Forward and Futures Prices with Bubbles

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Abstract

This paper extends and refines the Jarrow, Protter, Shimbo [12], [13] arbitrage free pricing theory for bubbles to characterize forward and futures prices. Some new insights are obtained in this regard. In particular, we: (i) provide a canonical process for asset price bubbles suitable for empirical estimation, (ii) discuss new methods to test empirically for asset price bubbles using both spot prices and call/put option prices on the spot commodity, (iii) show that futures prices always equal their fundamental values, (iv) relate forward and futures prices under bubbles, and (v) price options on futures with asset price bubbles.

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1 Introduction

Given the bursting of the (alleged) housing price bubble and its relation to the recent subprime mortgages related financial crisis, price bubbles are again of current interest to the financial and academic community.[1] Recently, Jarrow, Protter, Shimbo [12], [13] developed a comprehensive arbitrage-free pricing theory for bubbles in complete and incomplete markets. This paper builds on their abstract analysis to provide additional insights into the pricing of spot, forward and futures contracts, transforming their abstract structure into one that is more conducive to empirical investigation.

There are two different approaches to modeling futures prices. The first approach assumes that the underlying spot commodity trades, and it postulates an evolution for the spot price process. Futures prices are then characterized in terms of the underlying spot commodity's market price process. This is the usual approach (see Duffie [8] p. 173, Shreve [27] p. 244). In contrast, the second approach assumes that the underlying spot commodity does not trade, e.g. electricity or crude oil. Here, one models the evolution of the term structure of futures prices, analogously to a Heath, Jarrow, Morton [9] model, and the spot price process is characterized as the limit of the futures price as it approaches maturity (see Carr and Jarrow [5], and Rodríguez [21]). This paper concentrates on the first approach, but comments on the second near the end of the paper.

It is hoped that this paper will motivate a new wave of empirical research into asset price bubbles. Some new results relating to spot, forward and futures contracts are proved. In particular:

1. We show that all arbitrage-free futures prices must equal their fundamental values. This is due to the fact that the fundamental values implicitly incorporate the commodity's price bubble.

2. We relate forward and futures prices, extending the classical results. Under deterministic interest rates, we show that both the market and fundamental forward and futures prices must be equal. Under stochastic interest rates, we show that both the market and fundamental forward and futures prices need not be equal, and forward prices may contain a bubble component distinct from futures prices.

3. We provide a canonical stochastic process for price bubbles - an inverse Bessel process - that is suitable for empirical investigation.

4. Our asset price bubble characterization facilitates new approaches to empirically test for bubbles using both spot prices, and call/put option prices on the spot commodity. In contrast to conventional wisdom, the satisfaction of put call parity does not imply the nonexistence of bubbles (e.g., see Battalio and Schultz [3]).

5. We price call and put futures options with bubbles generating valuation formulas that can be empirically estimated.

[1] See, for example, *New York Times*, Sunday March 11, 2007, "Crisis Looms in Mortgages: Upbeat Analyses Add Air to Loan Balloons."

An outline for this paper is as follows. Section 2 presents the model. Sections 3 and 4 analyze forward and futures contracts, respectively. Section 5 relates forward and futures prices. The canonical bubble price process is presented in section 6. Section 7 values options on spot and futures, section 8 discusses an alternative method for studying futures prices, and section 9 concludes.

2 The Model

This section presents the basic model used to understand forward and futures price bubbles. It is a slight modification of that contained in Jarrow, Protter, Shimbo [13].

2.1 The Set Up

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space. We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the "usual hypotheses." (See Protter [20] for the definition of the usual hypotheses.)

Let τ be a stopping time which represents the maturity (or life) of a risky asset.

Let $D = (D_t)_{0\leq t<\tau}$ be a càdlàg semimartingale process adapted to $\mathbb{F}$ and representing the cumulative cash flow process of the risky asset. ΔD_t can be positive or negative depending on the sign of the cash flows (e.g. storage costs are negative, dividends are positive).

Let X_τ be an $\mathcal{F}_\tau$-measurable random variable representing the time τ terminal payoff or liquidation value of the asset. We assume that $X_\tau \geq 0$.

The *market price* of the risky asset is given by the non-negative càdlàg semimartingale $S = (S_t)_{0\leq t\leq\tau}$. Note that for t such that $\triangle D_t \neq 0$, S_t denotes a price *ex-cash flows*, since S is càdlàg.

Let r_t be a non-negative càdlàg semimartingale adapted to $\mathbb{F}$ representing the default free spot rate of interest. We define a money market account A_t by

$$A_t = \exp\left(\int_0^t r_u du\right). \tag{1}$$

Note that $A_t \geq 1$ is continuous and non-decreasing.

Let W be the wealth process on $t \in [0, \infty)$ associated with the market price of the risky asset, i.e.

$$W_t = S_t \mathbf{1}_{\{t<\tau\}} + A_t \int_0^{t\wedge\tau} \frac{1}{A_u} dD_u + A_t \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau\leq t\}}. \tag{2}$$

The market value of the wealth process is the position in the risky asset plus all accumulated cash flows, and the terminal payoff if $t \geq \tau$.[2] Note that the cash

[2] When considering non-financial commodities, this expression implicitly assumes that the risky asset is storable.

flows are invested in the money market account to keep the wealth process self-financing. We assume that (D, X_τ) are such that $W \geq 0$, i.e. holding the risky asset has non-negative value. This condition is needed to be consistent with the non-negativity of the risky asset's price process.

Assumption (No Arbitrage) *There exists a probability measure Q equivalent to P such that the wealth process $\frac{W}{A}$ is a Q local martingale. We call Q an Equivalent Local Martingale Measure (ELMM).*

First, as in Jarrow, Protter, Shimbo [12], p.104, it can be shown that $\frac{W}{A}$ is a uniformly integrable martingale.[3] Next, by the First Fundamental Theorem of Asset Pricing as in Jarrow, Protter, Shimbo [13], this implies that the market is arbitrage free in the sense of "no free lunch vanishing risk" (NFLVR).[4]

Second, we do not assume the ELMM is unique, hence, the market is *incomplete*. Instead, in order to uniquely identify the price of a derivative security, we assume that the market selects a unique ELMM from the collection of all possible ELMMs. For example, this will be the case if enough static trading in call options exist (see Jacod and Protter [10], Schweizer and Wissel [26], and Carmona and Nadtochiy [4] in this regard).

2.2 Market Price Operator

To study forward and futures contracts, we need the concept of a *market price* operator. For the remainder of the paper, let $T < \infty$ represent some fixed future time that exceeds the maturity dates of all relevant forward and futures contracts.

Let $\phi = A_T \int_t^T \frac{d\Delta_u}{A_u} + \Xi^T$ denote a time T payoff where: (i) $\Delta = (\Delta_t)_{0 \leq t \leq T}$ is an arbitrary càdlàg semimartingale adapted to $\mathbb{F}$ representing the asset's cumulative cash flow process, and (ii) $\Xi^T \in \mathcal{F}_T$ is a random variable which represents the asset's terminal payoff at time T. Note that both of these quantities may be negative. The payoff $\phi \in \mathcal{F}_T$.

Let Φ_0 represent the collection of all these $\mathcal{F}_T$ measurable random variables. Define $\Phi \equiv \{\phi \in \Phi_0 : E_Q(|\phi|) < \infty\}$ where $E_Q(\cdot)$ denotes expectation under Q. By construction, Φ is a linear space.

Define $\Phi_m \subset \Phi$ to be the linear combination of the random variables generated by all admissible and self-financing trading strategies involving the risky asset and money market account[5], and all static trading strategies involving

[3] Jarrow, Protter, Shimbo [12]'s assume that $(D, X_\tau) \geq 0$. The proof is still valid with $(W, X_\tau) \geq 0$ instead.

[4] By Jarrow, Protter, Shimbo [13] this structure also implies that the market is static and not dynamic in the sense of a shifting ELMM across time.

[5] An admissible trading strategy is defined in Jarrow, Protter, Shimbo [13]. Loosely speaking, it is a self-financing trading strategy involving continuous trading in the risky asset and money market account such that there exists a lower bound so that the value of the self-financing trading strategy's value is always greater than or equal to the lower bound for all times.

forward and futures contracts and European call and put options on the risky asset. Note that both $W_T, A_T \in \Phi$ where

$$W_T = 1_{\{T<\tau\}} S_T + A_T \int_0^{T\wedge\tau} \frac{1}{A_u} dD_u + A_T \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau \le T\}}.$$

As written, this expression extends the time domain of the risky asset wealth process beyond time τ.

We assume that we are given a unique *market price* operator $\Lambda_t : \Phi_m \to \mathbb{R}_+$ that gives for each $\phi \in \Phi_m$, its time t market price $\Lambda_t(\phi)$. Note that (in the presence of bubbles) the uniqueness of the market price operator is an additional assumption beyond the existence of an ELMM Q. We do not assume that Λ_t extends uniquely to the set Φ. For future reference, we note that by the definition of the market price operator, we have that both $\Lambda_t(A_T) = A_t$ and $\Lambda_t(W_T) = S_t$.

We need to impose two additional assumptions on the market price operator. Consistent with no arbitrage, the first is sometimes known as the "law of one price."

Assumption (Linearity) *Given $\phi', \phi \in \Phi_m$ and $a, b \in \mathbb{R}$, we have that $a\Lambda_t(\phi') + b\Lambda_t(\phi) = \Lambda_t(a\phi' + b\phi)$ for all t.*

That is, we assume that a portfolio of two assets trades for the same price as the cost of constructing the portfolio by trading in the individual assets themselves. Before stating our second assumption, we need a definition.

Definition (Dominance) *Let $\phi', \phi \in \Phi_m$. We say that ϕ' dominates ϕ if either of the following conditions holds*

(a) $Q(\phi' \ge \phi) = 1$ and $Q(\phi' > \phi) > 0$ and $\Lambda_t(\phi') \le \Lambda_t(\phi)$ for some t almost surely.

(b) $Q(\phi' = \phi) = 1$ and $\Lambda_t(\phi') < \Lambda_t(\phi)$ for some t almost surely.

If ϕ' dominates ϕ, then conceptually if one could short ϕ and go long ϕ', NFLVR would imply that no dominated assets exist in the economy. However, because of the admissibility condition, one cannot always short ϕ and hold it until time T. For example, one cannot short sell the risky asset and hold it until time T if the risky asset's price process is unbounded above. This is the reason that we need to assume no dominance directly.

Assumption (No Dominance) *There are no dominated assets in the market.*

The notion of no dominance was first used by Merton [16]. Intuitively, if one asset always has a higher payoff than a second asset, then it should trade for a higher price. It is trivial to show that no dominance is equivalent to the positivity of Λ_t.[6] We can now define the notions of a fundamental price and a price bubble.

Definition (Fundamental Price and Bubble) *Define the fundamental price* $\Lambda_t^* : \Phi_m \to \mathbb{R}_+$ *of* $\phi = A_T \int_t^T \frac{d\Delta_u}{A_u} + \Xi^T \in \Phi_m$ *by*

$$\Lambda_t^*(\phi) \equiv E_Q \left(\int_t^T \frac{d\Delta_u}{A_u} + \frac{\Xi^T}{A_T} \middle| \mathcal{F}_t \right) A_t, \text{ and} \tag{3}$$

define its bubble $\delta_t : \Phi_m \to \mathbb{R}$ *by*

$$\delta_t(\phi) \equiv \Lambda_t(\phi) - \Lambda_t^*(\phi). \tag{4}$$

Note that, by construction, δ_t is a linear function and $\delta_T(\phi) = 0$, i.e. any bubble disappears by time T. The linearity follows from the linearity of both Λ_t and Λ_t^*.

To study bubbles, we need to add more structure on the bubble itself. In an NFLVR economy, all discounted market prices must be sigma martingales.[7] Hence, without loss of generality, we assume

Assumption (Local Martingale Bubble) $\frac{\delta_t(\phi)}{A_t}$ *is a Q-sigma martingale.*

Lemma (Bounded Assets) *If* $\phi \in \Phi_m$ *is bounded, then* $\delta_t(\phi) = 0$.

Proof: *If* ϕ *is bounded, then there exists* $a > 0$ *such that* $\left| A_T \int_t^T \frac{d\Delta_u}{A_u} + \Xi^T \right| \leq a$. *Then, investing a dollars in the money market account implies by no dominance that* $\Lambda_t(\phi) \leq a\Lambda_t(A_T) = aA_t$. *This implies that the Q sigma martingale* $\frac{\Lambda_t(\phi)}{A_t}$ *is bounded, and hence a martingale (see[20]). By expression (4),* $\delta_t(\phi) = 0$. ■

This lemma will prove useful below.

2.3 Examples

This section provides some examples that are utilized in the pricing of forward and futures contracts.

[6] Let $a, b \in \mathbb{R}_+$ and $\phi', \phi \in \Phi$.
Positivity: if $Q(\phi' \geq \phi) = 1$ and $Q(\phi' > \phi) > 0$, then $\Lambda_t(\phi') > \Lambda_t(\phi)$ for all t.

[7] Sigma martingales are defined and discussed for example in [20] and [12]. When a sigma martingale is bounded below, it is a local martingale. Otherwise, in general, local martingales $\subset$ sigma martingales.

2.3.1 A Zero Coupon Bond

Consider $1_T \in \Phi_m$ where 1_T is a constant dollar payoff at time T. This is a default free zero coupon bond. Then,

$$p(t,T) \equiv \Lambda_t(1_T) = \Lambda_t^*(1_T) = E_Q\left(\left.\frac{1_T}{A_T}\right| \mathcal{F}_t\right) A_t. \tag{5}$$

The second equality follows by the lemma, which implies that a default free zero coupon bond price can have no bubble.

2.3.2 The Money Market Account

Consider $A_T \in \Phi_m$. This is the money market account. Here, by definition

$$A_t \equiv \Lambda_t(A_T) \text{ and}$$

$$\delta_t(A_T) = A_t - \Lambda_t^*(A_T) = A_t - E_Q\left(\left.\frac{A_T}{A_T}\right| \mathcal{F}_t\right) A_t = 0.$$

Therefore the money market account has no bubbles. This is intuitive in that, since the money market account is the numeraire, it can have no bubbles relative to itself.

2.3.3 A Risky Asset's Price

Consider $W_T = S_T \mathbf{1}_{\{T<\tau\}} + A_T \int_0^{T\wedge\tau} \frac{1}{A_u} dD_u + A_T \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau \leq T\}} \in \Phi_m$. This represents the time T payoff from buying the risky asset at time t. Then,

$$S_t \equiv \Lambda_t\left(S_T \mathbf{1}_{\{T<\tau\}} + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u + A_T \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau \leq T\}}\right).$$

Let us define some simpler notation. Let

$$\widehat{S}_T \equiv S_T \mathbf{1}_{\{T<\tau\}} + A_T \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau \leq T\}}$$

and

$$\mathrm{div}_{t,T} \equiv \Lambda_t(A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u).$$

These represent the payoff to the risky asset at time T (less cash flows prior to T) and the market price of the cash flow stream between $[t,T]$, respectively. Then, using linearity of the market price operator, we obtain

$$S_t = \Lambda_t(\widehat{S}_T) + \mathrm{div}_{t,T}. \tag{6}$$

Here, $\Lambda_t(\widehat{S}_T) = S_t - \mathrm{div}_{t,T}$ represents the time t market price of the payoff to the risky asset at time T.

Now, the payoff to the risky asset $\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u\right)$ has the bubble component given by

$$\begin{aligned}
&\delta_t\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u\right) \\
= \; &\Lambda_t\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u\right) - \Lambda_t^*\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u\right) \\
= \; &S_t - E_Q\left(\frac{\widehat{S}_T}{A_T} + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u \middle| \mathcal{F}_t\right) A_t. \qquad (7)
\end{aligned}$$

We can relate the time t bubble component of $\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{1}{A_u} dD_u\right)$ to the bubble of S_t as in Jarrow, Protter, Shimbo [13]. They define the fundamental price of the risky asset S_t^* by

$$S_t^* = E_Q\left(\int_t^{\tau} \frac{1}{A_u} dD_u + \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau<\infty\}} \middle| \mathcal{F}_t\right) A_t \qquad (8)$$

and the asset price bubble β by

$$\beta_t = S_t - S_t^*, \qquad (9)$$

when $S \geq 0$. They show that

Theorem (Bubbles [Jarrow, Protter, Shimbo [13]])

(1) *If there exists a non-trivial bubble $\beta \not\equiv 0$, then we have three possibilities depending on the asset's life:*
$\frac{\beta}{A}$ is a local martingale (which could be a uniformly integrable martingale) if $P(\tau = \infty) > 0$.
$\frac{\beta}{A}$ is a local martingale but not a uniformly integrable martingale if is unbounded, but with $P(\tau < \infty) = 1$.
$\frac{\beta}{A}$ is a strict Q-local martingale[8], if τ is a bounded stopping time.

(2) *The risky asset's price S admits a unique decomposition (up to an evanescent set)*

$$S = S^* + \beta = S^* + (\beta^1 + \beta^2 + \beta^3) \qquad (10)$$

where $\frac{\beta}{A} = (\frac{\beta_t}{A_t})_{t\geq 0}$ is a càdlàg local martingale, and

$\frac{\beta^1}{A}$ is a càdlàg non-negative uniformly integrable martingale with $\frac{\beta_t^1}{A_t} \to \frac{X_\infty}{A_\infty}$ almost surely,
$\frac{\beta^2}{A}$ is a càdlàg non-negative non-uniformly integrable martingale with $\frac{\beta_t^2}{A_t} \to 0$ almost surely,
$\frac{\beta^3}{A}$ is a càdlàg non-negative supermartingale (and strict local martingale) such that $E_Q\left(\frac{\beta_t^3}{A_t}\right) \to 0$ and $\frac{\beta_t^3}{A_t} \to 0$ almost surely. That is, $\frac{\beta^3}{A}$ is a potential.

[8] A strict local martingale is a local martingale that is not a martingale.

Note that in this theorem, $S_t \equiv \frac{X_\tau}{A_\tau} A_t$ *for* $t \geq \tau$.

This theorem provides a decomposition of a risky asset's bubble into three types (1, 2 and 3). The type 3 bubbles, with finite maturity, will be most relevant for the subsequent analysis. As a direct consequence of this theorem, we obtain the following corollary.

Corollary (Risky Asset Price Decomposition)

$$S_t = E_Q \left(\frac{\widehat{S}_T}{A_T} + \int_t^{T\wedge\tau} \frac{dD_u}{A_u} \middle| \mathcal{F}_t \right) A_t + \beta_t^3 - E_Q \left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t. \tag{11}$$

Proof: *By expression (10), we have* $\widehat{S}_T = S_T^* + \beta_T^1 + \beta_T^2 + \beta_T^3$ *where* $\widehat{S}_T = \mathbf{1}_{\{T<\tau\}} S_T + A_T \frac{X_\tau}{A_\tau} \mathbf{1}_{\{\tau \leq T\}}$. *Dividing by* A_T*, taking conditional expectations, and using the facts that* $\frac{S_t^*}{A_t} + \int_0^t \frac{dD_u}{A_u}, \frac{\beta_t^1}{A_t}, \frac{\beta_t^2}{A_b}$ *are martingales, yields* $E_Q \left(\frac{\widehat{S}_T}{A_T} \middle| \mathcal{F}_t \right) A_t = S_t^* - E_Q \left(\int_t^{T\wedge\tau} \frac{dD_u}{A_u} \middle| \mathcal{F}_t \right) A_t + \beta_t^1 + \beta_t^2 + E_Q \left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t$. *But,* $S_t = S_t^* + \beta_t^1 + \beta_t^2 + \beta_t^3$. *Subtracting, yields* $S_t - E_Q \left(\frac{\widehat{S}_T}{A_T} \middle| \mathcal{F}_t \right) A_t = E_Q \left(\int_t^{T\wedge\tau} \frac{dD_u}{A_u} \middle| \mathcal{F}_t \right) A_t + \beta_t^3 - E_Q \left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t$*, which gives the result.* ■

This expression clarifies why the risky asset price plus cash flow process is not a martingale. It differs from a martingale by the type 3 bubble embedded in the price process. Using expression (7) gives:

$$\delta_t \left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{dD_u}{A_u} \right) = \beta_t^3 - E_Q \left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t. \tag{12}$$

The time t bubble in the market price and accumulated cash flows received at time T, $\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{dD_u}{A_u} \right)$, is seen to be equal to the type 3 bubble present in S_t. The conditional expectation of the type 3 bubble's value at time T is subtracted in order that the bubble disappears by time T, i.e. $\beta_t^3 - E_Q \left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t = 0$ at time $t = T$ almost surely.

Note that in this corollary, $E_Q \left(\frac{\widehat{S}_T}{A_T} + \int_t^{T\wedge\tau} \frac{dD_u}{A_u} \middle| \mathcal{F}_t \right) A_t$ implicitly includes both the type 1 and 2 bubbles β^1, β^2. The type 1 and type 2 bubbles are embedded within the payoff $\left(\widehat{S}_T + A_T \int_t^{T\wedge\tau} \frac{dD_u}{A_u} \right)$ itself. We will use the observation repeatedly below to help clarify the difference between forward and futures prices.

We see that strict local martingales play a key role in the analysis of bubbles, in particular for type 3 bubbles, which are particularly germane to the study of forwards and futures. This fact is simultaneously both troubling and fascinating. It is troubling because as P.A Meyer [17] showed, strict local martingales do not exist in discrete time models (they are automatically genuine martingales). And, for those who obtain their intuition from discrete time models, this intuition will be misleading in the case of type 3 bubbles, *which exist only in continuous time models*. This observation explains why continuous time models, dating back to L. Bachelier's famous thesis in 1900 [1], and revived in the mid 20^{th} century by P. Samuelson (see, eg, [22][23][24][25]), are key to an understanding of financial derivatives pricing and hedging. Although discrete time models can often provide intuition and useful approximation techniques, they have this subtle but notable limitation. In fact, one consequence of our subsequent analysis is that continuous time models generate insights which are simply not available in discrete time models.

Commodities For the subsequent analysis, we will be interested in applying these insights to forward and futures contracts. For these contracts, we only consider underlying risky assets (commodities) whose liquidation dates exceed the maturity of the contract, e.g. gold, oil, a stock index. So, without loss of generality, we assume that $T < \tau$. Then, the above expressions simplify since $\widehat{S}_T = S_T$. We repeat the relevant expressions here for subsequent reference:

$$S_t = \Lambda_t(S_T) + \text{div}_{t,T} \tag{13}$$

and

$$S_t = E_Q\left(\frac{S_T}{A_T} + \int_t^T \frac{dD_u}{A_u}\middle| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T}\middle| \mathcal{F}_t\right) A_t. \tag{14}$$

2.3.4 European Call and Put Options

European call and put options on the risky asset give the owner (the long) the right to purchase or sell, respectively, the risky asset at a fixed price (called the strike price) on a fixed date (the maturity date). This section studies European call and put options on the risky asset S with strike price K and maturity T. Here, we assume that the risky asset is a commodity, i.e. $T < \tau$.

Consider a European put option on the risky asset with payoff $P_T^S = \max[K - S_T, 0] \in \Phi_m$. Then,

$$\begin{aligned} P_t^S &= \Lambda_t(\max[K - S_T, 0]) \\ &= \Lambda_t^*(\max[K - S_T, 0]) + \delta_t(\max[K - S_T, 0]) \\ &= E_Q\left(\frac{\max[K - S_T, 0]}{A_T}\middle| \mathcal{F}_t\right) A_t + \delta_t(\max[K - S_T, 0]). \end{aligned}$$

Since the put's payoff is bounded ($P_T^S \le K$), the lemma implies that the put has no bubble component, i.e. $\delta_t(\max[K - S_T, 0]) = 0$. Hence,

$$P_t^S = E_Q\left(\frac{\max[K - S_T, 0]}{A_T}\middle| \mathcal{F}_t\right) A_t. \tag{15}$$

This is an important result. It states that even in the presence of bubbles, a put option's price equals its risk neutral valuation.

Next, consider a European call option on the risky asset with payoff $C_T^S = \max[S_T - K, 0] \in \Phi_m$. Then,

$$\begin{aligned} C_t^S &= \Lambda_t(\max[S_T - K, 0]) \\ &= \Lambda_t^*(\max[S_T - K, 0]) + \delta_t(\max[S_T - K, 0]) \\ &= E_Q\left(\frac{\max[S_T - K, 0]}{A_T} \middle| \mathcal{F}_t \right) A_t + \delta_t(\max[S_T - K, 0]). \end{aligned} \tag{16}$$

Note that the market price of the option equals its fundamental value plus a bubble. To characterize the call's price bubble $\delta_t(\max[S_T - K, 0])$, we first derive put call parity.

Theorem (Put Call Parity)

$$S_t - \mathrm{div}_{t,T} - Kp(t,T) = C_t^S + P_t^S. \tag{17}$$

Proof: *Given $S_T - K \in \Phi_m$ we write $S_T - K = \max[S_T - K, 0] + \max[K - S_T, 0]$. Then, $\Lambda_t(S_T - K) = \Lambda_t(\max[S_T - K, 0] + \max[K - S_T, 0])$. By linearity of Λ_t we obtain: $\Lambda_t(S_T) - \Lambda_t(K) = \Lambda_t(\max[S_T - K, 0]) + \Lambda_t(\max[K - S_T, 0])$. But, by expressions (5) and (13) we have: $\Lambda_t(S_T) = S_t - \mathrm{div}_{t,T}$, $\Lambda_t(K) = Kp(t,T)$. Finally, by definition, $C_t^S = \Lambda_t(\max[S_T - K, 0])$ and $P_t^S = \Lambda_t(\max[K - S_T, 0])$. Substitution yields the result.* ■

This theorem shows that put call parity holds, regardless of the existence of bubbles.

Theorem (A Call's Bubble)

$$\delta_t(S_T) = \delta_t(\max[S_T - K, 0]). \tag{18}$$

Proof: *Given $S_T - K \in \Phi_m$ we write $S_T - K = \max[S_T - K, 0] + \max[K - S_T, 0]$. Then, $\delta_t(S_T - K) = \delta_t(\max[S_T - K, 0] + \max[K - S_T, 0])$. By linearity of δ_t we obtain: $\delta_t(S_T) - \delta_t(K1_T) = \delta_t(\max[S_T - K, 0]) + \delta_t(\max[K - S_T, 0])$. But, zero-coupon bonds and puts have no bubbles, yielding the result.* ■

This theorem states that the call's price inherits the bubble present in the market price of the risky asset at time T. The call does not inherit the bubble present in the cash flow process because a European call does not receive the cash flows paid over the life of the option.

From expression (12) we obtain

$$\delta_t(\max[S_T - K, 0]) = \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u} \right). \tag{19}$$

The call option's market price only inherits the portion of the commodity's type 3 price bubble present in the market price of the risky asset. For future reference, we note that

$$C_t^S = E_Q\left(\frac{\max[S_T - K, 0]}{A_T}\middle|\mathcal{F}_t\right)A_t + \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T}\middle|\mathcal{F}_t\right)A_t - \delta_t\left(A_T\int_t^T \frac{dD_u}{A_u}\right). \tag{20}$$

Unlike for the put option, the call's price does not equal its risk neutral valuation in the present of bubbles. This observation motivates a new testing methodology for bubbles.

2.4 Testing for Bubbles using Call and Puts

This section discusses the use of call and put option prices, expressions (15) and (20), in empirical testing for the existence of asset price bubbles. As shown in the previous section, under the no dominance assumption, put call parity holds, even in the presence of asset price bubbles. Consequently, tests for violations of put call parity do not test for the existence of price bubbles, but instead, only test for the contradiction of no dominance. This is contrast to the views expressed in the literature. For example, Battalio and Schultz [3] find no violations of put call parity during the alleged internet stock price bubble, and argue that this evidence is inconsistent with an internet stock price bubble.

In contrast, in conjunction, expressions (15) and (20) provide a method to test for the existence of type 3 asset price bubbles.[9] Indeed, a "correct" model for the price operator $E_Q(\cdot|\mathcal{F}_t)$, when applied to both call and put options on the same spot commodity, would give differential results in the presence of price bubbles. Puts would be priced correctly, but calls would not. If type 3 bubbles exist, this difference would be observable. Furthermore, the mispricings would be independent of the moneyness of the options, but dependent on the time to maturity. These testable implications of call and put pricing in the presence of asset price bubbles is an open area for future research.

As an aside, it is important to note that the existing empirical literature fitting alternative option pricing models implicitly assumes that type 3 bubbles do not exist, for example, see Bakshi, Cao and Chen [2] and Pan [19]. Furthermore, this implicit assumption may have some unintended consequences. In particular, in the presence of price bubbles, estimating the parameters for the pricing operator $E_Q(\cdot|\mathcal{F}_t)$ using both call and put prices would lead to a misspecified estimation procedure. The merging of call and put prices to facilitate empirical estimation is common practice, e.g. see Pan [19].

[9] This observation was originally contained in Jarrow, Protter, Shimbo [13].

3 Forward Prices

A forward contract is a financial contract written on a risky asset S that obligates the owner (the long) to purchase the risky asset on the delivery date T for a predetermined price, called the *forward price*. If the contract is written at time t, denote the forward price by $f_{t,T}$. The payoff to the forward contract at delivery is $[S_T - f_{t,T}] \in \Phi_m$. By market convention, the forward price is selected such that the forward contract has zero initial value.

Given these definitions, it is easy to prove the following theorem.

Theorem (Forward Price)

$$f_{t,T} \cdot p(t,T) = S_t - \mathrm{div}_{t,T} \tag{21}$$

Proof: *By definition of the contract* $0 = \Lambda_t(S_T - f_{t,T})$.
Linearity implies $0 = \Lambda_t(S_T) - f_{t,T}\Lambda_t(1_T)$.
Using expression (13) and the notation for the zero coupon bond yields the final result. $0 = S_t - \mathrm{div}_{t,T} - f_{t,T}p(t,T)$. ■

Theorem (Forward Price Bubbles)

(1) $f_{t,T} \cdot p(t,T) = S_t^* - \mathrm{div}_{t,T} + \beta_t$ *where* $\beta_t = S_t - S_t^*$.

(2) $f_{t,T} \cdot p(t,T) = E_Q\left(\left.\frac{S_T}{A_T}\right| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\left.\frac{\beta_T^3}{A_T}\right| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)$.

Proof: *By expression (21), we obtain* $f_{t,T} \cdot p(t,T) + \mathrm{div}_{t,T} = S_t$, and *Property 1 follows. Finally,* $f_{t,T} \cdot p(t,T) = \Lambda_t(S_T) = E_Q\left(\left.\frac{S_T}{A_T}\right| \mathcal{F}_t\right) A_t + \delta_t(S_T)$ $= E_Q\left(\left.\frac{S_T}{A_T}\right| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\left.\frac{\beta_T^3}{A_T}\right| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)$. *The last equality uses expression (12). This yields property 2.* ■

This theorem characterizes bubbles in the forward price process. Property 2 is particularly interesting as it shows that the discounted forward price represents the fundamental value of the risky asset's payoff at time T plus the type 3 bubble present in this asset's market price. As with a call option, it does not include any bubble present in the cash flow process.

4 Futures Prices

A futures contract is similar to a forward contract. It is a financial contract, written on the risky asset S, with a fixed maturity T. It represents the purchase of the risky asset at time T via a prearranged payment procedure. The prearranged payment procedure is called marking-to-market. Marking-to-market obligates the purchaser (long position) to accept a continuous cash flow stream equal to the continuous changes in the futures prices for this contract.

The time t *futures prices*, denoted $F_{t,T}$, are set (by market convention) such that newly issued futures contracts (at time t) on the same risky asset with the same maturity date T, have zero *market value*. Hence, futures contracts (by construction) have zero market value at all times, and a continuous cash flow stream equal to $dF_{t,T}$. At maturity, the last futures price must equal the asset's price $F_{T,T} = S_T$.

Let us construct a portfolio long one futures contract. The wealth process of this portfolio at time T is given by

$$A_T \int_0^T \frac{1}{A_u} dF_{u,T} \in \Phi_m. \tag{22}$$

Note that we do not *apriori* require futures prices $(F_{t,T})_{t \geq 0}$ to be non-negative, although we will see that they are in the theorem below.

Definition (Futures Price) *The futures price process* $(F_{t,T})_{t \geq 0}$ *is any* Q *sigma martingale process such that*

$$\Lambda_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) = 0 \textit{ for all } t \in [0,T] \quad \textit{and}$$

$$F_{T,T} = S_T.$$

Note that this is different from the definition in Jarrow, Protter, Shimbo [13], who define a futures price process independently of the market price operator. The original definition does not explicitly use the fact that the futures price is that price which makes the futures contract have zero value. In constrast, the new definition does. The explicit imposition of this fact makes this definition more precise and it enables us to obtain additional results.

Theorem (No Futures Price Bubbles) *Let* $E_Q\left([F_{\cdot,T}, F_{\cdot,T}]_t^{\frac{1}{2}}\right) < \infty$ *for all* $0 \leq t \leq T$. *Then, futures prices* $F_{t,T}$ *satisfy the following three properties:*

(a) $\delta_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t} = 0$ *all* t*, i.e. there are no futures price bubbles, and*

(b) $F_{t,T} = E_Q\left(S_T \middle| \mathcal{F}_t\right) \geq 0$.

(c) $t \mapsto F_{t,T}$ is a uniformly integrable martingale in $\mathcal{H}^1$ on $[0,T]$.

Proof: *By the definition of the futures price*
$\Lambda_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) = 0$.
Or, $\Lambda_t^*(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t} + \delta_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t} = 0$.
Since $\Lambda_t^*(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t}$ *is a martingale,* $\delta_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t}$ *must be a martingale.*
This implies $\delta_t(A_T \int_t^T \frac{1}{A_u} dF_{u,T}) \frac{1}{A_t} = 0$ *all* t.
Using the definition of a futures price process, this implies that

$E_Q\left(\int_t^T \frac{1}{A_u} dF_{u,T}\middle| \mathcal{F}_t\right) = 0$ *for all* t*. Hence,* $\int_0^t \frac{1}{A_u} dF_{u,T} \equiv M_t$ *is a martingale (compute the conditional expectation).*

Then, $Y_t \equiv \int_0^t A_u dM_u = F_{t,T} - F_{0,T}$ *is a martingale since* $E_Q\left([Y,Y]_t^{\frac{1}{2}}\right) < \infty$ *for all* $0 \leq t \leq T$. (See [20].) *This implies* $F_{t,T} = E_Q\left(S_T \middle| \mathcal{F}_t\right)$ is a uniformly integrable $\mathcal{H}^1$ martingale on $[0,T]$. ■

This theorem implies that futures contracts have no bubbles, and that the futures price equals its fundamental value. The risky asset's price bubble is implicitly embedded in the futures price via the risky asset's market payoff S_T at time T. This insight will be made explicit shortly. Note also that the theorem implies that futures prices are always non-negative. We record the following useful corollary.

Corollary (Alternative Characterization) *A futures price process* $(F_{t,T})_{t\geq 0}$ *is any càdlàg semimartingale process adapted to* $\mathbb{F}$ *satisfying* $E_Q\left([F_{\cdot,T}, F_{\cdot,T}]_t^{\frac{1}{2}}\right) < \infty$ *for all* $0 \leq t \leq T$ *such that*

$$F_{t,T} = E_Q\left(S_T \middle| \mathcal{F}_t\right). \tag{23}$$

Of course, this is the classical definition of the futures price, see Duffie [8] p. 143, Shreve [27] p. 244, although they do not make the extra hypothesis here that the quadratic variation is in $L^{\frac{1}{2}}$. Using this characterization, we can investigate the relationship between the futures price and the risky asset's price bubbles.

Theorem (Futures Price Bubbles)

(1) $F_{t,T} = p(t,T)\left(S_t^* - \mathrm{div}_{t,T}\right) + cov_Q\left(\frac{S_T}{A_T}, A_T \middle| \mathcal{F}_t\right)$
$+\beta_t - \left[\beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T}\middle| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)\right]$

(2) $F_{t,T} = p(t,T) E_Q\left(\frac{S_T}{A_T}\middle| \mathcal{F}_t\right) A_t + cov_Q\left(\frac{S_T}{A_T}, A_T \middle| \mathcal{F}_t\right)$

Proof: *First, algebra yields*
$F_{t,T} = E_Q\left(A_T \middle| \mathcal{F}_t\right) E_Q\left(\frac{S_T}{A_T}\middle| \mathcal{F}_t\right) + cov_Q\left(\frac{S_T}{A_T}, A_T \middle| \mathcal{F}_t\right)$.
Using the zero coupon bond, we can write this as
$F_{t,T} = p(t,T) E_Q\left(\frac{S_T}{A_T}\middle| \mathcal{F}_t\right) A_t + cov_Q\left(\frac{S_T}{A_T}, A_T \middle| \mathcal{F}_t\right)$. *This is property 2.*
Now, $\Lambda_t(S_T) = E_Q\left(\frac{S_T}{A_T}\middle| \mathcal{F}_t\right) A_t + \delta_t(S_T)$. *Hence,*
$F_{t,T} = p(t,T)\left(\Lambda_t(S_T) - \delta_t(S_T)\right) + cov_Q\left(\frac{S_T}{A_T}, A_T \middle| \mathcal{F}_t\right)$.
But, $\Lambda_t(S_T) = S_t - \mathrm{div}_{t,T}$, $S_t = S_t^* + \beta_t$, *and*
$\delta_t(S_T) = \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T}\middle| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)$.
Substitution yields property 1. ■

Property 1 shows that the futures price inherits the first two types of bubbles present in the risky asset price $\beta_t^1 + \beta_t^2$, but not the third β_t^3. It omits the type 3 bubble because the futures price is a bet on the market price of the risky asset S_T at time T. And, when viewed from time t, this market price already excludes $\left[\beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T}\middle|\mathcal{F}_t\right)A_t - \delta_t\left(A_T\int_t^T \frac{dD_u}{A_u}\right)\right]$. Property 2 is just the classical relationship between the futures and the spot price of the risky asset. We see that bubbles do not explicitly appear in this expression. Again, they are implicitly contained in the market price S_T.

5 Forward vs Futures Prices

This section relates forward and futures prices. In the classical literature (see Jarrow and Oldfield [11] and Cox, Ingersoll, Ross [7]), it is known that forward and futures prices are equal under deterministic interest rates, but unequal (in general) otherwise. To facilitate a comparison with the classical literature and to develop some intuition concerning forward and futures price bubbles, we first study an economy with deterministic interest rates before analyzing the general case.

5.1 Deterministic Interest Rates

For this subsection, we let the spot rate be a deterministic function of time. For this section only, we assume that $A_T(S_T - F_{0,T}) \in \Phi_m$.

Theorem (Deterministic Interest Rates)

$$F_{t,T} = f_{t,T} \text{ for all } t.$$

Proof: *(this logic is from Cox, Ingersoll, Ross [7])).*
Strategy 1.
Let us consider the following trading strategy.
At each time $t \in [0,T]$, go long $N(t)$ units of the futures contract.
At each $t+dt$, invest the proceeds from the futures contract into the money market account (if negative, short).
This implies we purchase $N(t)dF_{t,T}$ dollars of the money market account at time $t+dt$, or $\frac{N(t)dF_{u,T}}{A_{t+dt}}$ units. Note that A_t is continuous, so $A_{t+dt} = A_t$.
Hold this position until time T.
Because futures contracts always have zero value and reinvestment in the money market account has no cost, this strategy is self financing.
Let the value of this portfolio be denoted $G(t)$. Note $G(0) = 0$.
Then,
$G(t) = A_t \int_0^t \frac{N(u)}{A_u} dF_{u,T}$. *Of course, we are interested in time T.*
Now, choose $N(t) = A_t$.
Then,
$G(T) = A_T \int_0^T \frac{A_u}{A_u} dF_{u,T} = A_T(F_{T,T} - F_{0,T})$.

But, $F_{T,T} = S_T$. *Hence,*
$G(T) = A_T(S_T - F_{0,T})$. *Note that by assumption,* $A_T(S_T - F_{0,T}) \in \Phi_m$.
Strategy 2.
Consider the following trading strategy with a forward contract.
At time 0 *go long* $\frac{1}{p(0,T)}$ *forward contracts and hold until time* T.
This is self financing since it is a buy and hold position.
Let the value of this portfolio be denoted $H(t)$. *Note* $H(0) = 0$. *Then,*
$H(T) = \frac{1}{p(0,T)}(S_T - f_{0,T})$.
No Dominance.
Comparison at time T.

$$G(T) = A_T(S_T - F_{0,T}), \quad H(T) = \frac{1}{p(0,T)}(S_T - f_{0,T}).$$

Under deterministic interest rates $\frac{1}{A_T} = p(0,T)$. *Then, both these strategies give the same payoff at time* T.

To avoid no dominance, $0 = \Lambda_0(G(T)) = \Lambda_0(H(T))$. *Linearity of* Λ_0 *implies that* $F_{0,T} = f_{0,T}$. ∎

This implies that under deterministic interest rates, the classical relation holds.

5.2 Stochastic Interest Rates

We now consider the general case.

Theorem (Stochastic Interest Rates)

$$\begin{aligned} f_{t,T} &= F_{t,T} + cov_Q\left(S_T, \frac{1}{A_T} \middle| \mathcal{F}_t \right) \frac{A_t}{p(t,T)} \\ &\quad + \frac{\beta_t^3}{p(t,T)} - E_Q\left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) \frac{A_t}{p(t,T)} - \delta_t \left(A_T \int_t^T \frac{dD_u}{A_u} \right). \end{aligned} \tag{24}$$

Proof: *Using expression (21), we get:*

$f_{t,T} \cdot p(t,T) = E_Q\left(S_T \middle| \mathcal{F}_t\right) E_Q\left(\frac{1}{A_T} \middle| \mathcal{F}_t\right) A_t + cov_Q\left(S_T, \frac{1}{A_T} \middle| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)$.

Combined with expression (23) yields:

$f_{t,T} \cdot p(t,T) = F_{t,T} \cdot p(t,T) + cov_Q\left(S_T, \frac{1}{A_T} \middle| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t\right) A_t - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right)$.

Algebra generates the final result. ∎

This theorem relates forward prices to futures prices. The first covariance term is the classical difference between forward and futures prices. However, when the risky asset price has a bubble, there is an additional difference reflecting the type 3 bubble. The reason for this difference is that the (present value of the) forward price is "equivalent" to the spot commodity, and hence reflects all three types of bubbles. In contrast, the futures price is a bet on the market price S_T of the commodity at time T. When viewed from time t, this excludes the type 3 bubble component. Hence, expression (24).

6 Canonical Processes for Bubbles

This section provides a canonical process for asset price bubbles that is suitable for empirical investigation into their existence.

6.1 The Inverse Bessel Process

The *inverse Bessel process* is probably the best known example of a continuous strict local martingale. (See for example [18] for a close study of the inverse Bessel process as a strict local martingale, and related strict local martingales which could serve as bubble models.) To construct this process, let V_t be a one-dimensional Bessel process starting from a constant $v_0 > 0$. In other words (see [15, page 158] for the details), V is the (strong) solution of the stochastic differential equation:

$$dV_t = \frac{1}{V_t}dt + dB_t, \qquad V_0 = v_0 > 0 \tag{25}$$

where B is a standard one-dimensional Brownian motion.

We define the inverse Bessel process by[10]

$$\beta_t^3 = \frac{1}{V_t}. \tag{26}$$

One can show that $\{\beta_t^3,\ t \geq 0\}$ is not a martingale, but a strictly positive, strict local martingale (see [6, pages 20–21]). A simple application of Itô's formula gives

$$d\beta_t^3 = -(\beta_t^3)^2 dB_t \ \ \textit{with} \ \ \beta_0^3 \ \textit{a constant} > 0. \tag{27}$$

6.2 Market Price Processes

Using expressions (14) and (23), the canonical market price processes for the risky asset and futures price, including bubbles, are:

$$\frac{S_t}{A_t} = \frac{\overline{S}_t}{A_t} + \beta_t^3 - E_Q\left(\beta_T^3 \middle| \mathcal{F}_t\right) \text{ for } t \leq T \tag{28}$$

and

$$F_{t,T} = F_{t,T}^* \text{ for } t \leq T \tag{29}$$

[10]Recall that the superscript is not an exponent but the indication that this is a type 3 bubble.

where $\overline{S}_t = E_Q\left(\frac{S_T}{A_T} + \int_t^T \frac{dD_u}{A_u}\middle| \mathcal{F}_t\right) A_t$, $F^*_{t,T} = E_Q\left(S_T\middle|\mathcal{F}_t\right)$, and β^3_t comes from (27).

In these expressions, $\overline{S}_t$ is the time t fundamental value of the risky asset payoff $\left(S_T + A_T\int_t^T \frac{dD_u}{A_u}\right)$ at time T,[11] $F^*_{t,T}$ is the time t fundamental futures price, and $\beta^3_t - E_Q\left(\beta^3_T\middle|\mathcal{F}_t\right)$ is the type 3 bubble. Subtracting the conditional expectations from the β^3 process guarantees that $\beta^3_T - E_Q\left(\beta^3_t\middle|\mathcal{F}_t\right) = 0$ when $t = T$, i.e. the type 3 bubble disappears at time T.

To use these market price processes, one needs to assume stochastic processes for A_t and $\overline{S}_t$. In this regard, one can assume any one of the standard stochastic processes used in the derivatives literature for the risky asset fundamental price process $\overline{S}_t$ (e.g. geometric Brownian motion or a Lévy process) and any spot rate process consistent with an Heath Jarrow Morton [9] term structure model for A_t (e.g. an affine process).

6.3 Testing for Bubbles using Spot Prices

One can use expression (28) to empirically test for the existence of type 3 bubbles in the risky asset price process. If there is no price bubble, then the term $\beta^3_t - E_Q\left(\beta^3_T\middle|\mathcal{F}_t\right)$ does not appear in expression (28). The process for $\frac{S_t}{A_t}$ will have different first, second and higher order moments after the inclusion of the price bubble. These differences should be empirically discernible. However, just as in the existing literature, our testing procedure for bubbles is still dependent on a joint hypothesis for the evolution of (A_t and $\overline{S}_t$).

The following theorem facilitates this empirical testing.

Theorem (Bessel Processes)

$$E_P(\beta^3_t - E_Q(\beta^3_T|\mathcal{F}_t)) = \int_{bT^{-\frac{1}{2}}}^{bt^{-\frac{1}{2}}} e^{-x^2/2}dx \quad \textit{where} \tag{30}$$

$\beta^3_t = \frac{1}{b} \geq 0$ is the initial bubble at time t.

Proof: *If $V = (V_t)_{t\geq 0}$ is a BES(3) process starting from a constant $b > 0$, then V is the (strong) solution of (25). Let Y be the coordinate process on the canonical space, and let Q denote Wiener measure, so that under Q, Y is a Brownian motion. We take $Q(Y_0 = b) = 1$. We let τ_0 denote the first hitting time of zero:*

$$\tau_0 = \inf\{t \geq 0|Y_t = 0\}.$$

Under Q, $Y_{t\wedge\tau_0}$ is a martingale and $E_Q(Y_{t\wedge\tau_0}) = b$. We can define a new probability P by taking its derivative with respect to Q to be:

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = Y_{t\wedge\tau_0}.$$

[11] The difference between $\overline{S}_t$ and S^*_t is due to the possible existence of type 1 and type 2 bubbles.

It follows from Girsanov's theorem that Y is strictly positive a.s. under P and that moreover Y satisfies the SDE (25), hence under P we have that Y is a BES(3) process. Let

$$\beta_t^3 \equiv \frac{1}{Y_t}$$

so that β^3 is an inverse Bessel process under P. We can now easily calculate $E_P(\beta_t^3)$. We get that

$$\begin{aligned} E_P(\beta_t^3) &= E_P(\frac{1}{Y_t}) = E_Q(Y_{t\wedge\tau_0}\frac{1}{Y_t}) \\ &= E_Q(Y_t\frac{1}{Y_t}1_{\{\tau_0>t\}}) = E_Q(1_{\{\tau_0>t\}}) \\ &= 1 - Q(\tau_0 \le t) = 1 - \sqrt{\frac{2}{\pi}}\int_{bt^{-\frac{1}{2}}}^{\infty} e^{-x^2/2}dx \end{aligned}$$

where the last equality uses $Q(\tau_0 \le t) = P(T_b \le t)$ where T_b is the first passage time of level b for standard Brownian motion starting at 0. The calculation for $P(T_b \le t)$ is well known, and can be found for example in [15, page 80].

Given the above, we observe that in equation (28) the expectation of the term $\beta_t^3 - E_Q\left(\beta_T^3\middle|\mathcal{F}_t\right)$ is

$$\begin{aligned} E_P(\beta_t^3) - E_P(E_Q(\beta_T^3|\mathcal{F}_t)) &= E_P(\beta_t^3) - E_P(\beta_T^3) = Q(t < \tau_0 \le T) \\ &= \int_{bT^{-\frac{1}{2}}}^{bt^{-\frac{1}{2}}} e^{-x^2/2}dx. \end{aligned}$$

This completes the proof. ■

Using this theorem, if the bubble follows an inverse Bessel process, then we know the difference in expectation between $\frac{S_t}{A_t}$ with and without bubbles. One can perform a statistical test to see whether this expectation is zero, and therefore determine whether bubbles exist. Note that this approach is distinct from that used in the existing asset pricing literature testing for bubbles. The existing literature is based on infinite horizon, discrete time models, where type 3 bubbles (of the type just discussed) do not exist. The existing empirical literature really only tests for the existence of type 2 bubbles (see Jarrow, Protter, Shimbo [13] for a more detailed discussion).

7 Options

This section prices spot and futures options on the risky asset. For clarity, we use the stochastic processes given in expressions (28) and (29) where the bubbles Z_t comes from (27). The results, however, hold for an arbitrary bubble process.

7.1 Spot

Consider European call and put options on the risky asset S with maturity date T and strike price K. For the valuation of American options, we refer the interested reader to Jarrow, Protter, Shimbo [13]. For ease of reference, we note that the risky asset's price process is:

$$\frac{S_t}{A_t} = \frac{\overline{S}_t}{A_t} + \beta_t^3 - E_Q\left(\beta_T^3 \middle| \mathcal{F}_t\right) \text{ for } t \leq T. \tag{31}$$

From expression (15), we have that the put option's market price equals its fundamental value[12], i.e.

$$P_t^S = E_Q\left(\frac{\max[K - \overline{S}_T, 0]}{A_T} \middle| \mathcal{F}_t\right) A_t. \tag{32}$$

And, from expression (20), the call option's market price equals its fundamental value plus the portion of the type 3 bubble component in the risky asset not due to the cash flows, i.e.

$$\begin{aligned} C_t^S &= E_Q\left(\frac{\max[\overline{S}_T - K, 0]}{A_T} \middle| \mathcal{F}_t\right) A_t \\ &\quad + \beta_t^3 - E_Q\left(\beta_T^3 \middle| \mathcal{F}_t\right) - \delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right). \end{aligned}$$

Assuming that the cash flows are bounded over the life of the call (a reasonable first approximation), by the lemma in section 2, we have that $\delta_t\left(A_T \int_t^T \frac{dD_u}{A_u}\right) = 0$, or

$$C_t^S = E_Q\left(\frac{\max[\overline{S}_T - K, 0]}{A_T} \middle| \mathcal{F}_t\right) A_t + \beta_t^3 - E_Q\left(\beta_T^3 \middle| \mathcal{F}_t\right). \tag{33}$$

These expressions make computing the options' market prices straightforward. For example, if the fundamental price $\overline{S}_t$ follows a geometric Brownian motion, cash flows are zero, and the interest rate is deterministic, then the put and call fundamental values equal the corresponding Black-Scholes formulas for puts and calls. Given β_t^3 comes from (27), $E_Q\left(\beta_T^3 \middle| \mathcal{F}_t\right)$ is easily computed. Note that assuming the fundamental price $\overline{S}_t$ follows a geometric Brownian is distinct from assuming that the market price S_t follows a geometric Brownian motion. In the later case, the market would be complete (under our assumptions), and no bubbles would exist (see Jarrow, Protter, Shimbo [12])).

7.2 Futures

Consider European call and put options on the futures $F_{t,T}$ with maturity date $T^* \leq T$ and strike price K. For ease of reference, we note that the

[12] At time T, $S_T = \overline{S}_T$ because $\int_T^T \frac{dD_u}{A_u} = 0$ and $\beta_T^3 - E_Q\left(\beta_T^3 \middle| \mathcal{F}_T\right) = 0$.

futures price process is

$$F_{t,T} = F^*_{t,T} \text{ for } t \leq T. \tag{34}$$

Using the correspondence[13] between expression (31) and (34), we have

$$P^F_t = E_Q\left(\left.\frac{\max[K - F^*_{T^*,T}, 0]}{A_T}\right| \mathcal{F}_t\right) A_t, \tag{35}$$

and

$$C^F_t = E_Q\left(\left.\frac{\max[F^*_{T^*,T} - K, 0]}{A_T}\right| \mathcal{F}_t\right) A_t. \tag{36}$$

We see that the market prices for futures options equal their fundamental values and contain no bubbles. For example, if $F^*_{t,T}$ follows a geometric Brownian motion, and the interest rate is deterministic, then the put and call fundamental values equal the corresponding Blacks formulas for puts and calls (see Jarrow and Turnbull [14]).

For American call and put options, the analysis is similar to spot options, but with one well-known exception. Since the fundamental futures price process $F^*_{t,T}$ is a martingale, it has no drift. This makes early exercise of American calls on futures occur with a positive probability, even in the absence of cash flows, see Jarrow and Turnbull [14] in this regard.

In contrast to options on the spot commodity, expressions (35) and (36) do not provide a method for identifying price bubbles in the underlying spot commodity. As seen in these expressions, the price operator $E_Q(\cdot | \mathcal{F}_t)$ will correctly price both call and put futures options, even in the presence of bubbles.

8 Method 2 - No S_t Trading

As mentioned in the introduction, an alternative model for studying futures prices is appropriate when the spot price does not trade. In this case, one uses an HJM [9] type model as in Carr and Jarrow [5]. For this approach, the starting point is assuming a stochastic process for the futures price term structure

$$F_{t,T} = F^*_{t,T} \tag{37}$$

for all $t \leq T \leq \tau$ such that $F^*_{t,T}$ is a Q martingale.

The spot price of the risky asset is derived via the relation

$$S_t = \lim_{t \to T} F^*_{t,T}.$$

Of course, this second approach is just the classical methodology. The classical methodology works in the presence of bubbles due to the fact that futures prices always equal their fundamental values. As shown previously, the fundamental value implicitly includes the commodity's price bubbles.

[13] Identifying $\frac{S_t}{A_t}, \frac{\overline{S}_t}{A_t}$ with $F_{T^*,T}, F^*_{T^*,T}$ and noting that futures prices have no bubbles.

9 Conclusion

This paper extends and refines the Jarrow, Protter, Shimbo [12], [13] arbitrage free pricing theory for bubbles to characterize forward and futures prices. This extension enables us to obtain a more explicit characterization of the futures price. In particular, we: (i) provide a canonical process for asset price bubbles, (ii) show that futures prices always equal their fundamental values, (iii) relate forward and futures prices under bubbles, and (iv) price options on futures with asset price bubbles. It is our hope that this paper's characterization of price bubbles will better enable empiricists to test for the existence of bubbles in both the risky asset's and their traded derivative's price processes.

References

[1] L. Bachelier, 1900, *Théorie de la Spéculation*, *Annales Scientifiques de l'École Normale Supérieure*, 21–86.

[2] G. Bakshi, C. Cao and Z. Chen, 1997, Empirical Performance of alternative option pricing models, *Journal of Finance*, 52 (5), 2003 - 2049.

[3] R. Battalio and P. Schultz, 2006, Options and the bubble, *Journal of Finance*, 61 (5), 2071-2102.

[4] R. Carmona and S. Nadtochiy, 2007, Local volatility dynamic models, preprint.

[5] P. Carr and R. Jarrow, 1995, A discrete time synthesis of derivative security valuation using a term structure of futures prices," Jarrow, Maksimoviz, Ziemba, editors, *Finance: Handbook in Operations Research and Management Science*, North Holland.

[6] K. L. Chung and R. Williams, *Introduction to Stochastic Integration, Second Edition*, 1990, Birkhäuser: Boston.

[7] J. Cox, J. Ingersoll and S. Ross, 1981, The relationship between forward prices and futures prices, *Journal of Financial Economics*, 9, 4, 321- 346.

[8] D. Duffie, 2001, *Dynamic Asset Pricing Theory*, 3rd edition, Princeton University Press, Princeton, N.J.

[9] D. Heath, R. Jarrow, and A. Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology, *Econometrica,* 60, 77-105.

[10] J. Jacod and P. Protter. Risk neutral compatibility with option prices. Preprint, 2006.

[11] R. Jarrow and G. Oldfield, 1981, Forward contracts and futures contracts, *Journal of Financial Economics*, 9, 4, 373 - 382.

[12] R. Jarrow, P. Protter, and K. Shimbo, 2006, Asset price bubbles in a complete market, *Advances in Mathematical Finance*, In Honor of Dilip B. Madan: 105 - 130.

[13] R. Jarrow, P. Protter and K. Shimbo, 2008, Asset price bubbles in incomplete markets, forthcoming, *Mathematical Finance*.

[14] R. Jarrow and S. Turnbull, 2000, *Derivative Securities,* 2nd edition, SouthWestern Pub.

[15] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, second edition, Graduate Texts in Mathematics 113, Springer, 1991.

[16] R. C. Merton, 1973, Theory of rational option pricing, *Bell Journal of Economics*, 4 (1), 141–183.

[17] P.A. Meyer, Martingales and stochastic integrals, Lecture Notes in Math. **284**, Springer, Heidelberg, Berlin, New York, 1972.

[18] S. Pal and P. Protter, Analysis of strict local martingales via h-transforms, preprint, 2008.

[19] J. Pan, 2002, The jump-risk premia implicit in options: evidence from an integrated time-series study, *Journal of Financial Economics*, 63, 3-50.

[20] P. Protter, 2005, volume 21 of *Applications of Mathematics*, Springer-Verlag, Berlin, second edition, Version 2.1.

[21] J. F. Rodríguez, 2005, *A Modified Barlow Model Applied to Electricity Derivative Pricing*, PhD Thesis, Cornell University.

[22] Samuelson, P., 1965, *Rational Theory of Warrant Pricing*, *Industrial Management Review*, **6** 13–39.

[23] Samuelson, P., 1965, *Proof That Properly Anticipated Prices Fluctuate Randomly*, *Industrial Management Review*, **6** 41–49.

[24] Samuelson, P., 1973, *Mathematics of Speculative Price*, *SIAM Review*, **15** 1–42.

[25] Samuelson, P., 2002, *Modern finance theory within one lifetime*, *Mathematical Finance - Bachelier Congress 2000*, edited by Geman, H., Madan, D., Pliska, S.R., and T. Vorst; Springer-Verlag, Heidelberg, 41–46.

[26] M. Schweizer and J. Wissel, 2008, Term structures of implied volatilities: Absence of arbitrage and existence results, *Math. Finance*, 18, 77–114.

[27] S. Shreve, 2004, *Stochastic Calculus for Finance II: Continuous Time Models*, Springer Verlag, New York.