

# An Analysis of a Least Squares Regression Method for American Option Pricing

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## Abstract

Recently, various authors proposed Monte-Carlo methods for the computation of American option prices, based on least squares regression. The purpose of this paper is to analyze an algorithm due to Longstaff and Schwartz. This algorithm involves two types of approximation. Approximation one: replace the conditional expectations in the dynamic programming principle by projections on a finite set of functions. Approximation two: use Monte-Carlo simulations and least squares regression to compute the value function of approximation one. Under fairly general conditions, we prove the almost sure convergence of the complete algorithm. We also determine the rate of convergence of approximation two and prove that its normalized error is asymptotically Gaussian.

KEY WORDS: American options, optimal stopping, Monte-Carlo methods, least squares regression.

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## 1 Introduction

The computation of American option prices is a challenging problem, especially when several underlying assets are involved. The mathematical problem to solve is an optimal stopping

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problem. In classical diffusion models, this problem is associated with a variational inequality, for which, in higher dimensions, classical PDE methods are ineffective.

Recently, various authors introduced numerical methods based on Monte-Carlo techniques (see, among others, [1, 2, 3, 4, 5, 9, 12]). The starting point of these methods is to replace the time interval of exercise dates by a finite subset. This amounts to approximating the American option by a so called *Bermudan* option. A control of the error caused by this restriction to discrete stopping times is generally easy to obtain (see, for instance, [8], Remark 1.4). Throughout the paper, we concentrate on the discrete time problem.

The solution of the discrete optimal stopping problem reduces to an effective implementation of the dynamic programming principle. The conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte-Carlo techniques. One way of treating this problem is to use least squares regression on a finite set of functions as a proxy for conditional expectation. This idea (which already appeared in [5]) is one of the main ingredients of two recent papers by Longstaff and Schwartz [9], and by Tsitsiklis and Van Roy [12].

The purpose of the present paper is to analyze the least squares regression method proposed by Longstaff and Schwartz [9], which seems to have become popular among practitioners. In fact, we will consider a variant of their approach (see Remark 2.1). In order to present our results more precisely, we will distinguish two types of approximation in their algorithm. Approximation one: replace conditional expectations in the dynamic programming principle by projections on a finite set of functions taken from a suitable basis. Approximation two: use Monte-Carlo simulations and least squares regression to compute the value function of the first approximation. Approximation two will be referred to as *the Monte-Carlo procedure*. In practice, one chooses the number of basis functions and runs the Monte-Carlo procedure.

We will prove that the value function of approximation one approaches (with probability one) the value function of the initial optimal stopping problem as the number  $m$  of functions goes to infinity. We then prove that for a fixed finite set of functions, we have almost sure convergence of the Monte-Carlo procedure to the value function of the first approximation. We also establish a type of central limit theorem for the rate of convergence of the Monte-Carlo procedure, thus providing the asymptotic normalized error. We note that partial convergence results are stated in [9], together with excellent empirical results, but with no

study of the rate of convergence. On the other hand, convergence (but not the rate nor the error distribution) is provided in [12] for a somewhat different algorithm. We also refer to [12] for a discussion of accumulation of errors as the number of possible exercise dates rows. We believe that our methods could be applied to analyze the rate of convergence of the Tsitsiklis-Van Roy approach, but we will concentrate on the Longstaff-Schwartz method.

Mathematically, the most technical part of our work concerns the Central Limit Theorem for the Monte-Carlo procedure. One might think that the methods developed for the analysis of asymptotic errors in statistical estimation based on stochastic optimization (see, for instance, [6, 7, 11]) are applicable to our problem. However, the algorithm does not seem to fit in this setting for two reasons: the lack of regularity of the value function as a function of the parameters and the recursive nature of dynamic programming.

The paper is organized as follows. In Section 2, a precise description of the least squares regression method is given and the notation is established. In Section 3, we prove the convergence of the algorithm. In Section 4, we study the rate of convergence of the Monte-Carlo procedure.

## 2 The algorithm and notations

### 2.1 Description of the algorithm

As mentioned in the introduction, the first step in all probabilistic approximation methods is to replace the original optimal stopping problem in continuous time by an optimal stopping problem in discrete time. Therefore, we will present the algorithm in the context of discrete optimal stopping.

We will consider a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , equipped with a discrete time filtration  $(\mathcal{F}_j)_{j=0, \dots, L}$ . Here, the positive integer  $L$  denotes the (discrete) time horizon. Given an adapted payoff process  $(Z_j)_{j=0, \dots, L}$ , where  $Z_0, Z_1, \dots, Z_L$  are square integrable random variables, we are interested in computing

$$\sup_{\tau \in \mathcal{T}_{0,L}} \mathbb{E} Z_\tau,$$

where  $\mathcal{T}_{j,L}$  denotes the set of all stopping times with values in  $\{j, \dots, L\}$ .

Following classical optimal stopping theory (for which we refer to [10], chapter 6), we

introduce the Snell envelope  $(U_j)_{j=0,\dots,L}$  of the payoff process  $(Z_j)_{j=0,\dots,L}$ , defined by

$$U_j = \operatorname{ess-}\sup_{\tau \in \mathcal{T}_{j,L}} \mathbb{E}(Z_\tau \mid \mathcal{F}_j), \quad j = 0, \dots, L.$$

The dynamic programming principle can be written as follows:

$$\begin{cases} U_L = Z_L \\ U_j = \max(Z_j, \mathbb{E}(U_{j+1} \mid \mathcal{F}_j)), \quad 0 \leq j \leq L-1. \end{cases}$$

We also have  $U_j = \mathbb{E}(Z_{\tau_j} \mid \mathcal{F}_j)$ , with

$$\tau_j = \min\{k \geq j \mid U_k = Z_k\}.$$

In particular  $\mathbb{E}U_0 = \sup_{\tau \in \mathcal{T}_{0,L}} \mathbb{E}Z_\tau = \mathbb{E}Z_{\tau_0}$ .

The dynamic programming principle can be rewritten in terms of the optimal stopping times  $\tau_j$ , as follows:

$$\begin{cases} \tau_L = L \\ \tau_j = j \mathbf{1}_{\{Z_j \geq \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}} + \tau_{j+1} \mathbf{1}_{\{Z_j < \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}}, \quad 0 \leq j \leq L-1, \end{cases}$$

This formulation in terms of stopping rules (rather than in terms of value functions) plays an essential role in the least squares regression method of Longstaff and Schwartz.

The method also requires that the underlying model be a Markov chain. Therefore, we will assume that there is an  $(\mathcal{F}_j)$ -Markov chain  $(X_j)_{j=0,\dots,L}$  with state space  $(E, \mathcal{E})$  such that, for  $j = 0, \dots, L$ ,

$$Z_j = f(j, X_j),$$

for some Borel function  $f(j, \cdot)$ . We then have  $U_j = V(j, X_j)$  for some function  $V(j, \cdot)$  and  $\mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j) = \mathbb{E}(Z_{\tau_{j+1}} \mid X_j)$ . We will also assume that the initial state  $X_0 = x$  is deterministic, so that  $U_0$  is also deterministic.

The first approximation consists of approximating the conditional expectation with respect to  $X_j$  by the orthogonal projection on the space generated by a finite number of functions of  $X_j$ . Let us consider a sequence  $(e_k(x))_{k \geq 1}$  of measurable real valued functions defined on  $E$  and satisfying the following conditions:

**A<sub>1</sub>**: For  $j = 1$  to  $L-1$ , the sequence  $(e_k(X_j))_{k \geq 1}$  is total in  $L^2(\sigma(X_j))$ .

**A<sub>2</sub>**: For  $j = 1$  to  $L-1$  and  $m \geq 1$ , if  $\sum_{k=1}^m \lambda_k e_k(X_j) = 0$  a.s. then  $\lambda_k = 0$  for  $k = 1$  to  $m$ .

For  $j = 1$  to  $L - 1$ , we denote by  $P_j^m$  the orthogonal projection from  $L^2(\Omega)$  onto the vector space generated by  $\{e_1(X_j), \dots, e_m(X_j)\}$  and we introduce the stopping times  $\tau_j^{[m]}$ :

$$\begin{cases} \tau_L^{[m]} = L \\ \tau_j^{[m]} = j \mathbf{1}_{\left\{Z_j \geq P_j^m(Z_{\tau_{j+1}^{[m]}})\right\}} + \tau_{j+1}^{[m]} \mathbf{1}_{\left\{Z_j < P_j^m(Z_{\tau_{j+1}^{[m]}})\right\}}, \quad 1 \leq j \leq L - 1, \end{cases}$$

From these stopping times, we obtain an approximation of the value function:

$$U_0^m = \max \left( Z_0, \mathbb{E} Z_{\tau_1^{[m]}} \right). \quad (2.1)$$

Recall that  $Z_0 = f(0, x)$  is deterministic. The second approximation is then to evaluate numerically  $\mathbb{E} Z_{\tau_1^{[m]}}$  by a Monte-Carlo procedure. We assume that we can simulate  $N$  independent paths  $(X_j^{(1)}), \dots, (X_j^{(n)}), \dots, (X_j^{(N)})$  of the Markov chain  $(X_j)$  and we denote by  $Z_j^{(n)}$  ( $Z_j^{(n)} = f(j, X_j^{(n)})$ ) the associated payoff for  $j = 0$  to  $L$  and  $n = 1$  to  $N$ . For each path  $n$ , we then estimate recursively the stopping times  $(\tau_j^{[m]})$  by:

$$\begin{cases} \tau_L^{n,m,N} = L \\ \tau_j^{n,m,N} = j \mathbf{1}_{\left\{Z_j^{(n)} \geq \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}} + \tau_{j+1}^{n,m,N} \mathbf{1}_{\left\{Z_j^{(n)} < \alpha_j^{(m,N)} \cdot e^m(X_j^{(n)})\right\}}, \quad 1 \leq j \leq L - 1, \end{cases}$$

Here,  $x \cdot y$  denotes the usual inner product in  $\mathbb{R}^m$ ,  $e^m$  is the vector valued function  $(e_1, \dots, e_m)$  and  $\alpha_j^{(m,N)}$  is the least square estimator:

$$\alpha_j^{(m,N)} = \arg \min_{a \in \mathbb{R}^m} \sum_{n=1}^N \left( Z_{\tau_{j+1}^{n,m,N}}^{(n)} - a \cdot e^m(X_j^{(n)}) \right)^2,$$

Remark that for  $j = 1$  to  $L - 1$ ,  $\alpha_j^{(m,N)} \in \mathbb{R}^m$ . Finally, from the variables  $\tau_j^{n,m,N}$ , we derive the following approximation for  $U_0^m$ :

$$U_0^{m,N} = \max \left( Z_0, \frac{1}{N} \sum_{n=1}^N Z_{\tau_1^{n,m,N}}^{(n)} \right). \quad (2.2)$$

In the next section, we prove that, for any fixed  $m$ ,  $U_0^{m,N}$  converges almost surely to  $U_0^m$  as  $N$  goes to infinity, and that  $U_0^m$  converges to  $U_0$  as  $m$  goes to infinity. Before stating these results, we devote a short section to notation. We also mention that the above algorithm is not exactly the Longstaff-Schwartz algorithm as their regression involves only in-the-money paths (see Remark 2.1).

## 2.2 Notation

Throughout the paper we denote by  $|x|$  the Euclidean norm of a vector  $x$  in  $\mathbb{R}^d$ .

For  $m \geq 1$  we denote by  $e^m(x)$  the vector  $(e_1(x), \dots, e_m(x))$  and for  $j = 1$  to  $L - 1$  we define  $\alpha_j^m$  so that

$$P_j^m(Z_{\tau_{j+1}^{[m]}}) = \alpha_j^m \cdot e^m(X_j) \quad (2.3)$$

We remark that, under  $A_2$ , the  $m$  dimensional parameter  $\alpha_j^m$  has the explicit expression:

$$\alpha_j^m = (A_j^m)^{-1} \mathbb{E}(Z_{\tau_{j+1}^{[m]}} e^m(X_j)), \quad (2.4)$$

for  $j = 1$  to  $L - 1$ , where  $A_j^m$  is an  $m \times m$  matrix, with coefficients given by

$$(A_j^m)_{1 \leq k, l \leq m} = \mathbb{E}(e_k(X_j) e_l(X_j)). \quad (2.5)$$

Similarly, the estimators  $\alpha_j^{(m, N)}$  are equal to

$$\alpha_j^{(m, N)} = (A_j^{(m, N)})^{-1} \frac{1}{N} \sum_{n=1}^N Z_{\tau_{j+1}^{n, m, N}} e^m(X_j^{(n)}), \quad (2.6)$$

for  $j = 1$  to  $L - 1$ , where  $A_j^{(m, N)}$  is an  $m \times m$  matrix, with coefficients given by

$$(A_j^{(m, N)})_{1 \leq k, l \leq m} = \frac{1}{N} \sum_{n=1}^N e_k(X_j^{(n)}) e_l(X_j^{(n)}). \quad (2.7)$$

Note that  $\lim_{N \rightarrow \infty} A_j^{(m, N)} = A_j^m$  almost surely. Therefore, under  $A_2$ , the matrix  $A_j^{(m, N)}$  is invertible for  $N$  large enough. We also define  $\alpha^m = (\alpha_1^m, \dots, \alpha_{L-1}^m)$  and  $\alpha^{(m, N)} = (\alpha_1^{(m, N)}, \dots, \alpha_{L-1}^{(m, N)})$ .

Given a parameter  $a^m = (a_1^m, \dots, a_{L-1}^m)$  in  $\mathbb{R}^m \times \dots \times \mathbb{R}^m$  and deterministic vectors  $z = (z_1, \dots, z_L) \in \mathbb{R}^L$  and  $x = (x_1, \dots, x_L) \in E^L$ , we define a vector field  $F = (F_1, \dots, F_L)$  by:

$$\begin{aligned} F_L(a^m, z, x) &= z_L \\ F_j(a^m, z, x) &= z_j \mathbf{1}_{\{z_j \geq a_j^m \cdot e^m(x_j)\}} + F_{j+1}(a^m, z, x) \mathbf{1}_{\{z_j < a_j^m \cdot e^m(x_j)\}}, \text{ for } j = 1, \dots, L-1. \end{aligned}$$

We have

$$F_j(a^m, z, x) = z_j \mathbf{1}_{B_j^c} + \sum_{i=j+1}^{L-1} z_i \mathbf{1}_{B_j \dots B_{i-1} B_i^c} + z_L \mathbf{1}_{B_j \dots B_{L-1}},$$

with

$$B_j = \{z_j < a_j^m \cdot e^m(x_j)\}.$$

We remark that  $F_j(a^m, Z, X)$  does not depend on  $(a_1^m, \dots, a_{j-1}^m)$  and that we have

$$\begin{aligned} F_j(\alpha^m, Z, X) &= Z_{\tau_j^{[m]}} \\ F_j(\alpha^{(m,N)}, Z^{(n)}, X^{(n)}) &= Z_{\tau_j^{n,m,N}}^{(n)}. \end{aligned}$$

For  $j = 2$  to  $L$ , we denote by  $G_j$  the vector valued function

$$G_j(a^m, z, x) = F_j(a^m, z, x)e^m(x_{j-1}),$$

and we define the functions  $\phi_j$  and  $\psi_j$  by

$$\phi_j(a^m) = \mathbb{E}F_j(a^m, Z, X) \quad (2.8)$$

$$\psi_j(a^m) = \mathbb{E}G_j(a^m, Z, X). \quad (2.9)$$

Observe that with this notation, we have

$$\alpha_j^m = (A_j^m)^{-1}\psi_{j+1}(\alpha^m), \quad (2.10)$$

and similarly, for  $j = 1$  to  $L - 1$ ,

$$\alpha_j^{(m,N)} = (A_j^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^N G_{j+1}(\alpha^{(m,N)}, Z^{(n)}, X^{(n)}). \quad (2.11)$$

**Remark 2.1** In [9], the regression involves only in the money paths, i.e. samples for which  $Z_j^{(n)} > 0$ . This seems to be more efficient numerically. In order to stick to this type of regression, the above description of the algorithm should be modified as follows. Use

$$\hat{\tau}_j^{[m]} = j \mathbf{1}_{\{Z_j \geq \hat{\alpha}_j^m \cdot e(X_j)\} \cap \{Z_j > 0\}} + \hat{\tau}_{j+1}^{[m]} \mathbf{1}_{\{Z_j < \hat{\alpha}_j^m \cdot e(X_j)\} \cup \{Z_j = 0\}},$$

instead of  $\tau_j^{[m]}$  for  $j \leq L - 1$ , with

$$\hat{\alpha}_j^m = \arg \min_{a \in \mathbb{R}^m} \mathbb{E} \mathbf{1}_{\{Z_j > 0\}} \left( Z_{\hat{\tau}_j^{[m]}} - a \cdot e(X_j) \right)^2.$$

We analogously define  $\hat{\tau}_j^{n,m,N}$ ,  $\hat{\alpha}_j^{(m,N)}$ ,  $\hat{F}_j$  and  $\hat{G}_j$ . The convergence results still hold for this version of the algorithm (with similar proofs), provided assumptions  $A_1$  and  $A_2$  are replaced by

**$\hat{\mathbf{A}}_1$ :** For  $j = 1$  to  $L - 1$ , the sequence  $(e_k(X_j))_{k \geq 1}$  is total in  $L^2(\sigma(X_j), \mathbf{1}_{\{Z_j > 0\}} d\mathbb{P})$ .

**$\hat{\mathbf{A}}_2$ :** For  $1 \leq j \leq L - 1$  and  $m \geq 1$ , if  $\mathbf{1}_{\{Z_j > 0\}} \sum_{k=1}^m \lambda_k e_k(X_j) = 0$  a.s. then  $\lambda_k = 0$  for  $1 \leq k \leq m$ .

### 3 Convergence

The convergence of  $U_0^m$  to  $U_0$  is a direct consequence of the following result.

**Theorem 3.1** *Assume that  $A_1$  is satisfied, then, for  $j = 1$  to  $L$ , we have*

$$\lim_{m \rightarrow +\infty} \mathbb{E}(Z_{\tau_j^{[m]}} | \mathcal{F}_j) = \mathbb{E}(Z_{\tau_j} | \mathcal{F}_j),$$

in  $L^2$ .

*Proof:* We proceed by induction on  $j$ . The result is true for  $j = L$ . Let us prove that if it holds for  $j + 1$ , it is true for  $j$  ( $j \leq L - 1$ ). Since

$$Z_{\tau_j^{[m]}} = Z_j \mathbf{1}_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} + Z_{\tau_{j+1}^{[m]}} \mathbf{1}_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}},$$

for  $j \leq L - 1$ , we have

$$\begin{aligned} \mathbb{E}(Z_{\tau_j^{[m]}} - Z_{\tau_j} | \mathcal{F}_j) &= (Z_j - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)) \left( \mathbf{1}_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{Z_j \geq \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\}} \right) \\ &\quad + \mathbb{E}(Z_{\tau_{j+1}^{[m]}} - Z_{\tau_{j+1}} | \mathcal{F}_j) \mathbf{1}_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}}. \end{aligned}$$

By assumption, the second term of the right side of the equality converges to 0 and we just have to prove that  $B_j^m$  defined by

$$B_j^m = (Z_j - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)) \left( \mathbf{1}_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{Z_j \geq \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\}} \right),$$

converges to 0 in  $L^2$ . Observe that

$$\begin{aligned} |B_j^m| &= |Z_j - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)| \left| \mathbf{1}_{\{\mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j) > Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - \mathbf{1}_{\{\alpha_j^m \cdot e^m(X_j) > Z_j \geq \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\}} \right| \\ &\leq |Z_j - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)| \mathbf{1}_{\{|Z_j - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)| \leq |\alpha_j^m \cdot e^m(X_j) - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)|\}} \\ &\leq |\alpha_j^m \cdot e^m(X_j) - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)| \\ &\leq |\alpha_j^m \cdot e^m(X_j) - P_j^m(\mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j))| + |P_j^m(\mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)) - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)|. \end{aligned}$$

But

$$\alpha_j^m \cdot e^m(X_j) = P_j^m(Z_{\tau_{j+1}^{[m]}}) = P_j^m(\mathbb{E}(Z_{\tau_{j+1}^{[m]}} | \mathcal{F}_j)),$$

since  $P_j^m$  is the orthogonal projection on a subspace of the space of  $\mathcal{F}_j$ -measurable random variables. Consequently

$$\|B_j^m\|_2 \leq \|\mathbb{E}(Z_{\tau_{j+1}^{[m]}} | \mathcal{F}_j) - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\|_2 + \|P_j^m(\mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)) - \mathbb{E}(Z_{\tau_{j+1}} | \mathcal{F}_j)\|_2.$$

The first term of the right side of this inequality tends to 0 by the induction hypothesis and the second one by  $A_1$ .

□

In what follows, we fix the value  $m$  and we study the properties of  $U_0^{m,N}$  as  $N$  the number of Monte-Carlo simulations, goes to infinity. For notational simplicity, we drop the superscript  $m$  throughout the rest of the paper.

**Theorem 3.2** *Assume that for  $j = 1$  to  $L - 1$ ,  $\mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0$ . Then  $U_0^{m,N}$  converges almost surely to  $U_0^m$  as  $N$  goes to infinity. We also have almost sure convergence of  $\frac{1}{N} \sum_{n=1}^N Z_{\tau_j^{n,m,N}}^{(n)}$  towards  $\mathbb{E}Z_{\tau_j^{[m]}}$  as  $N$  goes to infinity, for  $j = 1, \dots, L$ .*

Note that with the notation of the preceding section, we have to prove that

$$\lim_N \frac{1}{N} \sum_{n=1}^N F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) = \phi_j(\alpha), \quad 1 \leq j \leq L. \quad (3.1)$$

The proof is based on the following lemmas.

**Lemma 3.1** *For  $j = 1$  to  $L - 1$ , we have :*

$$|F_j(a, Z, X) - F_j(b, Z, X)| \leq \left( \sum_{i=j}^L |Z_i| \right) \left( \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i - b_i \cdot e(X_i)| \leq |a_i - b_i| |e(X_i)|\}} \right).$$

*Proof :* Let  $B_j = \{Z_j < a_j \cdot e(X_j)\}$  and  $\tilde{B}_j = \{Z_j < b_j \cdot e(X_j)\}$ . We have :

$$\begin{aligned} F_j(a, Z, X) - F_j(b, Z, X) &= Z_j(\mathbf{1}_{B_j} - \mathbf{1}_{\tilde{B}_j}) \\ &\quad + \sum_{i=j+1}^{L-1} Z_i(\mathbf{1}_{B_j \dots B_{i-1} B_i^c} - \mathbf{1}_{\tilde{B}_j \dots \tilde{B}_{i-1} \tilde{B}_i^c}) \\ &\quad + Z_L(\mathbf{1}_{B_j^c \dots B_{L-1}^c} - \mathbf{1}_{\tilde{B}_j^c \dots \tilde{B}_{L-1}^c}). \end{aligned}$$

But

$$\begin{aligned} |\mathbf{1}_{B_j} - \mathbf{1}_{\tilde{B}_j}| &= \mathbf{1}_{\{a_j \cdot e(X_j) \leq Z_j < b_j \cdot e(X_j)\}} + \mathbf{1}_{\{b_j \cdot e(X_j) \leq Z_j < a_j \cdot e(X_j)\}} \\ &\leq \mathbf{1}_{\{|Z_j - b_j \cdot e(X_j)| \leq |a_j - b_j| |e(X_j)|\}} \end{aligned}$$

Moreover

$$\begin{aligned} |\mathbf{1}_{B_j \dots B_{i-1} B_i^c} - \mathbf{1}_{\tilde{B}_j \dots \tilde{B}_{i-1} \tilde{B}_i^c}| &\leq \sum_{k=j}^{i-1} |\mathbf{1}_{B_k} - \mathbf{1}_{\tilde{B}_k}| + |\mathbf{1}_{B_i^c} - \mathbf{1}_{\tilde{B}_i^c}| \\ &= \sum_{k=j}^i |\mathbf{1}_{B_k} - \mathbf{1}_{\tilde{B}_k}|, \end{aligned}$$

this gives

$$|F_j(a, Z, X) - F_j(b, Z, X)| \leq \sum_{i=j}^L |Z_i| \sum_{i=j}^{L-1} |\mathbf{1}_{B_i} - \mathbf{1}_{\tilde{B}_i}|.$$

Combining these inequalities, we obtain the result of Lemma 3.1. □

**Lemma 3.2** *Assume that for  $j = 1$  to  $L - 1$ ,  $\mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0$  then  $\alpha_j^{(N)}$  converges almost surely to  $\alpha_j$ .*

*Proof:* we proceed by induction on  $j$ . For  $j = L - 1$ , the result is a direct consequence of the law of large numbers. Now, assume that the result is true for  $i = j$  to  $L - 1$ . We want to prove that it is true for  $j - 1$ . We have

$$\alpha_{j-1}^{(N)} = (A_{j-1}^{(N)})^{-1} \frac{1}{N} \sum_{n=1}^N G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}).$$

By the law of large numbers, we know that  $A_{j-1}^{(N)}$  converges almost surely to  $A_{j-1}$  and it remains to prove that  $\frac{1}{N} \sum_{n=1}^N G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)})$  converges to  $\psi_j(\alpha)$ . From the law of large numbers, we have the convergence of  $\frac{1}{N} \sum_{n=1}^N G_j(\alpha, Z^{(n)}, X^{(n)})$  to  $\psi_j(\alpha)$  and it suffices to prove that :

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \left( G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \right) = 0.$$

We note  $G_N = \frac{1}{N} \sum_{n=1}^N \left( G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \right)$ . We have :

$$\begin{aligned} |G_N| &\leq \frac{1}{N} \sum_{n=1}^N |e(X_{j-1}^{(n)})| |F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_j(\alpha, Z^{(n)}, X^{(n)})| \\ &\leq \frac{1}{N} \sum_{n=1}^N |e(X_{j-1}^{(n)})| \sum_{i=j}^L |Z_i^{(n)}| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq |\alpha_i^{(N)} - \alpha_i| |e(X_i^{(n)})|\}}. \end{aligned}$$

Since, for  $i = j$  to  $L - 1$ ,  $\alpha_i^{(N)}$  converges almost surely to  $\alpha_i$ , we have for each  $\epsilon > 0$  :

$$\begin{aligned} \limsup |G_N| &\leq \limsup \frac{1}{N} \sum_{n=1}^N |e(X_{j-1}^{(n)})| \sum_{i=j}^L |Z_i^{(n)}| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq \epsilon |e(X_i^{(n)})|\}} \\ &= \mathbb{E} |e(X_{j-1})| \sum_{i=j}^L |Z_i| \sum_{i=j}^{L-1} \mathbf{1}_{\{|Z_i - \alpha_i \cdot e(X_i)| \leq \epsilon |e(X_i)|\}}, \end{aligned}$$

where the last equality follows from the law of large numbers. Letting  $\epsilon$  go to 0, we obtain the convergence to 0, since for  $j = 1$  to  $L - 1$ ,  $\mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0$ . □

The proof of Theorem 3.2 is similar to the proof of Lemma 3.2. Therefore, we omit it.

## 4 Rate of convergence of the Monte-Carlo procedure

### 4.1 Tightness

In this section we are interested in the rate of convergence of  $\frac{1}{N} \sum_{n=1}^N Z_{\tau_j^{n,N}}^{(n)}$ , for  $j = 1$  to  $L$ . Recall that  $m$  is fixed and that  $Z_j$  ( $1 \leq j \leq L$ ) and  $e(X_j)$  ( $1 \leq j \leq L - 1$ ) are square integrable variables.

We assume that :

**H<sub>1</sub>**:  $\forall j = 1, \dots, L - 1$ ,  $\limsup_{\epsilon \rightarrow 0} \frac{\mathbb{E} \bar{Y} \mathbf{1}_{\{|Z_j - \alpha_j \cdot e(X_j)| \leq \epsilon |e(X_j)|\}}}{\epsilon} < +\infty$ , where

$$\bar{Y} = \left( 1 + \sum_{i=1}^L |Z_i| + \sum_{i=1}^{L-1} |e(X_i)| \right) \left( 1 + \sum_{i=1}^{L-1} |e(X_i)| \right). \quad (4.1)$$

Note that  $H_1$  implies that  $\mathbb{P}(Z_j = \alpha_j \cdot e(X_j)) = 0$  and, consequently, under  $H_1$  we know from Section 3 that  $\frac{1}{N} \sum_{n=1}^N F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)})$  converges almost surely to  $\phi_j(\alpha)$ . Remark too that  $H_1$  is satisfied if the random variable  $(Z_j - \alpha_j \cdot e(X_j))$  has a bounded density near zero and the variables  $Z_j$  and  $e(X_j)$  are bounded.

**Theorem 4.1** *Under  $H_1$ , the sequences  $\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha)) \right)_{N \geq 1}$  ( $j = 1, \dots, L$ ) and  $(\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N \geq 1}$  ( $j = 1, \dots, L - 1$ ) are tight.*

The proof of Theorem 4.1 is based on the following Lemma.

**Lemma 4.1** *Let  $(U^{(n)}, V^{(n)}, W^{(n)})$  be a sequence of identically distributed random variables with values in  $[0, +\infty)^3$  such that*

$$\limsup_{\epsilon \rightarrow 0} \frac{\mathbb{E} \left( W^{(1)} \mathbf{1}_{\{U^{(1)} \leq \epsilon V^{(1)}\}} \right)}{\epsilon} < +\infty,$$

and  $(\theta_N)$  a sequence of positive random variables such that  $(\sqrt{N}\theta_N)$  is tight, then the sequence

$$\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N W^{(n)} \mathbf{1}_{\{U^{(n)} \leq \theta_N V^{(n)}\}} \right)_{N \geq 1} \text{ is tight.}$$

*Proof:* Let  $\sigma_N(\theta) = \frac{1}{\sqrt{N}} \sum_{n=1}^N W^{(n)} \mathbf{1}_{\{U^{(n)} \leq \theta V^{(n)}\}}$ . Observe that  $\sigma_N$  is a non decreasing function

of  $\theta$ . Let  $A > 0$ , we have

$$\begin{aligned}
\mathbb{P}(\sigma_N(\theta_N) \geq A) &\leq \mathbb{P}(\sigma_N(\theta_N) \geq A, \sqrt{N}\theta_N \leq B) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&\leq \mathbb{P}(\sigma_N(\frac{B}{\sqrt{N}}) \geq A) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&\leq \frac{1}{A} \mathbb{E}\sigma_N(\frac{B}{\sqrt{N}}) + \mathbb{P}(\sqrt{N}\theta_N > B) \\
&= \frac{\sqrt{N}}{A} \mathbb{E}(W^{(1)} \mathbf{1}_{\{U^{(1)} \leq \frac{B}{\sqrt{N}} V^{(1)}\}}) + \mathbb{P}(\sqrt{N}\theta_N > B).
\end{aligned}$$

From the assumption on  $(U^{(n)}, V^{(n)}, W^{(n)})$  and the tightness of  $(\sqrt{N}\theta_N)$ , we deduce easily the tightness of  $\sigma_N(\theta_N)$ . □

*Proof of Theorem 4.1:* We know from the classical Central Limit Theorem that the sequence  $(1/\sqrt{N}) \sum_{n=1}^N (F_j(\alpha, Z^{(n)}, X^{(n)}) - \phi_j(\alpha))$  is tight and it remains to prove the tightness of  $\frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_j(\alpha, Z^{(n)}, X^{(n)}))$ , for  $j = 1$  to  $L$ . Similarly, to prove the tightness of  $(\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N \geq 1}$ , for  $j = 1$  to  $L - 1$ , we just have to prove the tightness of  $\frac{1}{\sqrt{N}} \sum_{n=1}^N (G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}))$  (see Section 2 for the notation). We proceed by induction on  $j$ . The tightness of  $\frac{1}{\sqrt{N}} \sum_{n=1}^N (F_L(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_L(\alpha, Z^{(n)}, X^{(n)}))$  is obvious and that of  $(\sqrt{N}(\alpha_{L-1}^{(N)} - \alpha_{L-1}))$  follows from the Central Limit Theorem for the sequence  $(Z_L^{(n)} e(X_{L-1}^{(n)}))$  and the almost sure convergence of the sequence  $(A_{L-1}^{(N)})_{N \in \mathcal{N}}$ .

Assume that  $\frac{1}{\sqrt{N}} \sum_{n=1}^N (F_i(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_i(\alpha, Z^{(n)}, X^{(n)}))$  and  $(\sqrt{N}(\alpha_{i-1}^{(N)} - \alpha_{i-1}))$  are tight for  $i = j$  to  $L$ . We set

$$\bar{F}_N = \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_{j-1}(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - F_{j-1}(\alpha, Z^{(n)}, X^{(n)})).$$

Now from Lemma 3.1, we have :

$$|\bar{F}_N| \leq \frac{1}{\sqrt{N}} \sum_{n=1}^N \bar{Y}^{(n)} \sum_{i=j-1}^{L-1} \mathbf{1}_{\{|Z_i^{(n)} - \alpha_i \cdot e(X_i^{(n)})| \leq |\alpha_i - \alpha_i^{(N)}| \cdot |e(X_i^{(n)})|\}},$$

with

$$\bar{Y}^{(n)} = \left( 1 + \sum_{i=1}^L |Z_i^{(n)}| + \sum_{i=1}^{L-1} |e(X_i^{(n)})| \right) \left( 1 + \sum_{i=1}^{L-1} |e(X_i^{(n)})| \right). \quad (4.2)$$

From Lemma 4.1 and by the induction hypothesis, we deduce that  $\bar{F}_N$  is tight. In the same way, we prove that  $(\sqrt{N}(\alpha_{j-2}^{(N)} - \alpha_{j-2}))$  is tight. □

## 4.2 A central limit theorem

We prove in this section that under some stronger assumptions than in section 4.1, the vector  $\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N (Z_{\tau_j}^{(n),N} - \mathbb{E} Z_{\tau_j}^{[m]}) \right)_{j=1,\dots,L}$  converges weakly to a Gaussian vector. With the preceding notation, we have

$$\frac{1}{\sqrt{N}} \sum_{n=1}^N (Z_{\tau_j}^{(n),N} - \mathbb{E} Z_{\tau_j}^{[m]}) = \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha)). \quad (4.3)$$

In the following, we will denote by  $Y$  the pair  $(Z, X)$  and by  $Y^{(n)}$  the pair  $(Z^{(n)}, X^{(n)})$ . We will also use  $\bar{Y}$  and  $\bar{Y}^{(n)}$  as defined in (4.1) and (4.2).

We will need the following hypothesis:

**H<sub>1</sub>\***: For  $j = 1$  to  $L - 1$ , there exists a neighborhood  $V_j$  of  $\alpha_j$ ,  $\eta_j > 0$  and  $k_j > 0$  such that for  $a_j \in V_j$  and for  $\epsilon \in [0, \eta_j]$ ,

$$\mathbb{E}(\bar{Y} \mathbf{1}_{\{|Z_j - a_j \cdot e(X_j)| \leq \epsilon |e(X_j)|\}}) \leq \epsilon k_j.$$

**H<sub>2</sub>**: For  $j = 1$  to  $L$ ,  $Z_j$  and  $e(X_j)$  are in  $L^p$  for all  $1 \leq p < +\infty$ .

**H<sub>3</sub>**: For  $j = 1$  to  $L - 1$ ,  $\phi_j$  and  $\psi_j$  are  $\mathcal{C}^1$  in a neighborhood of  $\alpha$ .

Observe that  $H_1^*$  is stronger than  $H_1$ .

**Theorem 4.2** *Under  $H_1^*$ ,  $H_2$ ,  $H_3$ , the vector*

$$\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N (Z_{\tau_j}^{(n),N} - \mathbb{E} Z_{\tau_j}^{[m]}), \sqrt{N}(\alpha_j^{(N)} - \alpha_j) \right)_{j=1,\dots,L-1}$$

*converges in law to a Gaussian vector as  $N$  goes to infinity.*

For the proof of Theorem 4.2, we will use the following decomposition :

$$\begin{aligned} \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha^{(N)}, Y^{(n)}) - \phi_j(\alpha)) &= \\ \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha))) &+ \\ \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha, Y^{(n)}) - \phi_j(\alpha)) + \sqrt{N}(\phi_j(\alpha^{(N)}) - \phi_j(\alpha)). & \end{aligned}$$

From the classical Central Limit Theorem, we know that  $\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N (F_j(\alpha, Y^{(n)}) - \phi_j(\alpha)) \right)_j$  converges in law to a Gaussian vector. Moreover, we have

$$\begin{aligned}\sqrt{N}(\alpha_j^{(N)} - \alpha_j) &= (A_j^{(N)})^{-1} \frac{1}{\sqrt{N}} \sum_{n=1}^N \left( G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha) \right) \\ &\quad - (A_j)^{-1} \sqrt{N} (A_j^{(N)} - A_j) (A_j^{(N)})^{-1} \psi_{j+1}(\alpha),\end{aligned}$$

where  $A_j^{(N)}$  converges almost surely to  $A_j$  and  $\sqrt{N}(A_j^{(N)} - A_j)$  converges in law. We have the similar decomposition

$$\begin{aligned}\frac{1}{\sqrt{N}} \sum_{n=1}^N (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) &= \\ \frac{1}{\sqrt{N}} \sum_{n=1}^N \left( G_{j+1}(\alpha^{(N)}, Y^{(n)}) - G_{j+1}(\alpha, Y^{(n)}) - (\psi_{j+1}(\alpha^{(N)}) - \psi_{j+1}(\alpha)) \right) & \\ + \frac{1}{\sqrt{N}} \sum_{n=1}^N (G_{j+1}(\alpha, Y^{(n)}) - \psi_{j+1}(\alpha)) + \sqrt{N}(\psi_{j+1}(\alpha^{(N)}) - \psi_{j+1}(\alpha)). &\end{aligned}\tag{4.4}$$

Using these decompositions and Theorem 4.3 below, together with the differentiability of the functions  $\phi$  and  $\psi$ , Theorem 4.2 can be proved by induction on  $j$ .

**Theorem 4.3** *Under  $H_1^*$ ,  $H_2$ ,  $H_3$ , the variables*

$$\frac{1}{\sqrt{N}} \sum_{n=1}^N \left( F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha)) \right)$$

and

$$\frac{1}{\sqrt{N}} \sum_{n=1}^N \left( G_{j+1}(\alpha^{(N)}, Y^{(n)}) - G_{j+1}(\alpha, Y^{(n)}) - (\psi_{j+1}(\alpha^{(N)}) - \psi_{j+1}(\alpha)) \right)$$

converge to 0 in  $L^2$ , for  $j = 1$  to  $L - 1$ .

**Remark 4.1** If we try to compute the covariance matrix of the limiting distribution, we see from Theorem 4.3 and the above decompositions that it depends on the derivatives of the functions  $\phi_j$  and  $\psi_j$  at  $\alpha$ . This means that the estimation of this covariance matrix may prove difficult. Indeed, the derivation of an estimator for the derivative of a function is typically harder than for the function itself.

**Remark 4.2** Theorem 4.3 can be viewed as a way of centering the  $F_j$ -increments (resp. the  $G_{j+1}$ -increments) between  $\alpha$  and  $\alpha^{(N)}$  by the  $\phi_j$ -increments (resp. the  $\psi_{j+1}$ -increments). One way to get some intuition of the proof of Theorem 4.3 is to observe that if the sequence  $(\alpha^{(N)})_{N \in \mathbb{N}}$  were *independent* of the  $Y^{(n)}$ 's, the convergence in  $L^2$  would reduce to the convergence of  $(\alpha^{(N)})_{N \in \mathbb{N}}$  to  $\alpha$ . Indeed, we would have to consider expectations of the type

$\frac{1}{N} \mathbb{E} \left( \sum_{n=1}^N \xi_n \right)^2$  with variables  $\xi_n$  which are centered and iid, conditionally on  $\alpha^{(N)}$ . The main difficulty in the proof of Theorem 4.3 comes from the fact that  $\alpha^{(N)}$  is *not* independent of the  $Y^{(n)}$ 's. On the other hand, we do have identically distributed random variables and we will exploit symmetry arguments and the independence of  $\alpha^{(N-1)}$  and  $Y^{(N)}$ .

For the proof of Theorem 4.3, we need to control the increments of  $F_j$  and  $G_{j+1}$ . Lemma 4.2 relates these increments to indicator functions. Lemma 4.3 will enable us to localize  $\alpha^{(N)}$  near  $\alpha$ . We will then develop recursive techniques adapted to dynamic programming (see Lemma 4.3 and Lemma 4.5).

In the following, we denote by  $I(Y_i, a_i, \epsilon)$  the function

$$I(Y_i, a_i, \epsilon) = \mathbf{1}_{\{|Z_i - a_i \cdot e(X_i)| \leq \epsilon |e(X_i)|\}}.$$

Note that

$$I(Y_i, a_i, \epsilon) \leq I(Y_i, b_i, \epsilon + |b_i - a_i|). \quad (4.5)$$

The following Lemma is essentially a reformulation of Lemma 3.1.

**Lemma 4.2** *For  $j = 1$  to  $L - 1$ , and  $a, b$  in  $(\mathbb{R}^m)^{L-1}$ , we have*

$$\begin{aligned} |F_j(a, Y) - F_j(b, Y)| &\leq \bar{Y} \sum_{i=j}^{L-1} I(Y_i, a_i, |a_i - b_i|) \\ |G_j(a, Y) - G_j(b, Y)| &\leq \bar{Y} \sum_{i=j}^{L-1} I(Y_i, a_i, |a_i - b_i|) \end{aligned}$$

**Lemma 4.3** *Assume  $H_1^*$  and  $H_2$ , then for  $j = 1$  to  $L - 1$ , there exists  $C_j > 0$  such that for all  $\delta > 0$ ,*

$$\mathbb{P}(|\alpha_j^{(N)} - \alpha_j| \geq \delta) \leq \frac{C_j}{\delta^4 N^2}.$$

*Proof:* Let us recall that if  $(U_n)_n$  is a sequence of i.i.d. variables such that  $EU_1^4 < +\infty$ , we have

$$\forall \delta > 0, \quad \mathbb{P} \left( \left| \frac{1}{N} \sum_{n=1}^N U_n - \mathbb{E}U_1 \right| \geq \delta \right) \leq \frac{C}{\delta^4 N^2} \quad (4.6)$$

Observe that

$$\begin{aligned}\alpha_j^{(N)} - \alpha_j &= (A_j^{(N)})^{-1} \frac{1}{N} \sum_{n=1}^N (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) \\ &\quad - (A_j)^{-1} (A_j^{(N)} - A_j) (A_j^{(N)})^{-1} \psi_{j+1}(\alpha),\end{aligned}$$

We set  $\Omega_j^\epsilon = \{\|A_j^{(N)} - A_j\| \leq \epsilon\}$  and we choose  $\epsilon$  such that  $\|(A_j^{(N)})^{-1}\| \leq 2\|(A_j)^{-1}\|$  on  $\Omega_j^\epsilon$ . From (4.6), we know that  $\mathbb{P}((\Omega_j^\epsilon)^c) \leq \frac{C_j}{\epsilon^4 N^2}$ , for  $j = 1$  to  $L - 1$  and that, on  $\Omega_j^\epsilon$ ,

$$|\alpha_j^{(N)} - \alpha_j| \leq K' \left| \frac{1}{N} \sum_{n=1}^N (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) \right| + K\epsilon.$$

Now, since  $G_L(\alpha^{(N)}, Y^{(n)}) = Z_L^{(n)} e(X_{L-1}^{(n)})$ , we deduce from (4.6) applied with this choice of  $U_n$  that

$$\mathbb{P}(|\alpha_{L-1}^{(N)} - \alpha_{L-1}| \geq \delta) \leq \frac{C_{L-1}}{(\delta - K\epsilon)^4 N^2} + \frac{C_{L-1}}{\epsilon^4 N^2}.$$

Choosing  $\epsilon = \rho\delta$  with  $\rho$  small enough, we obtain:

$$\mathbb{P}(|\alpha_{L-1}^{(N)} - \alpha_{L-1}| \geq \delta) \leq \frac{C_{L-1}}{\delta^4 N^2}.$$

Assume now that the result of Lemma 4.3 is true for  $j + 1, \dots, L - 1$ . We will prove that  $\mathbb{P}(|\alpha_j^{(N)} - \alpha_j| \geq \delta) \leq \frac{C_j}{\delta^4 N^2}$ .

We have

$$\begin{aligned}\frac{1}{N} \sum_{n=1}^N (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) &= \frac{1}{N} \sum_{n=1}^N (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - G_{j+1}(\alpha, Y^{(n)})) \\ &\quad + \frac{1}{N} \sum_{n=1}^N (G_{j+1}(\alpha, Y^{(n)}) - \psi_{j+1}(\alpha)).\end{aligned}$$

From Lemma 4.2, we obtain on  $\Omega_j^\epsilon$ ,

$$\begin{aligned}|\alpha_j - \alpha_j^{(N)}| &\leq K\epsilon + \frac{1}{N} \sum_{n=1}^N \bar{Y}^{(n)} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|) \\ &\quad + \frac{K'}{N} \left| \sum_{n=1}^N (G_{j+1}(\alpha, Y^{(n)}) - \psi_{j+1}(\alpha)) \right|.\end{aligned}$$

The last term can be treated using (4.6). Therefore, it suffices to prove that

$$\forall \delta > 0, \quad \mathbb{P}(S_N \geq \delta) \leq \frac{C}{\delta^4 N^2},$$

where  $S_N = \frac{1}{N} \sum_{n=1}^N \bar{Y}^{(n)} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|)$ . But

$$\mathbb{P}(S_N \geq \delta) \leq \mathbb{P}\left(\frac{1}{N} \sum_{n=1}^N \bar{Y}^{(n)} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta\right) + \sum_{i=j+1}^{L-1} \mathbb{P}\left(|\alpha_i^{(N)} - \alpha_i| \geq \epsilon\right).$$

By assumption, for  $i = j + 1$  to  $L - 1$ , we have  $\mathbb{P}(|\alpha_i^{(N)} - \alpha_i| \geq \epsilon) \leq \frac{C_i}{\epsilon^4 N^2}$ . Moreover we know from  $H_1^*$  that  $\sum_{i=j+1}^{L-1} \mathbb{E} \bar{Y}^{(n)} I(Y_i^{(n)}, \alpha_i, \epsilon) \leq \epsilon K$ , with  $K = \sum_{i=j+1}^{L-1} k_j$ , so we have  $\delta - \sum_{i=j+1}^{L-1} \mathbb{E} \bar{Y}^{(n)} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta - \epsilon K$  and, using (4.6) again, we see that

$$\mathbb{P}\left(\frac{1}{N} \sum_{n=1}^N \bar{Y}^{(n)} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta\right) \leq \frac{C}{(\delta - \epsilon K)^4 N^2}.$$

Choosing  $\epsilon = \rho\delta$ , with  $\rho$  small enough, we obtain the result of Lemma 4.3.  $\square$

Before stating other technical results in preparation for the proof of Theorem 4.3, we introduce the following notations. Given  $k \in \{1, 2, \dots, L\}$ ,  $\lambda$  and  $\mu$  in  $\mathbb{R}^+$ , we define a sequence of random vectors  $(\mathcal{U}_i^{(N-k)}(\lambda, \mu), i = 1, \dots, L - 1)$ , by the recursive relations

$$\begin{aligned} \mathcal{U}_{L-1}^{(N-k)}(\lambda, \mu) &= \frac{\lambda}{N} \\ \mathcal{U}_i^{(N-k)}(\lambda, \mu) &= \frac{\lambda}{N} + \frac{\mu}{N} \sum_{n=1}^{N-k} \bar{Y}^{(n)} \sum_{j=i+1}^{L-1} I\left(Y_j^{(n)}, \alpha_j^{(N-k)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu)\right), \quad (1 \leq i \leq L - 2). \end{aligned}$$

With this definition,  $\mathcal{U}_i^{(N-k)}(\lambda, \mu)$  is obviously  $\sigma(Y^{(1)}, \dots, Y^{(N-k)})$ -measurable. We also observe that it is a *symmetric* function of  $(Y^{(1)}, \dots, Y^{(N-k)})$  (because  $\alpha^{(N-k)}$  depends symmetrically on  $Y^{(1)}, \dots, Y^{(N-k)}$ ). The next lemma establishes a useful relation between  $\mathcal{U}^{(N-k)}$  and  $\mathcal{U}^{(N-k-1)}$ .

**Lemma 4.4** *Assume  $H_2$ . There exist positive constants  $C, u, v$  such that for each  $N \in \mathbb{N}$ , one can find an event  $\Omega_N$  with  $\mathbb{P}(\Omega_N^c) \leq C/N^2$  and, on the set  $\Omega_N$  we have, for  $k \in \{1, \dots, L\}$  and  $i \in \{1, \dots, L - 1\}$ ,*

$$\mathcal{U}_i^{(N-k)}(\lambda, \mu) + |\alpha_i^{(N-k)} - \alpha_i^{(N-k-1)}| \leq \mathcal{U}_i^{(N-k-1)}\left(\lambda + (L\mu + u)\bar{Y}^{(N-k)}, v + \mu\right).$$

*Proof:* We have

$$\alpha_i^{(N-k)} = (A_i^{(N-k)})^{-1} \frac{1}{N-k} \sum_{n=1}^{N-k} G_{i+1}(\alpha^{(N-k)}, Y^{(n)}).$$

Since  $A_i^{(N)}$  is the mean of  $N$  iid random variables with moments of all orders and mean  $A_i$ , we can find  $\Omega_N$ , with  $\mathbb{P}(\Omega_N^c) = O(1/N^2)$ , on which  $\|(A_i^{(N-k)})^{-1}\| \leq 2\|A_i^{-1}\|$ , for  $k = 1, \dots, L$ ,  $i = 1, \dots, L-1$ . On this set, we have, for some positive constant  $C$ ,

$$\begin{aligned} |\alpha_i^{(N-k)} - \alpha_i^{(N-k-1)}| &\leq C \frac{\bar{Y}^{(N-k)}}{N-k} \left( 1 + \frac{1}{N-k-1} \sum_{n=1}^{N-k-1} \bar{Y}^{(n)} \right) + \\ &\frac{C}{N-k} \sum_{n=1}^{N-k-1} \left( G_{i+1}(\alpha^{(N-k)}, Y^{(n)}) - G_{i+1}(\alpha^{(N-k-1)}, Y^{(n)}) \right). \end{aligned} \quad (4.7)$$

Here we have used the inequality  $\|(N-k)A^{(N-k)} - (N-k-1)A^{(N-k-1)}\| \leq \bar{Y}^{(N-k)}$ . We may choose  $\Omega_N$  in such a way that  $\frac{1}{N-k-1} \sum_{n=1}^{N-k-1} \bar{Y}^{(n)}$  remains bounded on  $\Omega_N$ . Note that, for  $i = L-1$ , the last sum in (4.7) vanishes. Using Lemma 4.2 for  $i \leq L-2$ , we have, on  $\Omega_N$ ,

$$|\alpha_i^{(N-k)} - \alpha_i^{(N-k-1)}| \leq \frac{u\bar{Y}^{(N-k)}}{N} + \frac{v}{N} \sum_{n=1}^{N-k-1} \bar{Y}^{(n)} \sum_{j=i+1}^{L-1} I\left(Y_j^{(n)}, \alpha_j^{(N-k-1)}, |\alpha_j^{(N-k)} - \alpha_j^{(N-k-1)}|\right) \quad (4.8)$$

for some constants  $u$  and  $v$ .

To complete the proof of the lemma, we observe that

$$I\left(Y_j^{(n)}, \alpha_j^{(N-k)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu)\right) \leq I\left(Y_j^{(n)}, \alpha_j^{(N-k-1)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu) + |\alpha_j^{(N-k)} - \alpha_j^{(N-k-1)}|\right).$$

Now, for  $i \leq L-2$ , by going back to the recursive definition of  $\mathcal{U}_i^{(N-k)}(\lambda, \mu)$  and separating the  $N-k$ -th term of the sum, we obtain

$$\begin{aligned} \mathcal{U}_i^{(N-k)}(\lambda, \mu) &\leq \frac{\lambda + L\mu\bar{Y}^{(N-k)}}{N} + \\ &\frac{\mu}{N} \sum_{n=1}^{N-k-1} \bar{Y}^{(n)} \sum_{j=i+1}^{L-1} I\left(Y_j^{(n)}, \alpha_j^{(N-k-1)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu) + |\alpha_j^{(N-k)} - \alpha_j^{(N-k-1)}|\right) \end{aligned} \quad (4.9)$$

Now let  $V_i = \mathcal{U}_i^{(N-k)}(\lambda, \mu) + |\alpha_i^{(N-k)} - \alpha_i^{(N-k-1)}|$ . By combining (4.8) and (4.9) we get

$$V_i \leq \frac{\lambda + (L\mu + u)\bar{Y}^{(N-k)}}{N} + \frac{\mu + v}{N} \sum_{n=1}^{N-k-1} \bar{Y}^{(n)} \sum_{j=i+1}^{L-1} I\left(Y_j^{(n)}, \alpha_j^{(N-k-1)}, V_j\right)$$

□

**Lemma 4.5** Assume  $H_1^*$  and  $H_2$ . For all  $\varepsilon \in (0, 1]$  and for all  $\mu \geq 0$ , there exists a constant  $C_{\varepsilon, \mu}$  such that

$$\forall \lambda \geq 0, \quad \forall i \in \{1, \dots, L-1\}, \quad \mathbb{E} \mathcal{U}_i^{(N-2)}(\lambda, \mu) \leq \frac{C_{\varepsilon, \mu}(1 + \lambda)}{N^{1-\varepsilon}}.$$

*Proof:* We will prove by induction on  $k$  ( $= L, L-1, \dots, 2$ ) that

$$\sup_{k-1 \leq i \leq L-1} \mathbb{E} \mathcal{U}_i^{(N-k)}(\lambda, \mu) \leq \frac{C_{\varepsilon, \mu}(1 + \lambda)}{N^{1-\varepsilon}} \quad (4.10)$$

We obviously have (4.10) for  $k = L$ , since  $\mathcal{U}_{L-1}^{(N-L)}(\lambda, \mu) = \lambda/N$ .

We now assume that (4.10) holds for  $k+1$  and will prove it for  $k$ . For  $i \geq k-1$ , we have, using the symmetry of  $\mathcal{U}_j^{(N-k)}(\lambda, \mu)$  with respect to  $Y^{(1)}, \dots, Y^{(N-k)}$ ,

$$\mathbb{E} \mathcal{U}_i^{(N-k)}(\lambda, \mu) = \frac{\lambda}{N} + \frac{\mu(N-k)}{N} \mathbb{E} \left( \bar{Y}^{(N-k)} \sum_{j=i+1}^{L-1} I(Y_j^{(N-k)}, \alpha_j^{(N-k)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu)) \right)$$

For  $j = i+1, \dots, L-1$ , we have, using (4.5), Lemma 4.4, and the notation

$$\mathcal{V}_j^{(N-k-1)}(\lambda, \mu) = \mathcal{U}_j^{(N-k-1)}(\lambda + (L\mu + u)\bar{Y}^{(N-k)}, v + \mu),$$

$$\begin{aligned} & \mathbb{E} \left( \bar{Y}^{(N-k)} I(Y_j^{(N-k)}, \alpha_j^{(N-k)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu)) \right) \leq \\ & \mathbb{E} \bar{Y}^{(N-k)} I(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{U}_j^{(N-k)}(\lambda, \mu) + |\alpha_j^{(N-k)} - \alpha_j^{(N-k-1)}|) \\ & \leq \mathbb{E} \bar{Y}^{(N-k)} I(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)) + \mathbb{E} \bar{Y}^{(N-k)} \mathbf{1}_{\Omega_N^c} \end{aligned}$$

Note that  $\mathbb{E} \bar{Y}^{(N-k)} \mathbf{1}_{\Omega_N^c} \leq \|\bar{Y}\|_{L^2} \sqrt{\mathbb{P}(\Omega_N^c)} = O(1/N)$ .

At this point we would like to use  $H_1^*$  and the induction hypothesis. However, we have to be careful because  $\mathcal{V}_j^{(N-k-1)}(\lambda, \mu)$  depends on  $Y^{(N-k)}$ . For  $j \in \{i+1, \dots, L-1\}$ , we write

$$\mathbb{E} \bar{Y}^{(N-k)} I(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)) = \sum_{l=1}^{\infty} A_l,$$

with

$$\begin{aligned} A_l &= \mathbb{E} \bar{Y}^{(N-k)} \mathbf{1}_{\{l-1 \leq \bar{Y}^{(N-k)} < l\}} I(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)) \\ &\leq (l \mathbb{P}(\bar{Y}^{(N-k)} \geq l-1))^{1/p} \left( \mathbb{E} \bar{Y}^{(N-k)} \mathbf{1}_{\{\bar{Y}^{(N-k)} < l\}} I(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)) \right)^{1-\frac{1}{p}}, \end{aligned}$$

for all  $p \in (1, +\infty)$  (Hölder). Now,

$$\begin{aligned} \mathbb{E}\bar{Y}^{(N-k)} \mathbf{1}_{\{\bar{Y}^{(N-k)} < l\}} I\left(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)\right) \leq \\ \mathbb{E}\bar{Y}^{(N-k)} I\left(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{U}_j^{(N-k-1)}(\lambda + (L\mu + u)l, v + \mu)\right), \end{aligned}$$

and we may condition with respect to  $\sigma(Y^1, \dots, Y^{(N-k-1)})$  and use Lemma 4.3 and  $H_1^*$  to obtain

$$\begin{aligned} \mathbb{E}\bar{Y}^{(N-k)} \mathbf{1}_{\{\bar{Y}^{(N-k)} < l\}} I\left(Y_j^{(N-k)}, \alpha_j^{(N-k-1)}, \mathcal{V}_j^{(N-k-1)}(\lambda, \mu)\right) \leq \\ C\mathbb{E}\mathcal{U}_j^{(N-k-1)}(\lambda + (L\mu + u)l, v + \mu) + \frac{C}{N^2}. \end{aligned}$$

We can now apply the induction hypothesis, and we easily deduce (4.10) for  $k$ , using the fact that  $\mathbb{P}(\bar{Y}^{(N-k)} \geq l - 1) = o(1/l^m)$  for all  $m \in \mathbb{N}$ .

□

*Proof of Theorem 4.3:* We prove that

$$\lim_N \frac{1}{\sqrt{N}} \sum_{n=1}^N \left( F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha)) \right) = 0$$

in  $L^2$ . The proof is similar for the second term of the Theorem. We introduce the notation  $\Delta_j(a, b, Y) = F_j(a, Y) - F_j(b, Y) - (\phi_j(a) - \phi_j(b))$ . We have to prove that

$$\lim_N \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^N \Delta_j(\alpha^{(N)}, \alpha, Y^{(n)}) \right)^2 = 0.$$

Remark that for  $n = 1$  to  $N$ , the pairs  $(\alpha^{(N)}, Y^{(n)})$  and  $(\alpha^{(N)}, Y^{(1)})$  have the same law, and for  $n \neq m$ ,  $(\alpha^{(N)}, Y^{(n)}, Y^{(m)})$  and  $(\alpha^{(N)}, Y^{(N)}, Y^{(N-1)})$  have the same distribution. So we obtain

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^N \Delta_j(\alpha^{(N)}, \alpha, Y^{(n)}) \right)^2 &= \mathbb{E} \Delta_j^2(\alpha^{(N)}, \alpha, Y^{(1)}) \\ &\quad + (N-1) \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}). \end{aligned}$$

But  $|\Delta_j(\alpha^{(N)}, \alpha, Y^{(1)})| \leq 2(\bar{Y}^{(1)} + \mathbb{E}\bar{Y}^{(1)})$ . Since the sequence  $(\alpha^{(N)})$  goes to  $\alpha$  almost surely and  $\mathbb{P}(Z_j = \alpha_j \cdot e(X_j)) = 0$  for  $j = 1$  to  $L-1$  by assumption, we deduce that  $\Delta_j(\alpha^{(N)}, \alpha, Y^{(1)})$  goes to 0 almost surely. Consequently, we obtain that  $\mathbb{E} \Delta_j^2(\alpha^{(N)}, \alpha, Y^{(1)})$  tends to 0. It remains to prove that

$$\lim_N N \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) = 0. \quad (4.11)$$

We observe that

$$\mathbb{E} \left( \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N)}) | Y^{(1)}, \dots, Y^{(N-1)} \right) = 0,$$

since  $\mathbb{E} \left( F_j(\alpha^{(N-2)}, Y^{(N)}) | Y^{(1)}, \dots, Y^{(N-1)} \right) = \phi_j(\alpha^{(N-2)})$  almost surely. This gives

$$\mathbb{E} \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N)}) = 0,$$

and we just have to prove that

$$\lim_N N \mathbb{E} \left( \Delta_j(\alpha^{(N)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) - \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N)}) \right) = 0.$$

We have the equality

$$\begin{aligned} & \Delta_j(\alpha^{(N)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) - \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N)}) = \\ & \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) + \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}), \end{aligned}$$

We want to prove that

$$\lim_{N \rightarrow \infty} N \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) = 0 \quad (4.12)$$

and

$$\lim_{N \rightarrow \infty} N \mathbb{E} \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}) = 0. \quad (4.13)$$

Both equalities can be proved in a similar manner. We give the details for (4.13).

First, note that given any  $\eta > 0$ , we have, using  $H_2$  and Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left( \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}) \mathbf{1}_{\{|\alpha^{(N)} - \alpha| \geq N^{-\eta}\}} \right) \leq \\ C_p \left( \mathbb{P}(|\alpha^{(N)} - \alpha| \geq N^{-\eta}) \right)^{1/p}, \end{aligned}$$

for all  $p > 1$ . We know from Lemma 4.3 that  $\mathbb{P}(|\alpha^{(N)} - \alpha| \geq N^{-\eta}) \leq C/N^{2-4\eta}$ . Therefore, if  $\eta < 1/4$ ,

$$\lim_{N \rightarrow \infty} N \mathbb{E} \left( \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}) \mathbf{1}_{\{|\alpha^{(N)} - \alpha| \geq N^{-\eta}\}} \right) = 0.$$

On the other hand, we have, using Lemma 4.2,

$$\begin{aligned} \left| \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \right| & \leq \bar{Y}^{(N-1)} \sum_{i=j}^{L-1} I \left( Y_i^{(N-1)}, \alpha_i^{(N-2)}, \left| \alpha_i^{(N)} - \alpha_i^{(N-2)} \right| \right) + \\ & \left| \phi_j(\alpha^{(N)}) - \phi_j(\alpha^{(N-2)}) \right|, \end{aligned}$$

and

$$\left| \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) \right| \leq \bar{Y}^{(N)} \sum_{i=j}^{L-1} I\left(Y_i^{(N)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|\right) + \left| \phi_j(\alpha) - \phi_j(\alpha^{(N)}) \right|.$$

By the same reasoning as in the proof of Lemma 4.4, we have positive constants  $C$ ,  $s$  and  $t$  such that, for each  $N \in \mathbb{N}$ , one can find a set  $\Omega_N$  with  $\mathbb{P}(\Omega_N^c) \leq C/N^2$ , on which

$$\left| \alpha_i^{(N)} - \alpha_i^{(N-2)} \right| \leq \mathcal{U}_i^{(N-2)}\left(s(\bar{Y}^{(N-1)} + \bar{Y}^{(N)}), t\right), \quad i = j, \dots, L-1.$$

Using Lemma 4.3 and  $H_3$ , we may also assume that, on  $\Omega_N$ ,

$$\left| \phi_j(\alpha^{(N)}) - \phi_j(\alpha^{(N-2)}) \right| \leq K \sum_{i=j}^{L-1} \left| \alpha_i^{(N)} - \alpha_i^{(N-2)} \right|,$$

and

$$\left| \phi_j(\alpha^{(N)}) - \phi_j(\alpha) \right| \leq K \sum_{i=j}^{L-1} \left| \alpha_i^{(N)} - \alpha_i \right|$$

for some positive constant  $K$ .

We now have, for  $\eta < 1/4$ ,

$$\begin{aligned} \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) = \\ \sum_{l=j}^{L-1} \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \left( \bar{Y}^{(N)} I\left(Y_l^{(N)}, \alpha_l, N^{-\eta}\right) + KN^{-\eta} \right) + o(1/N). \end{aligned}$$

In order to prove (4.12), it now suffices to show that, for  $j \leq i, l \leq L-1$ ,

$$\lim_{N \rightarrow \infty} N \mathbb{E} \left( \bar{Y}^{(N-1)} I\left(Y_i^{(N-1)}, \alpha_i^{(N-2)}, V_i^{(N-2)}\right) + V_j^{(N-2)} \right) \left( \bar{Y}^{(N)} I\left(Y_l^{(N)}, \alpha_l, N^{-\eta}\right) + N^{-\eta} \right) = 0,$$

with the notation

$$V_i^{(N-2)} = \mathcal{U}_i^{(N-2)}\left(s(\bar{Y}^{(N-1)} + \bar{Y}^{(N)}), t\right).$$

We will only prove that

$$\lim_{N \rightarrow \infty} N \mathbb{E} \left( \bar{Y}^{(N-1)} I\left(Y_i^{(N-1)}, \alpha_i^{(N-2)}, V_i^{(N-2)}\right) \bar{Y}^{(N)} I\left(Y_l^{(N)}, \alpha_l, N^{-\eta}\right) \right) = 0,$$

since the other terms are easier to control. For  $m \in \mathbb{N}$ , let

$$A_m = \{m-1 \leq \bar{Y}^{(N-1)} + \bar{Y}^{(N)} < m\}.$$

We have

$$\begin{aligned} \mathbb{E} \left( \bar{Y}^{(N-1)} I \left( Y_i^{(N-1)}, \alpha_i^{(N-2)}, V_i^{(N-2)} \right) \bar{Y}^{(N)} I \left( Y_l^{(N)}, \alpha_l, N^{-\eta} \right) \mathbf{1}_{A_m} \right) &\leq \\ &\mathbb{E} \left( \bar{Y}^{(N-1)} I \left( Y_i^{(N-1)}, \alpha_i^{(N-2)}, \mathcal{U}_i^{(N-2)}(sm, t) \right) \bar{Y}^{(N)} I \left( Y_l^{(N)}, \alpha_l, N^{-\eta} \right) \right) \\ &= \mathbb{E} \bar{Y}^{(N-1)} I \left( Y_i^{(N-1)}, \alpha_i^{(N-2)}, \mathcal{U}_i^{(N-2)}(sm, t) \right) \mathbb{E} \bar{Y}^{(N)} I \left( Y_l^{(N)}, \alpha_l, N^{-\eta} \right). \end{aligned}$$

Here we have used the fact that  $Y^{(N)}$  is independent of  $(Y^{(1)}, \dots, Y^{(N-1)})$ . We now condition with respect to  $\sigma(Y^{(1)}, \dots, Y^{(N-2)})$  in the first expectation and we use  $H_1^*$  and Lemma 4.5 to obtain

$$\mathbb{E} \left( \bar{Y}^{(N-1)} I \left( Y_i^{(N-1)}, \alpha_i^{(N-2)}, V_i^{(N-2)} \right) \bar{Y}^{(N)} I \left( Y_l^{(N)}, \alpha_l, N^{-\eta} \right) \mathbf{1}_{A_m} \right) \leq C_\varepsilon \frac{1+m}{N^{1-\varepsilon+\eta}}.$$

Here  $\varepsilon$  is an arbitrary positive number and, by taking  $\varepsilon < \eta < 1/4$ , summing up over  $m$  and using Hölder's inequality we easily complete the proof. □

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