and  $Z$  predictable that solve (33).

Given  $Z$ , one can consider a more general version of (33) of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s. \quad (34)$$

We wish to consider an even more general equation than (34): Backward Stochastic Differential Equations where the solution  $Y$  is forced to stay above an obstacle. This can be formulated as follows (here we follow [20]):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s$$

where  $Y_t \geq U_t$  ( $U$  is optional),

$K$  is a continuous, increasing, adapted,  $K_0 = 0$ , and  $\int_0^T (Y_t - U_t) dK_t = 0$ .

The obstacle process  $U$  is given, as are the random variables  $\xi$  and the function  $f$ , and the unknowns are  $(Y, Z, K)$ . Once again it is  $Z$  that makes both  $Y$  and  $K$  adapted.

**Theorem 22 (EKPPQ)** *Let  $f$  be Lipschitz in  $(y, z)$  and assume  $E\{\sup_{0 \leq t \leq T} (U_t^+)^2\} < \infty$ . Then there exists a unique solution  $(Y, Z, K)$  to equation (5).*

Two proofs are given in [EKPPQ]: one uses the Skorohod problem, a priori estimates and Picard iteration; the other uses a penalization method.

Now let us return to American type derivatives. Let  $S$  be the price process of a risky security and let us take  $R_t \equiv 1$ . For an American put option, by definition, the payoff takes the form  $(K - S_\tau)^+$  where  $K$  is a strike price and the exercise rule  $\tau$  is a stopping time with  $0 \leq \tau \leq T$ . Thus, we should let  $U_t = (K - S_t)^+$ , and if  $X$  is the Snell envelope of  $U$ , we see from Section 4.1 that a rational exercise policy is

$$\tau^* = \inf\{t > 0 : X_t = U_t\}$$

and that the price is  $V_0^* = V_0(\tau^*) = E^*\{U_{\tau^*} | \mathcal{F}_0\} = E^*\{(K - S_{\tau^*})^+\}$ . Therefore, finding the price of an American put option is related to finding the Snell envelope of  $U$ . Recall that the Snell envelope is a supermartingale such that

$$X_\tau = \text{ess sup}_{\nu \geq \tau} E\{U_\nu | \mathcal{F}_\tau\}$$

where  $\nu$  is also a stopping time.

We consider the situation where  $U_t = (K - S_t)^+$  and  $\xi = (K - S_T)^+$ . We then have

**Theorem 23 (EKPPQ)** *Let  $(Y, K, Z)$  be the solution of (5). Then*

$$Y_t = \operatorname{ess\,sup}_{\nu \text{ a stopping time}; t \leq \nu \leq T} E \left\{ \int_t^\nu f(s, Y_s, Z_s) ds + U_\nu | \mathcal{F}_t \right\}.$$

**Proof:** [Sketch] In this case

$$Y_t = U_T + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s,$$

hence

$$Y_\nu - Y_t = - \int_t^\nu f(s, Y_s, Z_s) ds + (K_t - K_\nu) + \int_t^\nu Z_s dB_s$$

and since  $Y_t \in \mathcal{F}_t$  we have

$$\begin{aligned} Y_t &= E \left\{ \int_t^\nu f(s, Y_s, Z_s) ds + Y_\nu + (K_\nu - K_t) | \mathcal{F}_t \right\} \\ &\geq E \left\{ \int_t^\nu f(s, Y_s, Z_s) ds + U_\nu | \mathcal{F}_t \right\}. \end{aligned}$$

Next let  $\gamma_t = \inf\{t \leq u \leq T : Y_u = U_u\}$ , with  $\gamma_t = T$  if  $Y_u > U_u$ ,  $t \leq u \leq T$ . Then

$$Y_t = E \left\{ \int_t^{\gamma_t} f(s, Y_s, Z_s) ds + Y_{\gamma_t} + K_{\gamma_t} - K_t | \mathcal{F}_t \right\}.$$

However on  $[t, \gamma_t)$  we have  $Y > U$ , and thus  $\int_t^{\gamma_t} (Y_s - U_s) dK_s = 0$  implies that  $K_{\gamma_t^-} - K_t = 0$ ; however  $K$  is continuous by assumption, hence  $K_{\gamma_t} - K_t = 0$ . Thus (using  $Y_{\gamma_t} = U_{\gamma_t}$ ):

$$Y_t = E \left\{ \int_t^{\gamma_t} f(s, Y_s, Z_s) ds + U_{\gamma_t} | \mathcal{F}_t \right\}$$

and we have the other implication.  $\square$

The next corollary shows that we can obtain the price of an American put option via reflected backwards stochastic differential equations.

**Theorem 24 (Corollary)** *The American put option has the price  $Y_0$ , where  $(Y, K, Z)$  solves the reflected obstacle backwards SDE with obstacle  $U_t = (K - S_t)^+$  and where  $f = 0$ .*

**Proof:** In this case the previous theorem becomes

$$Y_0 = \operatorname{ess\,sup}_{\nu \text{ a stopping time } 0 \leq \nu \leq T} E\{U_\nu | \mathcal{F}_t\},$$

and  $U_\nu = (K - S_\nu)^+$ .  $\square$

The relationship between the American put option and backwards SDEs can be exploited to numerically price an American put option, see Jin Ma, Jaime San Martín, Soledad Torres and the second author ([43]), as well as work of V. Bally and G. Pagès ([3]), and the more recent and very promising work of Lemor et al: E. Gobet, J-P. Lemor, and X. Warin ([24] and also see [41]). More traditional methods are to use numerical methods associated with variational partial differential equations, Monte Carlo simulation, or lattice (binomial) type approximations.

We note that one can generalize these results to American Game Options (sometimes called Israeli options), using Forward-Backward Reflected Stochastic Differential Equations. See, eg, [42] or the “Game Options” introduced by Y. Kifer [39].

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