Option Pricing with Liquidity Risk

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Abstract

This paper provides a model for pricing options in an economy with liquidity risk. Liquidity risk is modeled as a stochastic supply curve for the underlying stock that depends on the size of a trade. Consistent with the market microstructure literature, large buys increase the purchase price while large sales decrease the selling price. The arbitrage free cost of constructing an option is derived and the liquidity cost characterized. An extended Black Scholes economy is utilized to illustrate the theory and to provide some empirical estimates of the impact of liquidity costs on the option's value.

1 Introduction

Risk management is concerned with controlling three financial risks: market risk, credit risk, and liquidity risk. Market risk is the risk of price fluctuations in financial securities due to changes in either equity prices, interest rates, commodities, or foreign currencies. Credit risk is the risk of price fluctuations in financial contracts due to default, and liquidity risk is the risk of price fluctuations in financial securities due to supply/demand considerations related to the quantity impact of a trade's size on the price obtained.[1] Starting with the Black-Scholes-Merton option pricing formula in the nineteen seventies, risk management theory has successfully modeled both market and credit risk. Although practice and research is still refining the use of the various models in this area, the abstract theory is well understood (see Duffie [4], Bielecki and Rutkowski [2]). Only liquidity risk is an unsolved modeling problem in finance from a financial engineering perspective.

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[1]Two other risks are often discussed in the literature: operational risk and model risk. Operational risk is the risk of changes in a portfolio's value due to malfeasance or miscommunication in the execution of a trade. Model risk is the risk of changes in a portfolio's value due to model error. These risks are due to the legal system and model usage, respectively, and as such can be argued to be non-financial.

From a risk management perspective, the need is paramount for a simple yet robust method that incorporates liquidity risk into arbitrage pricing theory. The market microstructure literature (see Kyle [14], Glosten and Milgrom [5], Grossman and Miller [6]), although conceptually useful, is lacking in this regard. As a first solution to this problem, liquidity risk has recently been incorporated into arbitrage pricing theory using the notion of a convenience yield (see Jarrow and Turnbull [11], Jarrow [10]). Convenience yields have a long history in the context of commodity pricing. This solution to the problem successfully captures that component of liquidity risk due to inventory considerations. And, more importantly, it retains the price taking condition so that classical arbitrage pricing theory can still be applied (although in the context of an incomplete market). Nonetheless, this convenience yield approach to the inclusion of liquidity risk has an important omission. This approach doesn't explicitly capture the impact of different trade sizes on the price. Consequently, there is no notion of a bid/ask spread in this model structure. This is a significant omission because all markets experience price inelasticities (quantity impacts) and bid/ask spreads.

Recently, Cetin, Jarrow and Protter [3] have provided a general methodology for modeling liquidity risk that overcomes the limitations of the convenience yield approach. They hypothesize the existence of a stochastic supply curve for a security's price as a function of trade size, for which traders act as price takers. Analogous to Heath, Jarrow and Morton [7], they study conditions on the supply curve such that there are no arbitrage opportunities in the economy (appropriately defined). Given an arbitrage free evolution, conditions for a complete market are characterized and the pricing of derivatives studied.

The purpose of this paper is to utilize this liquidity risk arbitrage pricing theory to price and hedge a European call option on a stock. We study an extended Black-Scholes economy to illustrate the approach. In this economy, stock prices follow a geometric Brownian motion, subject to a quantity impact of trade size on the price. To illustrate the theory and estimation, the simplest quantity impact on the stock price is imposed. In this extended Black-Scholes economy, it is shown that the market is (approximately) complete and that in pricing options, liquidity costs comprise a significant percentage of the option's value. Although we calibrate the model's parameters with market data, this is by no means an empirical paper. Both the stochastic supply curve and the empirical procedure need to be significantly generalized to obtain a more realistic model that incorporates both stochastic bid/ask spreads and, perhaps, stochastic volatilities. This generalization and its empirical evaluation await future research.

An outline for this paper is as follows. Section 2 describes the model. Section 3 prices a European call option in an extended Black-Scholes economy. Section 4 estimates the magnitude of the liquidity costs associated with an option's value, and section 5 concludes the paper.

2 The Model

This section presents the basics of the liquidity risk pricing model. We are given a filtered probability space $(\Omega, F, (F_t)_{0\leq t\leq T}, \mathbf{P})$ satisfying the usual conditions where T is a fixed time. $\mathbf{P}$ represents the statistical or empirical probability measure. We consider a market for a stock, although the model applies equally well to interest rates, bonds, commodities, or foreign currencies. We will initially assume that ownership of the stock has no cash flows associated with it. Also traded is a money market account that accumulates value at the spot rate of interest.

2.1 The Supply Curve

Following Cetin, Jarrow, Protter [3], we consider an arbitrary trader who acts as a price taker with respect to an exogenously given supply curve for shares bought or sold of this stock. More formally, let $S(t,x)$ represent the stock price, *per share*, at time $t \in [0,T]$ that the trader pays/receives for an order of size x normalized by the value of a money market account starting at time 0.[2] A positive order ($x > 0$) represents a buy, a negative order ($x < 0$) represents a sale, and a zero order ($x = 0$) corresponds to the marginal trade. This is the classical structure often imposed when studying asset pricing theory with one exception. Rather than the trader facing a horizon supply curve as in the classical theory (the same price for any order size), the trader now faces a supply curve that depends on his order size.[3]

We assume that the stochastic supply curve is non-decreasing in x (i.e. $x \leq y$, $S(t,x) \leq S(t,y)$), F_t-measurable and non-negative, C^2 in its second argument, $\partial S(t,x)/\partial x$ is uniformly bounded for all x and continuous in t, $\partial^2 S(t,x)/\partial x^2$ is continuous in t, and that $S(t,x)$ is a continuous semi-martingale for all x.[4] Except for the first property, these restrictions are all technical in nature. The essence of the first property is that the larger the purchase (or sale), the larger the price impact that occurs on the share price. This is the usual situation faced in asset pricing markets, where the quantity impact on the price is due to either information effects or supply/demand imbalances (see Kyle [14], Glosten and Milgrom [5], Grossman and Miller [6]). This structure can be viewed as an extension of the model in Jouini [12] and Jouini and Kallal [13] where bid and ask prices follow separate stochastic processes, to a continuum of stochastic processes indexed by trade size.

The techniques for handling arbitrage opportunities and complete markets follow the standard approach (e.g. Duffie [4]) with some notable exceptions. We will concentrate on these exceptions in the subsequent discussion.

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[2]Formally, let s_t be the stock price in dollars and $b_t = e^{\int_0^t r_u du}$ be the value of a money market account with r_t the spot rate of interest, then $S_t = s_t/b_t$. This is equivalent to an economy with $r_t = 0$.

[3]In contrast, the trader is assumed to have no quantity impact due to his trades in the money market account.

[4]Cetin, Jarrow, Protter [3] work under a weaker set of hypotheses with $S(t,x)$ a general semi-martingale.

2.2 Trading Strategies

A *trading strategy* is a triplet $((X_t, Y_t : t \in [0,T]), \tau)$ where X_t represents the trader's aggregate stock holding at time t (units of the stock), Y_t represents the trader's aggregate money market account position at time t (units of the money market account), and τ represents the liquidation time of the stock position, subject to the following restrictions: (a) X_t and Y_t are predictable and optional processes, respectively, with $X_{0-} \equiv Y_{0-} \equiv 0$ and $X_T = 0$, and (b) $X = H1_{[0,\tau)}$ for some predictable process $H(t,\omega)$ where τ is a predictable $(F_t : 0 \leq t \leq T)$ stopping time with $\tau \leq T$.

Condition (a) states that a trading strategy must be a predictable process that starts with zero units of both the stock and money market account, and returns to zero units of the stock at time T. Given a quantity impact on prices, liquidation is essential to determine the value of a trading strategy because paper wealth and real wealth are distinct notions in this context (see Jarrow [8], [9] for related discussion). This insight explains condition (b) where the stock position must be liquidated prior to time T at a predictable stopping time τ (i.e. $X_\tau = 0$). This is the first exception to the standard techniques.

We are interested in a particular type of trading strategy - those that are self-financing. By definition, a self-financing trading strategy generates no cash flows for all times $t \in [0,T)$. That is, purchase/sales of the stock must be obtained via borrowing/investing in the money market account. The second exception to the standard approach occurs in constructing the self-financing condition.

In the standard approach, the self-financing condition is based on the portfolio's time t value, which is uniquely defined given a trading strategy. However, given quantity impacts on the price, a trading strategy no longer has a unique time t value. Indeed, any price on the supply curve (up to the current share holdings) gives a plausible price for use in valuation. For example, the value of a trading strategy under immediate liquidation differs from the value of a trading strategy under slow liquidation. This non-uniqueness invalidates the standard self-financing condition. What is unique, however, is the cost of forming the current stock position via a trading strategy. This cost is used to define the self-financing condition.

A *self-financing trading strategy* (s.f.t.s.) is a trading strategy (X, Y, τ) where X_t is cadlag with square integrable jumps ($\sum_{0 \leq u \leq T} \Delta X_u^2 < \infty$) and finite quadratic variation ($[X,X]_T^c < \infty$), (b) $Y_0 = -X_0 S(0, X_0)$, and (c)

$$Y_t = Y_0 + X_0 S(0, X_0) + \int_0^t X_{u-} dS(u,0) - X_t S(t,0) \tag{1}$$

$$- \sum_{0 \leq u \leq t} \Delta X_u [S(u, \Delta X_u) - S(u,0)] - \int_0^t \frac{\partial S}{\partial x}(u,0) d[X,X]_u^c \quad for\ 0 < t \leq T$$

Condition (a) imposes restrictions on the class of acceptable trading strategies. The reasons for these restrictions are so that expression (1) is finite given liquidity costs. Under X_t cadlag, the right side of expression (1) is well-defined, but the last two terms (always being non-positive) may be negative infinity. The classical theory, under frictionless and competitive markets, does not need these restrictions. Condition (b) follows because any holdings in the stock at time 0 must be financed from the money market account, i.e.[5]

$$Y_0 = -[X_0 - X_{0-}]S(0, X_0 - X_{0-}) = -X_0 S(0, X_0).$$

Condition (c) is the self-financing condition at time $t > 0$. The money market account equals its value at time 0 (the first two terms), plus the accumulated trading gains (evaluated at the marginal trade), less the cost of attaining this position evaluated at the marginal trade, less the price impact costs of discrete changes in share holdings, and less the price impact costs of continuous changes in the share holdings. The price impact costs of discrete changes capture the quantity impact of the share size on the price through the quantity $[S(u, \Delta X_u) - S(u, 0)]$. The analogous quantity impact is captured for continuous changes by the quantity $\frac{\partial S}{\partial x}(u, 0)$.

2.3 Liquidity Costs

To enable the characterization of the liquidity cost of a trading strategy, we need to define the marked-to-market value of a non-zero initial investment s.f.t.s. (X, Y, τ) at time t. The *marked-to-market value* equals the value of the money-market account plus the value of the stock position evaluated at the price of the marginal trade, i.e.

$$Z_t \equiv Y_t + X_t S(t, 0). \tag{2}$$

This is a purely hypothetical notion because it does not reflect the value of the stock position under any possible liquidation strategy. Nonetheless, it is a useful construct for defining the liquidity costs of a trading strategy.

To define the liquidity cost, we first need to construct the hypothetical value of a money market account for a self-financing trading strategy as if it has always been executed at the marginal trade. Given X, this hypothetical s.f.t.s. (X, Y^0, τ) is defined by:

$$Y_0^0 \equiv -X_0 S(0, 0)$$

and

$$Y_t^0 \equiv \int_0^t X_{u-} dS(u, 0) - X_t S(t, 0) \quad \text{for} \quad 0 < t \leq T.$$

[5]When studying complete markets, condition (b) of the self-financing trading strategy is removed so that $Y_0 + X_0 S(0, X_0) \neq 0$.

Continuing, we analogously define the marked-to-market value of this hypothetical s.f.t.s. (X, Y^0, τ) as

$$Z_t^0 \equiv Y_t^0 + X_t S(t, 0).$$

The liquidity cost of a s.f.t.s. (X, Y, τ) is defined to be the difference between these two constructs, i.e.

$$L_t \equiv Z_t^0 - Z_t.$$

Combined with expressions (1) and (2) this gives:

$$L_t = \sum_{0 \le u \le t} \Delta X_u [S(u, \Delta X_u) - S(u, 0)] + \int_0^t \frac{\partial S}{\partial x}(u, 0) d[X, X]_u^c \ \ge 0 \qquad (3)$$

where $L_{0-} = 0$, $L_0 = X_0[S(0, X_0) - S(0, 0)]$ and L_t is non-decreasing in t.

Expression (3) shows that the liquidity cost is non-negative and non-decreasing in t. It consists of two components. The first is due to discontinuous changes in the share holdings. The second is due to the continuous component. Note that because $X_{0-} = Y_{0-} = 0$, $\Delta L_0 = L_0 - L_{0-} = L_0 > 0$ is possible. Also, if the trading strategy is continuous, then first term in (3) always equals $X_0[S(0, X_0) - S(0, 0)]$ after time 0. If the trading strategy is also of finite variation, then the second term in (3) is zero. Thus, if one can use a trading strategy that is both continuous and of finite variation, then the entire liquidity cost of the strategy is due to forming the initial position, $L_0 = X_0[S(0, X_0) - S(0, 0)]$. Because continuous and finite variation trading strategies can approximate more general trading strategies, this insight can be used to minimize liquidity costs in the replication of contingent claims.

2.4 Arbitrage Opportunities

An arbitrage opportunity can now be defined in the standard way, but using the liquidation value at time T, Y_T, in the definition. An *arbitrage opportunity* is a s.f.t.s. (X, Y, τ) such that

$$\mathbf{P}\{Y_T \ge 0\} = 1 \ \ \textit{and} \ \ \mathbf{P}\{Y_T > 0\} > 0 \ \ \textit{where} \ \ Y_0 + X_0 S(0, X_0) = 0. \qquad (4)$$

Using this definition, Cetin, Jarrow, Protter [3] show that an extended first fundamental theorem of asset pricing holds under the above formulation.

For our implementation, we utilize the following theorem proved therein

Theorem 1 *Given a mild restriction on the class of self-financing trading strategies*[6]*, the supply curve is arbitrage free if there exists a* $\mathbf{Q} \sim \mathbf{P}$ *such that* $S(\cdot, 0)$ *is a Q-martingale.*

[6] Let $(X_- \cdot s)_t \equiv \int_0^t X_{u-} dS(u, 0)$, $\mathcal{H}^1 \equiv \{s.f.t.s.(X, Y, \tau) \mid (X_- \cdot s)_t$ is an integrable process with respect to $\mathbf{Q}\}$ and $\Theta_\alpha \equiv \{s.f.t.s.(X, Y, \tau) \in \mathcal{H}^1 \mid (X_- \cdot s)_t \ge -\alpha$ for all t almost surely for some non-negative constant $\alpha\}$. The restriction is that $(X, Y, \tau) \in \Theta_\alpha$.

As seen, this theorem reduces the condition for an arbitrage-free economy with liquidity risk to the standard condition that is well-understood and well-utilized in the existing literature.

2.5 Market Completeness

In an illiquid market, market completeness is also defined in the standard way, but again utilizing the liquidation value at time T, Y_T, in the definition. Indeed, let Q be an equivalent martingale measure for the marginal stock price, then a market is *complete* if given any F_T- measurable random variable C with $E^Q(C^2) < \infty$, there exists a non-zero initial investment[7] s.f.t.s. (X, Y, τ) such that $Y_T = C$. The random variable C is called a *contingent claim*. Unfortunately, Cetin, Jarrow, Protter show that market with illiquidities are, in general, incomplete. However, to reclaim the existing theory, they extend market completeness to the notion of an approximately complete market.

Definition 2 *The market is said to be approximately complete if given any contingent claim C, there exists a sequence of non-zero initial investment s.f.t.s. (X^n, Y^n, τ^n) such that $Y_T^n \to C$ as $n \to \infty$ in $L^2(dQ)$.*[8]

Using the notion of an approximately complete market, Cetin, Jarrow, Protter [3] prove the following theorem.

Theorem 3 *If a unique martingale measure exists for $S(t, 0)$, then the market is approximately complete.*

Again, this theorem reduces the condition for an approximately complete market with liquidity risk to the standard condition that is well-understood and well-utilized in the existing literature.

In particular, for our implementation, Cetin, Jarrow, Protter also show that if the martingale measure is unique, then given any contingent claim C, there exists a predictable X such that $C = c + \int_0^T X_u ds_u$ and a sequence of non-zero initial investment s.f.t.s. (X^n, Y^n, τ^n) with X^n continuous, of finite variation and with $X_0^n = X_0$ for all n such that $Y_T^n \to C$ in $L^2(dQ)$. We call this non-zero initial investment s.f.t.s. (X^n, Y^n, τ^n) the *approximate replicating portfolio* for C. Note that in the statement of the approximate replicating portfolio, the trading strategy utilized is both continuous and of finite variation. As such, the liquidity costs of this approximate replicating portfolio are equal to the initial liquidity cost of constructing the position, i.e. $L_t = L_0$. The approximation to the contingent claim's cash flows at time T is given by

$$Y_T^n = Y_0 + X_0 S(0, X_0) - L_0 + \int_0^T X_{u-}^n dS(u, 0). \tag{5}$$

[7] See footnote 4 in this regard.

[8] The space $L^2(dQ)$ is the set of F_T- measurable random variables that are square integrable using the probability measure Q.

This follows from expression (1) with $X_T^n = 0$ and $L_T^n = L_0$. The quality of this approximation is determined by the position in the stock X_t^n. Cetin, Jarrow, Protter use the approximating trading strategy determined by

$$X_t^n = n \int_{t-\frac{1}{n}}^{t} X_s ds.$$

Note that this position in the stock is continuous and of finite variation (even if X_t is not) due to the smoothing action of the integral.

2.6 Contingent Claim Valuation

Given this characterization of the liquidity cost, Cetin, Jarrow, Protter [3] characterize the time 0 price of any contingent claim in an approximately complete market. The characterization result is as follows.

Theorem 4 *Suppose there exists a unique* $\mathbf{Q} \sim \mathbf{P}$ *such that* $S(\cdot, 0)$ *is a Q-martingale. Let* X_t *be the predictable process such that* $C = c + \int_0^T X_u ds_u$ *, then the time* 0 *value of a long position in a contingent claim* C *is equal to*

$$E^Q(C) + L_0 = Y_0 + X_0 S(0, X_0) \tag{6}$$

and a short position in a contingent claim C *is equal to*

$$E^Q(C) - \overline{L}_0 = Y_0 + X_0 S(0, -X_0) \tag{7}$$

where $L_0 = X_0[S(0, X_0) - S(0, 0)]$ *and* $\overline{L}_0 = -X_0[S(0, -X_0) - S(0, 0)]$ *.*

This pricing theorem is used to price a European call option in the next section. The approximate replicating portfolio for this contingent claim (X^n, Y^n, τ^n) has X^n being continuous and of finite variation in order to avoid all liquidity costs after time 0. Recall that in an approximately complete market, given any contingent claim C, there exists a predictable X such that $C = c + \int_0^T X_u ds_u$ and a sequence of non-zero initial investment s.f.t.s. (X^n, Y^n, τ^n) with X^n continuous, of finite variation and with $X_0^n = X_0$ for all n such that $Y_T^n \to C$ in $L^2(dQ)$. The initial stock holdings, X_0^n, is selected to match the initial position X_0 that would have been used to replicate the contingent claim in a world with no liquidity costs. Furthermore, this construction generates a terminal value Y_T^n for the portfolio that is arbitrarily close to C, with a given initial cost of L_0. Using expressions (5) and (6), we have that for a long position in the contingent claim the terminal value of the approximate replicating portfolio is:

$$Y_T^n = E^Q(C) + \int_0^T X_{u-}^n dS(u, 0). \tag{8}$$

The approximation error between Y_T^n and C is determined by the difference between X_t^n and X_t. Indeed, we can write

$$C = E^Q(C) + \int_0^T X_u dS(u, 0)$$

so that

$$C - Y_T^n = \int_0^T (X_u - X_{u-}^n) dS(u, 0). \tag{9}$$

An explicit example of this approximation is presented in the next section.

3 European Call Option Valuation

We apply the above model to an extended Black-Scholes economy that depends on only one additional parameter α. To obtain this economy, we let the supply curve satisfy

$$S(t, x) = e^{\alpha x} S(t, 0) \quad \textit{with } \alpha > 0 \tag{10}$$

and

$$S(t, 0) \equiv \frac{s_t}{e^{rt}} = \frac{s_0 e^{\mu t + \sigma W_t}}{e^{rt}} \tag{11}$$

where μ, σ are constants and W_t is a standard Brownian motion initialized at zero. We let the spot rate of interest be constant and equal r per unit time. The normalization by the money market account's value is made explicit in expression (11).

We discuss these expressions in reverse order. Expression (11) states that the marginal stock price follows a geometric Brownian motion. Geometric Brownian motion is the stochastic process underlying the standard Black-Scholes economy. The extended Black-Scholes economy is obtained by adding expression (10). This form of the supply function was chosen for simplicity, in order to illustrate the theory in the simplest setting. Note that the supply curve is increasing in x and it satisfies the technical properties required in Cetin, Jarrow and Protter [3]. This supply curve could easily be generalized to make α into a stochastic process, implying a stochastic bid/ask spread. As an intermediate step, different α parameters could be estimated for share purchases or sales. These generalizations, however, are left to subsequent research.

Under this supply curve, there exists a unique martingale measure for $S(t, 0) = s_t$, see Duffie [4]. Hence, we know that the market is arbitrage-free and approximately complete. Therefore, we can invoke expression (6) for valuation and hedging.

Consider an European call option on the stock with a strike price of K and maturity $T_0 \leq T$. To value the option, we need to understand its payoff at expiration given the supply curve for the underlying stock. There are two cases to consider: cash delivery and physical delivery.

1. If the option has cash delivery, then the synthetic option position must be liquidated at time T_0 to match the cash settlement on the option. This implies that the boundary condition for the option will be $C \equiv$

$\max[S(T_0,-1) - Ke^{-rT_0}, 0]$.[9] Replicating a cash settled option incurs a liquidity cost at time T_0 due to the sale of the stock ($X_{T_0} = -1$).

2. If the option has physical delivery, then the synthetic stock position should match the physical delivery in the option contract. The stock itself less K dollars is needed if the option ends up in-the-money. In this case, the stock position at time T_0 is not liquidated. The boundary condition will be $C \equiv \max[S(T_0,0) - Ke^{-rT_0}, 0]$. The stock price of the marginal trade is used in the option's payoff and there is no final liquidity cost.

We will value the option under physical delivery, corresponding to the standard provision for exchange traded options. Given physical delivery, as noted, the payoff to the option at time T is $C_T = \max[S(T,0) - Ke^{-rT}, 0]$. Under this structure, by expression (6), with a non zero interest rate r

$$\begin{aligned} C_0 &= e^{-rT}E^Q(\max[s_T - K, 0]) + L_0 \quad \text{where} \\ L_0 &= X_0[S(0,X_0) - S(0,0)]. \end{aligned} \tag{12}$$

It is well-known that the expectation in this expression is the Black-Scholes-Merton formula:

$$E^Q(C_T) = e^{-rT}E^Q(\max[s_T - K, 0]) = s_0 N(h_0) - Ke^{-rT}N(h_0 - \sigma\sqrt{T}) \tag{13}$$

where $N(\cdot)$ is the standard cumulative normal distribution function, $h_t \equiv \frac{\log(s_t) - \log K + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}$. At this point, a numerical example is best to illustrate the deviation from the Black Scholes framework.

Example 5 *Consider a European call option with 30 days to maturity struck at-the-money with $S(0,0)$ and K being \$20. The prevailing interest rate is 5% and the underlying volatility of the equity is 30%. Furthermore, assume that the liquidity parameter α associated with the equity is $.00005$ which corresponds to bid and ask quotes of \$19.90 and \$20.10, respectively, for 100 shares.*

To begin with, the Black Scholes value of this option on 100 underlying shares equals \$72.64 and does not depend on the α parameter. The hedge position X_0 corresponding to this option equals 53.62 shares.

The additional cost of illiquidity is measured as $L_0 = X_0[S(0,X_0) - S(0,0)]$ which equals $(e^{\alpha X_0} - 1)X_0 S(0,0)$ or \$2.88. This corresponds to a 3.96% increase in the price of the call option.

The Black-Scholes hedging strategy, represented by $X_t = N(h_t)$ is continuous, but not of finite variation. Although the call's value is the Black-Scholes formula plus liquidity costs, the standard hedging strategy will not attain this value. Indeed, using the Black-Scholes hedging strategy leads to the following liquidity costs [10](proof in the appendix):

[9] The strike price is normalized by the value of the money market account to be consistent with the previous construct.

[10] Both L_T and Y_T^n are already normalized by the value of the money market account.

$$L_T = X_0(S(0, X_0) - S(0,0)) + \int_0^T \frac{\alpha \left(N'\left(h_u\right)\right)^2 s_u}{T-u} du$$

whose time 0 value are infinite.

A hedging strategy that is continuous and of finite variation, and that avoids these liquidity costs, yet approximates the value in expression (12), is the sequence of non-zero initial investment s.f.t.s. (X^n, Y^n, τ^n) with

$$X_t^n = n \int_{t-\frac{1}{n}}^t N(h_s) ds.$$

This trading strategy has $Y_T^n = E^Q(C_T) + \int_0^T X_{u-}^n dS(u,0) \to C_T = \max[S(T,0) - Ke^{-rT}, 0]$ in $L^2(dQ)$. This trading strategy is a "smoothing" of the Black-Scholes hedging strategy. Indeed, the fact that the trading strategy can be represented by an integral of a continuous function implies that it is of bounded variation and continuous.

The "smoothed" hedge parameters are illustrated graphically below in Figure 1 with rebalancing done daily, twice daily, hourly, and finally the limiting Black Scholes case without any smoothing. The remainder of this section does not require the α parameter, only functions of the original Black Scholes hedge parameters are considered. Hence results are presented for an option on a single stock for simplicity.

The next set of graphs in Figure 2 illustrate the approximation error between Y_T^n and $\max[S(T,0) - Ke^{-rT}, 0]$ for various values of n as detailed in equation (9). As expected, the approximation error distribution becomes more compact as hedging is performed more frequently.

The root mean squared error[11] (RMSE) and range of the absolute dollar differences is presented below in Table 1. based on N=5,000 simulations for three separate hedging frequencies: daily, twice daily, and hourly. For a \$20 stock, the "average" daily hedging error over the life of the option is 37.05 cents, or about 1.85 percent of the initial stock's value.

This observation helps clarify industry practice of using the Black-Scholes formula to price call options, but using a different hedging strategy to replicate the call options. Indeed, as proven above, the Black-Scholes option value lies between the purchase and selling price of the option, considering the liquidity costs L_0. This is consistent with industry usage of the Black-Scholes formula. Also, the Black-Scholes hedging strategy, due to liquidity costs, will not attain this value. To attain this value, one must modify the Black-Scholes hedge (after time 0) to avoid liquidity costs. This modification of the Black-Scholes hedging strategy is also consistent with industry usage of the model.

To explicitly illustrate the implementation of this theory, the next section estimates the α parameter for the supply curve and investigates the impact of liquidity on actual market prices.

[11] RMSE $= \sqrt{\frac{1}{N} \sum_{i=1}^N \left(Y_T^n - C\right)^2}$

4 Empirical Illustration

The TAQ database is used for estimating the supply curve, in particular the information contained in the quotes file. The TAQ database provides bid and ask quotes as well as the sizes of the respective quotes. For illustrative purposes, we selected five well known companies trading on the NYSE: American Airlines (AMR), Metro Goldwyn (MGM), Philip Morris (MO), Reebok (RBK), and Toys R US (TOY) during the month of May, 2000. All of the firms have options traded on the CBOE. In addition, the firms did not pay dividends over the maturities of their traded options in May, 2000. Therefore, for call options, a comparison with the Black Scholes model is justified as European and American options would be identical. This enables a direct application of the Black Scholes theory to the options data.

4.1 Estimation of the Supply Curve

To estimate the supply curve, we concentrated on using bid and ask quotes for one round lot. Denote the ask price by $S(t, x_a)$ and the bid price by $S(t, x_b)$ where $S(t, x_a) > S(t, x_b)$ and $x_a = 100$ and $x_b = -100$. Quotes were filtered so that only bid and ask quotes corresponding to bid and ask sizes of one round lot of 100 shares were included in the estimation of α. This is reasonable since for call options on 100 shares, the delta hedge parameter is constrained to lie within this interval[12]. Using expressions (10) and (11), we can obtain estimates for a and $S(t, 0)$):

$$\frac{S(t, x_a)}{S(t, x_b)} = e^{\alpha(x_a - x_b)} \quad \text{or} \quad \alpha = \frac{\log\left(\frac{S(t,x_a)}{S(t,x_b)}\right)}{x_a - x_b} \tag{14}$$

and

$$\begin{aligned} \log S(t, x_a) + \log S(t, x_b) &= 2 \log S(t, 0)) \quad \text{or} \\ S(t, 0) &= e^{[\log S(t,x_a) + \log S(t,x_b)]/2} \end{aligned} \tag{15}$$

These parameter estimates are computed for the five companies and summarized in Table 2.

The α estimates contain very little variation and when inserted into equation (10) mirror the observed bid-ask spread surrounding the average $S(t, 0)$. To estimate σ, a historical time series of $S(t, 0)$ observations could be used based on standard techniques. However, our ultimate goal is to ascertain what percentage of the observed market option prices can be attributed to illiquidity. Hence, differences between implied volatilities and historical volatilities would confound this comparison. Therefore, implied Black Scholes volatilities with liquidity

[12] For less than 100 shares, the supply curve is a useful abstraction. For portfolios of multiple options, the issue of higher prices for odd lots is resolved and viewing the supply curve as upward sloping becomes reasonable.

based on equation (12) are computed. This procedure also allows a comparison of options with different strike prices on any given day, incorporating *smile* effects, which would also bias results based on a single historical estimate.

4.2 Computing Option Values

Closing daily option prices for the five companies were obtained from Prophet Financial Systems,[13] Inc with interest rate data obtained from the Federal Reserve Bank of St. Louis[14]. The call options data was filtered to remove options with zero volume, option prices that violated their intrinsic value, or options prices below $.50. To illustrate the magnitude of illiquidity, every available option price was inverted to obtain the implied volatility. In particular, using the average α for the company under consideration, equation (12) was inverted to obtain an implied Black Scholes volatility including liquidity. This estimate is necessarily lower than the standard Black Scholes implied volatility, given that L_0 is positive. Computations were performed for call options based on 100 shares of the underlying stock. This estimate of σ is then used to compute the liquidity cost L_0. With this, the percentage of the market price of the option attributed to illiquidity is easily computed. Table 3. below summarizes the average dollar and percentage value of the option's market price attributed to illiquidity. The averages are over time to maturity and strike prices. The table also contains minimum and maximum percentage values.

As expected, within the sample of trade options, those with higher X_0 values (larger initial hedges) had a greater proportion of their market price influenced by illiquidity. For example, illiquidity exerts a larger influence on in-the-money options as compared to out-of-the-money options everything else constant.

5 Conclusion

This paper provides a model for pricing options in an economy with liquidity risk. An extended Black Scholes economy is utilized to illustrate the theory and to provide estimates of the impact of liquidity costs on the option's value. Liquidity risk is modeled as a stochastic supply curve for the underlying stock that depends on the size of a trade. Consistent with the market microstructure literature, large buys increase the purchase price while large sales decrease the selling price. The arbitrage free cost of constructing an option is derived and the liquidity cost characterized. This implementation demonstrates that liquidity costs comprise a significant percentage of the option's value.

[13] www.prophetfinance.com
[14] www.stls.frb.org/fred

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Appendix

The liquidity cost of the Black-Scholes hedge is

$$
\begin{aligned}
L_T &= X_0(S(0,X_0)-S(0,0)) + \int_0^T \alpha s_u d[N(h),N(h)]_u^c \\
&= X_0(S(0,X_0)-S(0,0)) + \int_0^T \alpha s_u \left(N'(h_u)\right)^2 d\,[h,h]_u^c \\
&= X_0(S(0,X_0)-S(0,0)) + \int_0^T \alpha s_u \left(N'(h_u)\right)^2 \frac{1}{s_u^2\sigma^2(T-u)} d\,[s,s]_u \\
&= X_0(S(0,X_0)-S(0,0)) + \int_0^T \frac{\alpha\left(N'(h_u)\right)^2}{\sigma^2 s_u(T-u)}\sigma^2 s_u^2 du \\
&= X_0(S(0,X_0)-S(0,0)) + \int_0^T \frac{\alpha\left(N'(h_u)\right)^2 s_u}{T-u} du.
\end{aligned}
$$

The time 0 value of these liquidity costs is:

$$
X_0(S(0,X_0)-S(0,0)) + E^Q\left(\int_0^T \frac{\alpha\left(N'(h_u)\right)^2 s_u}{T-u} du\right).
$$

We next evaluate $E^Q\left(\int_0^T \frac{\alpha\left(N'(h_u)\right)^2 s_u}{T-u} du\right)$. With $r=0$, the h_u term becomes

$$
\begin{aligned}
h_u &= \frac{\ln\left(\frac{s_u}{K}\right)+\frac{\sigma^2}{2}(T-u)}{\sigma\sqrt{T-u}} \\
h_u^2 &= \frac{1}{\sigma^2(T-u)}\left[\left(\ln\left(\frac{s_u}{K}\right)\right)^2 + \ln\left(\frac{s_u}{K}\right)\sigma^2(T-u) + \frac{\sigma^4}{4}(T-u)^2\right] \\
&= \frac{\left(\ln\left(\frac{s_u}{K}\right)\right)^2}{\sigma^2(T-u)} + \ln\left(\frac{s_u}{K}\right) + \frac{\sigma^2}{4}(T-u)
\end{aligned}
$$

making

$$
\begin{aligned}
\left(N'(h_u)\right)^2 &= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h_u^2}\right)^2 = \frac{1}{2\pi}e^{-h_u^2} \\
&= \frac{1}{2\pi}e^{-\frac{\left(\ln\left(\frac{s_u}{K}\right)\right)^2}{\sigma^2(T-u)}}\frac{K}{s_u}e^{-\frac{\sigma^2}{4}(T-u)}
\end{aligned}
$$

Therefore, the integrand reduces to

$$
\frac{\alpha K e^{-\frac{\left(\ln\left(\frac{s_u}{K}\right)\right)^2}{\sigma^2(T-u)}} e^{-\frac{\sigma^2}{4}(T-u)}}{2\pi(T-u)}
$$

The only random term left is $s_u = s_0 e^{\sigma W_u}$ in the exponent of

$$\begin{aligned}
-\left[\frac{\left(\ln\left(\frac{s_u}{K}\right)\right)^2}{\sigma^2(T-u)}\right] &= -\left[\frac{\ln\left(\frac{s_0}{K}\right)+\sigma W_u}{\sigma\sqrt{(T-u)}}\right]^2 \\
&= -\left[\mathcal{N}\left(\frac{\ln\left(\frac{s_0}{K}\right)}{\sigma\sqrt{(T-u)}}, \frac{u}{2(T-u)}\right)\right]^2 \\
&= -\left[\sqrt{\frac{u}{2(T-u)}}\mathcal{N}\left(\sqrt{\frac{2}{\sigma^2 u}}\ln\left(\frac{s_0}{K}\right), 1\right)\right]^2 \\
&= -\frac{u}{2(T-u)}\left[\mathcal{N}\left(\sqrt{\frac{2}{\sigma^2 u}}\ln\left(\frac{s_0}{K}\right), 1\right)\right]^2
\end{aligned}$$

Using the non central chi-squared moment generating function with one degree of freedom evaluates the squared normal:

$$E^Q\left[\exp\left\{-tU\right\}\right] = \frac{1}{\sqrt{1-2t}}\exp\left\{\frac{\beta^2 t}{1-2t}\right\}$$

where U is a squared normal with mean β and unit variance. With $\beta = \sqrt{\frac{2}{\sigma^2 u}}\ln\left(\frac{s_0}{K}\right)$ and $t = \frac{u}{2(T-u)}$, the expectation evaluates to

$$\frac{1}{\sqrt{1-\frac{u}{(T-u)}}}\exp\left\{\frac{\left(\ln\left(\frac{s_0}{K}\right)\right)^2\frac{1}{\sigma^2(T-u)}}{1-\frac{u}{(T-u)}}\right\}$$

$$\begin{aligned}
&= \frac{\alpha K\exp\left\{\frac{\left(\ln\left(\frac{s_0}{K}\right)\right)^2\frac{1}{\sigma^2(T-u)}}{1-\frac{u}{(T-u)}}\right\}\exp\left\{-\frac{\sigma^2}{4}(T-u)\right\}}{2\pi(T-u)\sqrt{1-\frac{u}{(T-u)}}} \\
&= \frac{\alpha K\exp\left\{\frac{\left(\ln\left(\frac{s_0}{K}\right)\right)^2}{\sigma^2(T-2u)}\right\}\exp\left\{-\frac{\sigma^2}{4}(T-u)\right\}}{2\pi\sqrt{(T-u)(T-2u)}}
\end{aligned}$$

The final expression for the expectation is

$$E^Q\left(\int_0^T \frac{\alpha\left(N'(h_u)\right)^2 s_u}{T-u}du\right) = \frac{\alpha K}{2\pi}\int_0^T \frac{\exp\left\{\frac{\left(\ln\left(\frac{s_0}{K}\right)\right)^2}{\sigma^2(T-2u)}\right\}\exp\left\{-\frac{\sigma^2}{4}(T-u)\right\}}{\sqrt{(T-u)(T-2u)}}du.$$

This integral is infinite.

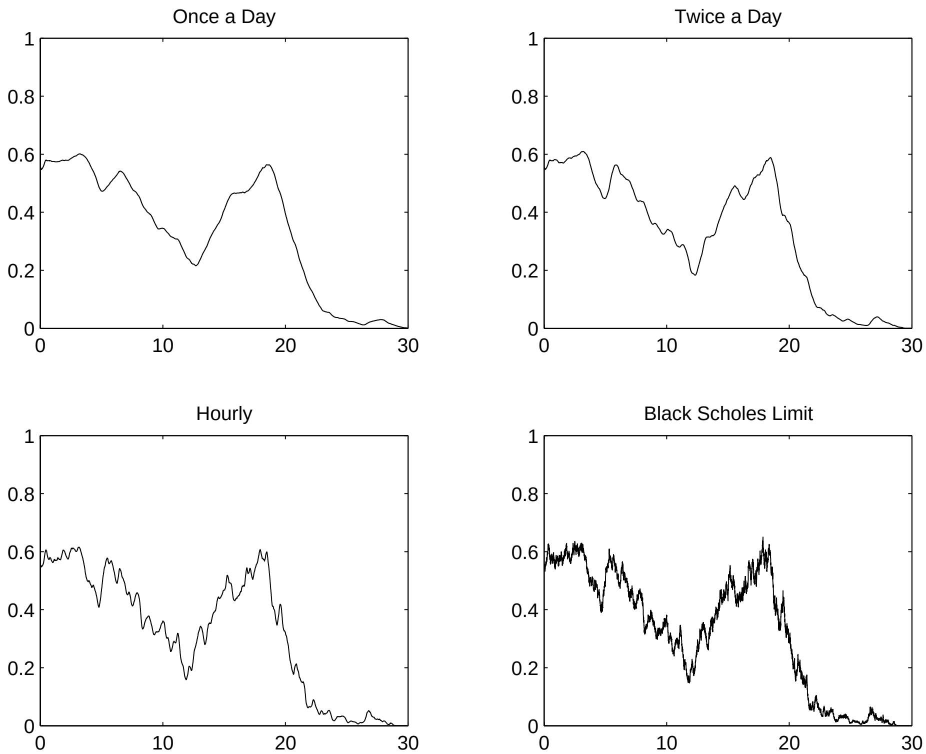

Figure 1: Hedge parameters over time for various hedging frequencies. Hedge parameters for 30 day at-the-money call option with $S(0,0) = K = \$20$ on underlying stock with 30% annualized volatility and zero percent interest rate. The 30 day period is discretized into 10,000 intervals for simulation. Note that the α parameter is not required to generate these hedge parameters.

Frequency	RMSE	Minimum	Maximum
Daily	.3705	-2.1589	1.5176
Twice Daily	.2631	-1.3327	1.2184
Hourly	.1388	-0.5711	0.6497

Table 1: Average dollar value and range of error between replicating portfolio and call option. Call option is on one unit of the underlying replicated using "smoothed" hedge parameters associated with daily, twice daily, and hourly rebalancing for 30 day at-the-money option with $S(0,0) = K =$\$20 on underlying stock with 30% annualized volatility and zero percent interest rate. The 30 day period is discretized into 10,000 intervals for simulation while 5,000 paths are generated to produce 5,000 terminal call option and replicating portfolio payoffs. Note that the α parameter is not required to generate these errors.

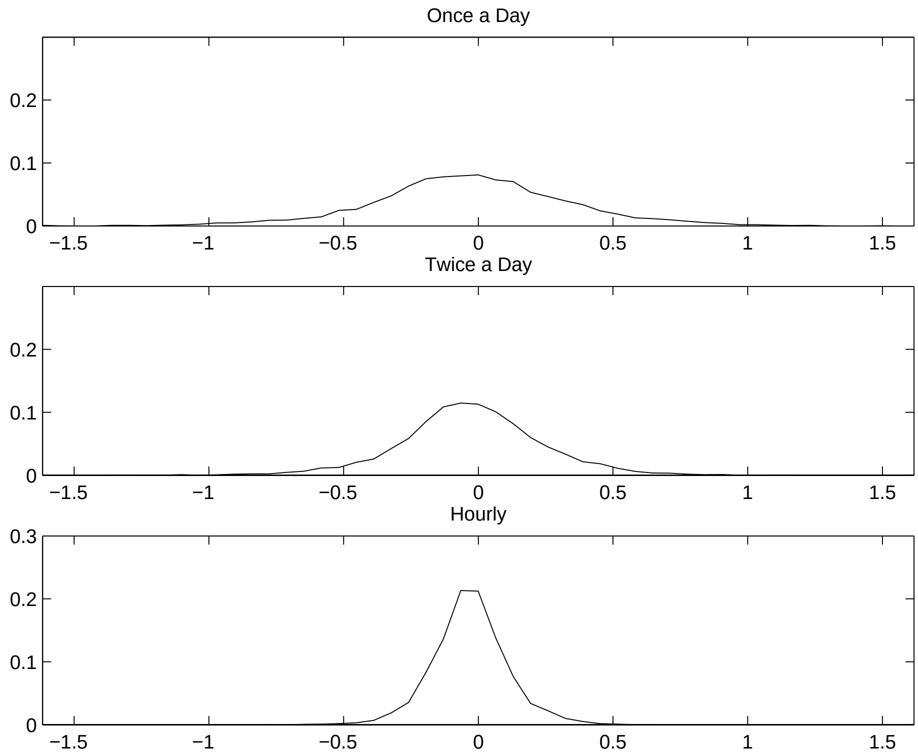

Figure 2: Convergence of replicating portfolio to call option payoff for various hedging frequencies. The difference or error between the replicating portfolio Y_T^n and the call option payoff based on one unit of the underlying for daily, twice daily, and hourly rebalancing. Underlying option is 30 day at-the-money call option with $S(0,0) = K = \$20$ on underlying stock with 30% annualized volatility and zero percent interest rate. The 30 day period is discretized into 10,000 intervals for simulation while 5,000 paths are generated to produce 5,000 terminal call option and replicating portfolio payoffs. Note that the α parameter is not required to generate these errors.

Company	Sample Size	Mean	Standard Deviation	Minimum	Maximum
AMR	31,591	0.000067546	0.000018243	0.000034443	0.00012836
MGM	9,218	0.000090752	0.000020675	0.000053591	0.00024226
MO	19,034	0.000074530	0.000021313	0.000022371	0.00013454
RBK	13,567	0.000132600	0.000030890	0.000074351	0.00027734
TOY	8,822	0.000125700	0.000040374	0.000040161	0.00035529

Table 2: α estimates. Summary statistics for firm specific α parameter estimated using May, 2000 bid and ask quote data for one round lot obtained from TAQ database.

Company	Sample Size	Percentage Value	Minimum	Maximum
AMR	63	2.93 %	0.82 %	10.47 %
MGM	9	4.36 %	2.11 %	8.66 %
MO	150	2.53 %	0.82 %	8.89 %
RBK	15	6.57 %	2.55 %	27.00 %
TOY	43	5.00 %	2.05 %	12.73 %

Table 3: Impact of illiquidity on call option prices. Average dollar value of illiquidity for May 2000 traded options across all strike prices and time to maturities along with average percentage (and range) of market prices attributed to illiquidity.