Analysis of continuous strict local martingales via h-transforms

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Abstract

We study strict local martingales via h-transforms, a method which first appeared in Delbaen-Schachermayer [7]. Although nonnegative local martingales are supermartingales, we identify a class of convex functions which when applied to such local martingales produce submartingales. We extend part of this analysis to conformal local martingales in dimensions three and higher. An interesting connection with Kelvin transforms emerges which helps us to identify subharmonic functions which when applied to conformal strict local martingales produce submartingales. From the perspective of mathematical finance, the functions $(K-x)^+$ and $(x-K)^+$ are studied. The former is a nice function in the sense mentioned above, the latter is not. We show by various examples that strict local martingales do not behave uniformly when the function $(x-K)^+$ is applied to them. Implications to the recent literature on financial bubbles are discussed.

1 Introduction

An elementary and classic result is that a convex function of a martingale is a submartingale. This result does not necessarily hold for local martingales. For example, positive local martingales might lose their martingale property and become supermartingales. Local martingales which are not martingales (known as "strict" local martingales) arise naturally in the Doob-Meyer decomposition and in multiplicative functional decompositions, as well as in

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stochastic integration theory, they are nevertheless often considered to be anomalies, processes that need to be maneuvered by localization.

Our goal in this paper is to demonstrate another method to deal with such processes, namely, via applications of Doob's h-transforms. For positive local martingales, such a method was originally identified by Delbaen and Schachermayer in [7]. We extend that analysis and identify a class of convex functions (Proposition 3) which when applied to any local martingale (strict or not) will produce a submartingale. This is the class of all nonnegative convex functions that grow sublinearly at infinity. For example, this class contains all nonnegative convex functions which have compact support. We also give *en passant* a new proof of a recent result of Madan and Yor [28, Theorem 1].

Next we deal with a multi-dimensional version of the same question. We restrict our analysis to conformal local martingales in several dimensions in order to use tools from harmonic function theory. Convex functions in one-dimension are replaced by subharmonic functions. More importantly, positive local martingales are replaced by local martingales that avoid the open unit ball. An application of Kelvin transforms allows us to identify a class of subharmonic functions which when applied to any conformal local martingales will result in a submartingale. This is the content of Proposition 6 which is a generalization of the one-dimensional analogue and is an application of the idea *harmonicity at infinity*. The technique of proof is based on constructing a suitable h-transform.

From the standpoint of applications we are interested in two specific functions $x \mapsto (k-x)^+$ and $x \mapsto (x-k)^+$, for some positive k. Both are convex. However, the former one is sublinear at infinity while the latter is not. Thus, the former falls under the domain of Proposition 3. In subsection 4.1, we demonstrate through examples that $(x-k)^+$ is somehow a critical function, for which there is no uniform behavior among strict local martingales. In Proposition 7 we analyze its effect on the inverse 3-dimensional Bessel process (the canonical continuous strict local martingale, much as Brownian motion is the canonical continuous martingale) and demonstrate a curious phase transition phenomenon.

The rest of the paper discusses the implication of our results in mathematical finance. A natural source for local martingales in mathematical finance is the condition of No Free Lunch with Vanishing Risk. Roughly, it states that in a financial market the no arbitrage condition is equivalent (in the case of continuous paths) to the existence of an (equivalent) "risk neutral" probability measure Q which turns the price process into either a martingale or a strict local martingale.

The implications of our results can be readily understood if we assume that the risk neutral measure produces a (one-dimensional) price process $X = (X_t)_{t\geq 0}$ that is a strict local martingale. In that case the process $Y_t = (X_t - K)^+$ *need not be a submartingale*, and the function $t \mapsto E\{(X_t - K)^+\}$ need no longer be increasing, contradicting the usual wisdom in the theory. The original purpose of this paper was to understand this phenomenon better, motivated in particular by the role local martingales play in financial bubbles (cf [21] and [22]). We are able to construct an example where Merton's famous mathematical finance "no early exercise" theorem [29] does not hold, and we explain in detail exactly what does hold in its place. We connect this to recent results concerning Financial Bubbles; our results indicate that one theoretically possible way to detect a bubble is to analyze the behavior of European call prices through time.

2 h transforms in one dimension

We start with an example. The 3-dimensional Bessel process (denoted by BES(3)) is the process of the Euclidean norm of a three dimensional Brownian motion. Due to the radial symmetry of Brownian motion, this process turns out to be a strong Markov process. In fact, if X_t is a BES(3) process starting from a nonnegative constant x_0, then (see [25, page 158] for the details), X is the (strong) solution of the stochastic differential equation:

$$dX_t = \frac{1}{X_t} dt + d\beta_t, \qquad X_0 = x_0, \tag{1}$$

where β is an one-dimensional standard Brownian motion. Since the origin is polar to the three dimensional BM, the reciprocal of this process is well-defined throughout. This reciprocal process, known as the inverse Bessel process, serves as a prototypical example of a local martingale which is not a martingale, a property commonly referred to as being a *strict local martingale*. The *strictness* holds in spite of the fact that the family of random variables $\{1/X_t,\ t \geq 0\}$ is uniformly integrable on the entire range of t.

It's a strange coincidence that the BES(3) process also turns out to be important in various descriptions of Brownian path decompositions (see, e.g., [32, page 318]). The reason here being that there is yet another description of the same process law. To describe this let us call as the *canonical space*, the space of continuous function $C[0, \infty)$ together with the right-continuous filtration obtained from the natural filtration generated by the coordinate process. The laws of all continuous stochastic processes are probability measures on this space.

Theorem 1. *Let X_t denote the coordinate process on the canonical space, and let Q denote the Wiener measure such that $Q(X_0 = 1) = 1$. Let τ_0 denote the first hitting time of zero, i.e., $\tau_0 = \inf\{t \geq 0,\ X_t = 0\}$. Then $X_{t\wedge\tau_0}$ is a martingale under Q and $E(X_{t\wedge\tau_0}) = 1$. Define a probability measure P by the domination relation*

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = X_{t\wedge\tau_0}. \tag{2}$$

Then, under P, the law of the coordinate process X_t is BES(3) with $X_0 = 1$.

Proof. Use Girsanov's theorem ([32, page 327]) to see that under P the coordinate process satisfies the relation

$$X_t = \beta_t + \int_0^t \frac{ds}{X_{s\wedge\tau_0}},$$

where β is a Brownian motion. It then follows that X must be strictly positive throughout and satisfy SDE (1). Since the function $f(x) = 1/x$ is locally Lipschitz on $(0, \infty)$, we have the uniqueness of the solution of the SDE, and the theorem follows. □

This theorem is one of the earliest instances of Doob's h-transforms and encapsulates the idea that a 3-dimensional Bessel process is a one dimensional Brownian motion conditioned to remain positive.

We have already remarked that the inverse BES(3) process is a canonical example of a strict local martingale. The previous theorem establishes a relationship between a pair of laws: a non-negative strict local martingale (i.e., inverse BES(3)) on the one hand, and a non-negative *true* martingale (i.e., BM absorbed at zero) on the other. It turns out that this relationship holds much more generally. In fact, all strict local martingales which remain strictly positive throughout can be obtained as the reciprocal of a martingale under an h-transform.

This was essentially proved by Delbaen and Schachermayer [7] in 1995 in their analysis of arbitrage possibilities in Bessel processes. We replicate their theorem below. The construction is related to the Föllmer measure of a positive supermartingale [14]. The technique of h-transform itself has also been applied to several other problems, for example in the analysis of Brownian meander by Biane and Yor [3].

Before we state the result we need a technique which adds an extra absorbing point *infinity* to the state space $\mathbb{R}^+$. Our treatment is inspired by the work of P. A. Meyer [30], and we follow the notation used in [7] closely. The space of trajectories is the space $C_\infty[0,T]$ or $C_\infty[0,\infty)$ of continuous paths ω defined on the time interval $[0,T]$ or $[0,\infty)$ with values in $[0,\infty]$ with the extra property that if $\omega(t)=\infty$, then $\omega(s)=\infty$ for all $s>t$. The topology endowed is the one associated with local uniform convergence. The coordinate process is denoted by L, i.e., $L(t)=\omega(t)$. Their are some delicate issues with the filtration involved in order to be completely rigorous; see [7] for the details.

Theorem 2 (Delbaen and Schachermayer, Theorem 4 in [7]). *If R is a measure on $C[0,1]$ such that L is a strictly positive strict local martingale, then*

(i) there is a probability measure R^* on $C_\infty[0,1]$ such that $M=1/L$ is an R^* martingale.

(ii) We may choose R^* in such a way that the measure R is absolutely continuous with respect to R^* and its Radon-Nikodým derivative is given by $dR = M_1 dR^*$.

The following result is a corollary.

Proposition 1. *Let N_t, $t\ge 0$, be a continuous strictly positive local martingale such that $N_0=1$. Then there exists a nonnegative martingale M with law Q on the canonical space such that the following holds.*

(i) The probability measure defined by

$$P(A) := E\left(M_t 1_A\right), \quad \forall\, A\in\mathcal{F}_t,\ t\ge 0, \tag{3}$$

is the law of the process $\{1/N_t,\ t\ge 0\}$.

(ii) N is a strict local martingale if and only if $Q(\tau_0<\infty)>0$, where $\tau_0=\inf\{t\ge 0:\ M_t=0\}$ is the first hitting time of zero for the martingale M.

Conversely, let $\{M_t,\ t\ge 0\}$ be a nonnegative martingale on the canonical space starting from one. Let Q denote the law of M, and assume that $Q(\tau_0<\infty)>0$. Consider the change of measure

$$P(A) := E\left(M_{t\wedge\tau_0} 1_A\right), \quad \forall\, A\in\mathcal{F}_t,\ t\ge 0.$$

Then, under P, the process $1/M_t$ is a positive strict local martingale.

Proof. The only difference in the first part of this proposition with the previous theorem is that the construction is on the entire space $C[0,\infty)$. Consider the law of the local martingale L on the canonical space. Note that, by scaling time, Theorem 2 holds for any time interval $[0,T]$, $T=1,2,\ldots$. In other words, for every positive integer T, there is a martingale $M_t(T)$, $0\le t\le T$, which satisfies the two conditions in Theorem 2 in time interval $[0,T]$. Let Q_T be the law of $M(T)$ on the σ-algebra $\mathcal{F}_T$ generated by the coordinate process up to time T. Once we demonstrate that this tower of probability measures is consistent, it follows from standard arguments that they induce a probability measure Q on the entire space $C_\infty[0,\infty)$ with the required properties holding locally.

However, consistency is immediate once both the properties (i) and (ii) in Theorem 2 hold for each interval $[0,T]$.

Part (ii) follows once we show that $E(N_\tau)=Q(\tau_0>\tau)$ for all bounded stopping times τ. Indeed, if $Q(\tau_0=\infty)=1$, the claim shows that $E(N_\tau)=1$ for all bounded stopping times τ establishing its martingale property. To show $E(N_\tau)=Q(\tau_0>\tau)$, note that, since τ is bounded, the absolute continuity relation (3) holds on $\mathcal{F}_\tau$, and we immediately derive

$$E(N_\tau)=E\left(\frac{1}{M_\tau}M_\tau 1_{\{\tau_0>\tau\}}\right)=Q(\tau_0>\tau). \tag{4}$$

The converse is easier to show. A detailed account can be found in the proof of Theorem 1 in [7]. In short, one shows $1/M$ is a local martingale under P by using Girsanov's theorem. Since it is nonnegative, it must hence be a supermatingale. That it is not a martingale follows from what we just showed: $E(N_t)=Q(\tau_0>t)$. □

Other than the example of BES(3) and Brownian motion that we have encountered, several other examples of Lemma 1 can be derived. In particular, for any Bessel process X of dimension $\delta>2$, it is well-known that $X^{2-\delta}$ is a strict local martingale. The law of its reciprocal can be derived in a procedure similar to Lemma 1 (see [28]) using the martingale $Y^{\delta-2}$, where Y is a Bessel process of (possibly negative) dimension $(4-\delta)$.

Inspired by the previous representation theorem, we make the following definition:

Definition. We will call an ordered pair of continuous processes (N,M) a **Girsanov pair** if

1. $\{N_t, t\ge 0\}$ is a positive strict local martingale starting from one;

2. $\{M_t, t \geq 0\}$ is a nonnegative martingale;

3. The laws of M and N are related by Proposition 1. That is, the law of the process $1/N$ in any time interval $[0,T]$ is absolutely continuous with respect to the law of M during that time interval with a Radon-Nikodým derivative given by M_T.

The advantage of Proposition 1 is that it allows us to transport stochastic calculus with respect to strict local martingales to that with actual martingales via a change of measure.

Proposition 2. *Let (N, M) be a Girsanov pair with their corresponding laws denoted by P and Q respectively. Also let τ_0 denote the hitting time of zero.*

Consider any nonnegative function $h : (0,\infty) \to \mathbb{R}^+$. For any bounded stopping time τ, we get

$$E\left(h(N_\tau)\right) = Eg(M_\tau)1_{\{\tau_0 > \tau\}}. \tag{5}$$

Here g is the function $g(x) = xh(1/x)$, for all $x > 0$.

Now suppose $\lim_{x\to 0} g(x) = \eta < \infty$. Define a map $\bar{g} : [0,\infty) \to \mathbb{R}$ by extending g continuously, i.e., $\bar{g}(x) = g(x)$ for $x > 0$, and $\bar{g}(0) = \eta$. Then we have

$$E\left(h(N_\tau)\right) = E\bar{g}(M_\tau) - \eta Q\left(\tau_0 \leq \tau\right). \tag{6}$$

Before we prove the statement above, as an example note that when $h(x) = (x-a)^+$ for some $a \geq 0$, we get

$$E\left(N_\tau - a\right)^+ = E\left(1 - aM_\tau\right)^+ - Q(\tau_0 \leq \tau). \tag{7}$$

Proof of Proposition 2. This is immediate from the absolute continuity relationship between the laws of the two processes. Note that, by nonnegativity of the martingale M, we have $M_\tau = M_{\tau\wedge\tau_0} = M_\tau 1_{\{\tau_0>\tau\}}$. One gets

$$Eh(N_\tau) = EM_\tau h\left(\frac{1}{M_\tau}\right)1_{\{\tau_0>\tau\}} = Eg(M_\tau)1_{\{\tau_0>\tau\}}.$$

For the second assertion assume that one can define $\bar{g}(0) = \eta$ by continuously extending g. Now for any nonnegative path ω which gets absorbed upon hitting zero, the following is an algebraic identity:

$$g(\omega_\tau)1_{\{\tau_0>t\}} = g(\omega_\tau) - \eta 1_{\{\tau_0\leq\tau\}}.$$

In particular this identity holds pathwise when ω is a path of a nonnegative martingale M. Taking expectation on both sides of the last equation with respect to the law of M, we obtain

$$Eg(M_\tau)1_{\{\tau_0>t\}} = E\bar{g}(M_\tau) - \eta Q(\tau_0 \le t).$$

This proves the proposition. □

We immediately derive the following curious fact which will be important for us later in the text.

Corollary 1. *Let $h:(0,\infty)\to(0,\infty)$ be a function which is sublinear at infinity, i.e.,*

$$\lim_{x\to\infty}\frac{h(x)}{x}=0. \tag{8}$$

Then, for all bounded stopping times τ, one has

$$Eh(N_\tau) = E\bar{g}(M_\tau), \quad \bar{g}(x) = xh(1/x),\ x>0,\ \bar{g}(0)=0. \tag{9}$$

Proof. The first part follows directly from (6) since

$$\eta = \lim_{x\to 0} xh\left(\frac{1}{x}\right) = \lim_{x\to\infty}\frac{h(x)}{x} = 0.$$

The second conclusion is obvious. □

The previous corollary has some interesting consequences. We will soon show that when h is convex, so is $\bar{g}$. And thus both $h(N)$ and $\bar{g}(M)$ are submartingales by (9). This is in spite of the strictness in the local martingale property of N. Additionally, if h is symmetric with respect to inverting x, i.e. $h(x) = xh(1/x)$, then $\bar{g} = h$. Hence strictness of local martingales have no effect when these functions are applied. For example, $E\sqrt{N_\tau} = E\sqrt{M_\tau}$.

Proposition 3. *Let N be a positive strict local martingale ($N_0 = 1$). Let L_t^a be the semimartingale local time of the process N at a. Then for any $a>0$, the process*

$$(N_t - a)^+ - \frac{1}{2}L_t^a - N_t, \quad t \ge 0, \tag{10}$$

is a true martingale.

Moreover, if h is a nonnegative convex function on $(0,\infty)$ which satisfies sublinear growth at infinity as in (8), then

$$h(N_t) - \frac{1}{2}\int_0^\infty L_t^a \mu(da), \quad t \ge 0, \tag{11}$$

is a martingale. Here μ is the positive measure which is the second derivative in the sense of distributions for the function h.

In particular, $h(N_t)$ is a submartingale.

The proof below uses the occupation time formula involving the local time for general continuous semimartingales. We refer the reader to [31, p. 220] for a discussion of the relevant concepts.

Proof. Let (N, M) be a Girsanov pair and let $L^a_\cdot(N)$ and $L^a_\cdot(M)$ denote the process of local times at $a > 0$ for N and M respectively. We first show that for any $a > 0$, we have

$$aE\left(L^{1/a}_\tau(M)\right) = E\left(L^a_\tau(N)\right), \quad \tau - \text{bounded stopping time.} \tag{12}$$

We use the absolute continuity relationship between M and N as obtained from Lemma 1. Note that for any measurable compactly supported function $f : (0, \infty) \to (0, \infty)$ we use the occupation time formula to obtain

$$\int_0^\infty f(a)L^a_\tau(N)da = \int_0^\tau f(N_s)d\langle N_s\rangle = \int_0^\tau f\left(\frac{1}{M_s}\right)\frac{1}{M_s^4}d\langle M_s\rangle$$

The change from $\langle N\rangle$ to $\langle M\rangle$ can be easily justified by Girsanov's theorem.

Now, by the change of measure formula (3), we obtain

$$\begin{aligned} E\left[\int_0^\infty f(a)L^a_\tau(N)da\right] &= E\left[M_\tau\int_0^\tau f\left(\frac{1}{M_s}\right)\frac{1}{M_s^4}d\langle M_s\rangle\right] \\ &= E\left[\int_0^{\tau\wedge\tau_0} M_s f\left(\frac{1}{M_s}\right)\frac{1}{M_s^4}d\langle M_s\rangle\right]. \end{aligned} \tag{13}$$

The final equality is due to the integration by parts formula and the fact that M is a martingale. Using now the occupation time formula for the martingale M, we get

$$\begin{aligned} E\left[\int_0^{\tau\wedge\tau_0} f\left(\frac{1}{M_s}\right)\frac{1}{M_s^3}d\langle M_s\rangle\right] &= E\int_0^\infty f\left(\frac{1}{b}\right)L^b_\tau(M)\frac{db}{b^3} \\ &= E\int_0^\infty f(a)\, aL^{1/a}_\tau(M)da, \quad a = 1/b. \end{aligned} \tag{14}$$

Combining (13) and (14) and using Fubini-Tonelli for nonnegative integrands to interchange expectations and integrals, we obtain

$$\int_0^\infty f(a)E\left(L^a_\tau(N)\right)da = \int_0^\infty f(a)\, aE(L^{1/a}_\tau(M))da.$$

Since this holds for all nonnegative functions with compact support in $(0,\infty)$, we have proved (12).

To prove (10), consider any positive a. Using Tanaka's formula applied to the martingale M, we get the following semimartingale decomposition

$$(1-aM_t)^+ = (1-aM_0)^+ - a\int_0^t 1_{\{M_s<1/a\}}dM_s + \frac{a}{2}L_t^{1/a}(M). \tag{15}$$

Taking expectation on both sides of (15) at a bounded stopping time τ, we obtain

$$E(1-aM_\tau)^+ = (1-aM_0)^+ + \frac{a}{2}E\left(L_\tau^{1/a}(M)\right) = (1-a)^+ + \frac{1}{2}EL_\tau^a(N).$$

Combining this with (7), we get

$$E(N_\tau - a)^+ = (1-a)^+ + \frac{1}{2}EL_\tau^a(N) - (1-EN_\tau)\,.$$

Here we use the fact that $EN_\tau = Q(\tau_0 > \tau)$.

Since the above equality holds for all bounded stopping times, this immediately proves that the following process is a martingale:

$$(N_t - a)^+ - \frac{1}{2}L_t^a(N) - N_t = -N_t \wedge a - \frac{1}{2}L_t^a(N).$$

This proves (10) and provides an alternative proof of Theorem 1 in Madan and Yor [28].

To show (11) we use equality (9). But first we show that if h is convex and sublinear at infinity, then $\bar{g}$ is convex in $[0,\infty)$. To see this, note that for any convex function h, up to an affine term we have the representation

$$h(x) = \int_0^\infty (x-a)^+\mu(da), \quad x > 0,$$

where μ is a positive Borel measure on $(0,\infty)$. It is immediate that for $x > 0$,

$$\bar{g}(x) = xh(1/x) = \int_0^\infty (1-ax)^+\mu(da).$$

Taking the left derivative of $\bar{g}$, we get

$$\bar{g}'_-(x) = -\int_0^{1/x} a\mu(da). \tag{16}$$

This is clearly seen to be an increasing function of x from which the convexity of $\bar{g}$ follows in $(0,\infty)$. Since we have extended $\bar{g}(0)=0$ by continuity, $\bar{g}$ is convex in the entire set $[0,\infty)$.

By Tanaka's formula, for any bounded stopping time τ one gets

$$
\begin{aligned}
E\bar{g}(M_\tau) &= \frac{1}{2}\int_0^\infty EL_\tau^b(M)\bar{g}''(db) \\
&= \frac{1}{2}E\int_0^\infty L_\tau^b(M)b^{-3}\mu(d(1/b)), \quad \text{from (16)}, \\
&= \frac{1}{2}E\int_0^\infty aL_\tau^{1/a}(M)\mu(da) = \frac{1}{2}E\int_0^\infty L_\tau^a(N)\mu(da).
\end{aligned}
$$

Thus, combining the above equality with (9), we conclude that

$$
Eh(N_\tau) = \frac{1}{2}E\int_0^\infty L_\tau^a(N)h''(da),
$$

which proves (11). □

3 A multidimensional analogue

In the last section we see that when the local martingale remains positive throughout time, there is a class of convex functions which when applied to them produces true submartingales regardless of their strictly local property. Can this be generalized in higher dimensions ? The answer to this question has two stumbling blocks. One is obviously the definition of a multidimensional strict local martingale. The other is a more subtle point. What is the correct multidimensional analogue of *being positive* in one dimension ?

We take the following approach. Let $d>2$ (we will mention the case of $d=2$ separately) be the dimension of the underlying space. Let $X=(X_1,X_2,\ldots X_d)$ be a d-dimensional conformal local martingale, i.e., each coordinate X_i is a local martingale and

$$
\langle X_i, X_j\rangle = \langle X_1\rangle\, 1\{i=j\}, \quad \text{for all } 1\le i,j\le d.
$$

In other words, each coordinate process has the same quadratic variation as the others, and is orthogonal to the others. We consider it to be *strictly local* if at least one of its coordinate processes is a strict local martingale. This settles the first stumbling block as discussed above.

For the second, we do not let this process to enter the open unit ball in dimension d. This is a non-trivial restriction on the process for which the following analysis applies.

Lemma 4. *Let $|\cdot|$ denote the Euclidean norm in dimension d. Let τ_1 be the hitting time of the ball of radius of radius one, i.e.,*

$$D = \{x : \; |x| \le 1\}, \quad \tau_1 = \inf\{t \ge 0 : \; |X_t| \in D\}.$$

Then, the process

$$|X_{t\wedge\tau_1}|^{2-d}, \quad t \ge 0,$$

is a martingale.

Proof. The function $|x|^{2-d}$ is harmonic in $\mathbb{R}^d$. Thus $|X_t|^{2-d}$ is a local martingale itself. Since it is bounded in D^c, it must be a true martingale. □

Suppose $X_0 = x_0 \in D^c$. We can change the law of X by using $|X_{t\wedge\tau_1}|^{2-d}$ as a Radon-Nikodým derivative (after rescaling). Thus we have the following multidimensional version of Proposition 3.

Proposition 5. *Suppose $X_0 = x_0$ such that $|x_0| > 1$. Let us call the law of the process X by Q. We change Q by using the positive martingale*

$$\phi(X_t) = |X_{t\wedge\tau_1}|^{2-d} / |x_0|^{2-d}$$

as a Radon-Nikodým derivative. Call this measure P. Under P, the process

$$Y_t = \frac{X_{t\wedge\tau_1}}{|X_{t\wedge\tau_1}|^2} \tag{17}$$

is again a d-dimensional conformal local martingale such that every coordinate process Y is a true martingale.

Before we go on to the proof, let us remark that the procedure above of mapping $x \mapsto x/|x|^2$ is known as inversion with respect to the unit sphere in harmonic analysis. In the proofs below we use the technique of *Kelvin transforms* and we refer the reader to the excellent book on harmonic function theory [1, Chapter 4] by Axler, Bourdon, and Ramey.

The Kelvin transform K is an operator acting on the space of real functions u on a subset of $\mathbb{R}^d \backslash \{0\}$. Let u be a C^2 function on an open subset Ω of $\mathbb{R}^d \backslash \{0\}$. Let Ω^* be the image of Ω under the inversion map $x \mapsto x/|x|^2$. For such a u, we define a function $K[u] : \Omega^* \to \mathbb{R}$ by the formula

$$K[u](y) = |y|^{2-d} u\left(y/|y|^2\right). \tag{18}$$

The most striking property of this transform is that K *commutes* with the Laplacian ([1, page 62]). That is, at any point $y \in \Omega^*$, we have

$$\Delta K[u](y) = K\left[|x|^4 \Delta u(x)\right](y). \tag{19}$$

In particular, if u is harmonic in Ω (i.e., $\Delta u = 0$), then $K[u]$ is harmonic in Ω^*. More importantly, if u is subharmonic (i.e., $\Delta u \geq 0$), then so is $K[u]$.

Proof of Proposition 5. To show that the ith coordinate process $Y(i)$ is a local martingale, we use the harmonic function

$$u(x) = x_i \quad \text{on} \quad \Omega = \mathbb{R}^n \backslash \{0\}.$$

By (19), its Kelvin transform is also harmonic. Hence the process

$$|X_t|^{2-d} u\left(X_t / |X_t|^2\right) = |X_t|^{-d} X_t(i)$$

is a local martingale under Q. But, since $|X_t|^{2-d}$ (until τ_1) is used as a Radon-Nikodým derivative after being scaled, by Bayes rule [25, page 193], the process $u\left(X_{t\wedge\tau_1} / |X_{t\wedge\tau_1}|^2\right)$ is a local martingale under the changed measure P. But this implies that $Y(i)$ is a local martingale under P. But, since every $Y(i)$ is bounded, being inside the unit disc, they must be true martingales.

To show that Y is conformal, we use the harmonic function $u(x) = x_i x_j$ again on the full domain $\mathbb{R}^n \backslash \{0\}$ for any pair of coordinates $i \neq j$. Exactly as in the previous paragraph, we infer from (19) that $Y_t(i) Y_t(j)$ is a local martingale under P. But this implies $\langle Y(i), Y(j)\rangle \equiv 0$. That their quadratic variations must be the same follows from symmetry. □

Proposition 6. *Let X be any conformal local martingale in $\mathbb{R}^d$ ($d > 2$) absorbed upon hitting the boundary of D. Let U be a sub-harmonic function in a region containing the complement of the disc, i.e., $\Delta U \geq 0$ on $\Omega \supseteq \mathbb{R}^d \backslash D$.*

Assume more over that U is harmonic at a neighborhood of infinity, i.e., $\Delta U = 0$ in $\{x : |x| > R\}$ for some large R, and is "harmonic at infinity", i.e.,

$$\lim_{x \to 0} |x|^{2-d} U\left(x / |x|^2\right) = L < \infty, \tag{20}$$

then $U(X)$ is a submartingale even if X is not a martingale. Equivalently, condition (20) can be replaced by

$$\lim_{x \to \infty} U(x) = 0. \tag{21}$$

Finally if U is actually harmonic throughout Ω, then $U(X)$ is a martingale.

Proof. To prove this, we will again use Kelvin transforms but in the reverse direction. We are given that U is harmonic in $\Omega \supseteq \{x : |x| > 1\}$. It is clear that under inversion map $x \mapsto x/|x|^2$, the image of Ω is given by Ω^* which contains $\{x : |x| < 1\}$ except $\{0\}$.

Now we note from (18) that K takes nonnegative functions to nonnegative functions. Thus if $\Delta U \geq 0$ in Ω, then

$$\Delta K[U] = K\left[|x|^4 \Delta U\right] \geq 0.$$

In other words, U sub-harmonic in Ω implies that $K[U]$ is sub-harmonic in Ω^*, which is harmonic in a punctured neighborhood around zero. The condition (20), via Theorem 2.3 [1, page 32], implies that the singularity at zero for $K[U]$ is removable. Thus one can extend $K[U]$ to an actual bounded subharmonic function in the entire region $\Omega^* \cup \{0\}$ which is also harmonic in a neighborhood of zero. By Cauchy's estimates [1, page 33], $K[U]$ has bounded derivatives inside the unit disc.

We now use the process Y defined in (17). As we have already shown Y is a conformal martingale under the changed measure P. Since $K[U]$ is subharmonic with bounded derivatives it follows that $K[U](Y_t)$ is a submartingale under P. The following is well-known.

• Suppose Z is a process such that for all pairs of bounded stopping times (σ, τ) such that $\sigma \leq \tau$ almost surely, one has $E(Z_\sigma) \leq E(Z_\tau)$. Then Z is a submartingale.

The proof of the above follows by considering any $s < t$ and considering any set A measurable with respect to $\mathcal{F}_s$. Define the stopping time

$$\tau = \begin{cases} t & \text{if } \omega \in A^c \\ s & \text{otherwise.} \end{cases}$$

One can verify that $\tau \leq t$ is a stopping time. Thus, by assumption,

$$E(X_t) \geq EX_\tau = EX_t(1 - 1_A) + EX_s 1_A = EX_t - E(X_t - X_s)1_A.$$

This proves the result.

Now we have shown that $K[U](Y_t)$ is a submartingale. We use below the fact that K is its own inverse, i.e., $K[K[u]] = u$. Thus, for any two stopping times $\sigma \le \tau$, we have

$$
\begin{aligned}
0 &\le E^P K[U](Y_\tau) - E^P K[U](Y_\sigma) \\
&= E^Q \frac{|X_\tau|^{2-d}}{|x_0|^{2-d}} K[U](X_\tau / |X_\tau|^2) - E^Q \frac{|X_\sigma|^{2-d}}{|x_0|^{2-d}} K[U](X_\sigma / |X_\sigma|^2) \\
&= |x_0|^{d-2} \left[EU(X_\tau) - EU(X_\sigma) \right].
\end{aligned}
$$

By the argument in the previous paragraph, we have proved that $U(X_t)$ is a submartingale.

The equivalence of conditions (20) and (21) is a theorem which can be found in [1, Theorem 4.8, page 64].

Settling the case when U is harmonic, is essentially repeating the previous proofs by replacing submartingales with martingales. This completes the proof. □

Before we move on let us remark that dimension 2 should be treated very differently from the preceding analysis. The primary reason being the property of conformal invariance of local martingales. As an example, any local martingale which avoids the open unit disc can be made a true martingale under the function $z \mapsto 1/z$ *without* taking recourse to a change of measure. For this reason an analogue of Proposition 6 in two dimensions is obvious. However, the picture is not complete since we feel that avoiding the disc is not the right restriction to use, and a much more general result should be possible. However, at this point it is not clear what the scope of such a result might be and we leave it for the future.

4 Applications to financial bubbles

4.1 A curious property of the inverse Bessel process

We have seen in Proposition 3 that if N is a strict local martingale, and h is a convex function sublinear at infinity, then $h(N_t)$ is again a submartingale. This is in contrast to functions which are linear at infinity. For example, in the case of $f(x) = x$, the process is actually a supermartingale. Here we demonstrate another example, for the function $(x - K)^+$ with $K > 0$.

Proposition 7. *Let X_t be a BES(3) process starting from one. For any real $K \in [0, 1/2]$, the function $t \mapsto E\{(1/X_t - K)^+\}$ is strictly decreasing for all $t \in (0, \infty)$. However, if $K > 1/2$, the function $t \mapsto E\{(1/X_t - K)^+\}$*

is initially increasing and then strictly decreasing for

$$t \geq \left(K \log \frac{2K+1}{2K-1} \right)^{-1}.$$

*R*emark: Note that the bound on the right hand becomes zero when $K = 1/2$ which demonstrates its sharpness.

Proof of Proposition 7. The proof uses the change of measure technique in Theorem 1. From the change of measure relationship (2) we deduce the following identity

$$\begin{aligned} h(t) := E^P \left\{ \left(\frac{1}{X_t} - K \right)^+ \right\} &= E^Q \left\{ X_{t \wedge \tau_0} \left(\frac{1}{X_t} - K \right)^+ \right\} \\ = E \left[(1 - KB_t)^+ 1_{\{\tau_0 > t\}} \right] &= E(1 - KB_t)^+ - P(\tau_0 \leq t). \end{aligned}$$

where B denotes a Brownian motion starting from one, and τ_0 is the hitting time of zero for the Brownian motion B.

If we take derivatives with respect to t in the equation above, we get

$$\begin{aligned} h'(t) &= \frac{d}{dt} E(1 - KB_t)^+ \frac{d}{dt} P(\tau_0 \leq t) \\ &= \frac{K}{2} \frac{d}{dt} E L_t^{1/K} + \frac{d}{dt} P(\tau_0 \leq t). \end{aligned} \tag{22}$$

The second equality above is due to Tanaka formula (15).

Now the second term on the right side of (22) above is the density of the first hitting time of zero, which we know ([25, page 80]) to be

$$\frac{1}{\sqrt{2\pi t^3}} e^{-1/2t}. \tag{23}$$

To compute the first term on the right of (22) we have the following claim.

Lemma 8. *Suppose* $\{X_t,\ t \geq 0\}$ *is a continuous nonnegative local martingale which satisfies the following SDE*

$$dX_t = \sigma(t, X_t) d\beta_t, \quad t \in [0, \infty), \quad X_0 = 1. \tag{24}$$

Here β *is a one-dimensional standard Brownian motion and* $\sigma(t, x)$ *is some measurable nonnegative function on* $\mathbb{R}^+ \times \mathbb{R}^+$.

Further assume that the process X_t admits a continuous marginal density at each time t at every strict positive point y which is given by

$$p_t(y) = P\left(X_t \in dy \;\middle|\; X_0 = 1\right), \quad y > 0.$$

Let L_t^a denote the local time of X at level $a > 0$ and at time t. Then

$$\frac{d}{dt} E\left(L_t^a\right) = \sigma^2(t,a) p_t(a). \tag{25}$$

Using the previous lemma, we can explicitly compute the right side of equation (22). Recall (see [25, page 97]) that for x, y, and t strictly positive the transition function of Brownian motion absorbed at zero is given by

$$p(t,x,y) := \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right)\right].$$

Thus, combining (22), (23), and (25), we get

$$h'(t) = \frac{K}{2\sqrt{2\pi t}}\left[e^{-(1-1/K)^2/2t} - e^{-(1+1/K)^2/2t}\right] - \frac{1}{\sqrt{2\pi t^3}} e^{-1/2t}. \tag{26}$$

Thus, $h'(t) < 0$ if and only of

$$\begin{aligned} \frac{2}{Kt} &> e^{1/2t}\left[e^{-(1-1/K)^2/2t} - e^{-(1+1/K)^2/2t}\right] \\ &= \exp\left[\frac{(2K-1)}{2K^2 t}\right] - \exp\left[-\frac{(2K+1)}{2K^2 t}\right]. \end{aligned} \tag{27}$$

We need to do a bit more work. Let $t = 1/y$. Consider the function on the right side of the last inequality. We need to consider two separate cases. First suppose $K > 1/2$. Then both $2K-1$ and $2K+1$ are positive. If for two positive parameters $\lambda_2 > \lambda_1 > 0$, we define a function q by $q(y) = \exp(\lambda_1 y) - \exp(-\lambda_2 y)$, $y > 0$, it then follows that

$$\begin{aligned} q'(y) &= \lambda_1 e^{\lambda_1 y} + \lambda_2 e^{-\lambda_2 y}, \quad q'(0) = \lambda_1 + \lambda_2, \\ q''(y) &= \lambda_1^2 e^{\lambda_1 y} - \lambda_2^2 e^{-\lambda_2 y}. \end{aligned} \tag{28}$$

Note that $q''(y) < 0$, for all

$$0 \le y < \frac{2\log\left(\lambda_2/\lambda_1\right)}{\lambda_1 + \lambda_2}. \tag{29}$$

Since $q'(y)$ is always positive, it follows that q is an increasing concave function starting from zero in the interval given by (29). Thus it also follows that, in that interval,

$$q(y) = q(y) - q(0) < yq'(0) = y\left(\lambda_1 + \lambda_2\right). \tag{30}$$

Take $\lambda_1 = (2K-1)/2K^2$ and $\lambda_2 = (2K+1)/2K^2$. Then $\lambda_1 + \lambda_2 = 2/K$. By (29) we get that if

$$y \le C_1 := K \log \frac{2K+1}{2K-1},$$

then, from (30) it follows

$$K\left\{\exp\left[\frac{(2K-1)y}{2K^2}\right] - \exp\left[-\frac{(2K+1)y}{2K^2}\right]\right\} < 2y.$$

That is, by (27), $h'(t) < 0$, i.e., h is strictly decreasing for all

$$t > \left(K \log \frac{2K+1}{2K-1}\right)^{-1}.$$

The case when $0 < K \le 1/2$ can handled similarly. Suppose $0 < \lambda_1 < \lambda_2$ are positive constants. Consider the function

$$r(y) = -\lambda_1 y + \lambda_2 y - e^{-\lambda_1 y} + e^{-\lambda_2 y}, \quad y \in [0, \infty).$$

Then $r(0) = 0$, and

$$r'(y) = -\lambda_1\left(1 - e^{-\lambda_1 y}\right) + \lambda_2\left(1 - e^{-\lambda_2 y}\right) > 0, \quad y \in [0, \infty),$$

because $\lambda_1 < \lambda_2$. Thus, for all positive y, we have $r(y) > 0$, i.e.,

$$e^{-\lambda_1 y} - e^{-\lambda_2 y} < (-\lambda_1 + \lambda_2)y.$$

We use this for $\lambda_1 = (1-2K)/2K^2$ and $\lambda_2 = (1+2K)/2K^2$. Note that, as before

$$(-\lambda_1 + \lambda_2)y = 2y/K.$$

From (27) it follows that $h'(t) > 0$ for all $t \in (0, \infty)$. Thus we have established that if $K \le 1/2$, the function $t \mapsto E(1/X_t - K)^+$ is strictly decreasing for all $t \in (0, \infty)$. This completes the proof of the proposition. □

Proof of Lemma 8. To prove this, we again use the occupation time formula involving the local time for general continuous semimartingales.

For any smooth nonnegative function $f : \mathbb{R} \to \mathbb{R}^+$ with compact support contained in $(0, \infty)$, we have the following identity

$$\int_{\mathbb{R}^+} f(a) L_t^a da = \int_0^t f(X_s) d\langle X \rangle_s = \int_0^t f(X_s)\sigma^2(s, X_s) ds,$$

where the final identity follows from (24). Now taking expectations on both sides, we obtain

$$\begin{aligned} E\left[\int_{\mathbb{R}^+} f(a) L_t^a da\right] &= E\int_0^t f(X_s)\sigma^2(s, X_s) ds = \int_0^t E\left[f(X_s)\sigma^2(s, X_s)\right] ds \\ &= \int_0^t \left[\int_{\mathbb{R}^+} f(a)\sigma^2(s, a) p_s(a) da\right] ds. \end{aligned} \tag{31}$$

The second equality above is due to Fubini-Tonelli for nonnegative integrands. The final equality is by definition of the marginal density and the fact that the support of f is in $(0, \infty)$.

Applying Fubini-Tonelli repeatedly and interchanging the orders of integration on both sides of (31), we get

$$\begin{aligned} \int_{\mathbb{R}^+} f(a) E(L_t^a) da = E\left[\int_{\mathbb{R}^+} f(a) L_t^a da\right] &= \int_0^t \left[\int_{\mathbb{R}^+} f(a)\sigma^2(s, a) p_s(a) da\right] ds \\ &= \int_{\mathbb{R}^+} f(a) \left[\int_0^t \sigma^2(s, a) p_s(a) ds\right] da. \end{aligned}$$

Since this holds for all smooth nonnegative functions f with compact support in $(0, \infty)$, it follows that

$$E(L_t^a) = \int_0^t \sigma^2(s, a) p_s(a) ds, \quad \forall\, a > 0.$$

The conclusion of the lemma follows.

□

For mathematical completeness we show below that a similar result can be proved for the Bessel process starting from zero, although in this case there is no dependence on K. The proof is much simpler and essentially follows by a scaling argument. Note that, even in this case the reciprocal of the Bessel process is well-defined for all times except at time zero. Hence

$1/X_t$, $t \in (0,\infty)$, can be thought as a Markov process with an *entrance distribution*, i.e., a pair consisting of a time-homogenous Markov transition kernel $\{P_t\}$, $t > 0$, and a family of probability measures $\{\mu_s\}$, $s > 0$, satisfying the constraint $\mu_s * P_t = P_{t+s}$. Here $*$ refers to the action of the kernel on the measure.

Proposition 9. *Let X_t be a 3-dimensional Bessel process, BES(3), such that $X_0 = 0$. For any two time points $u > t > 0$, and for $K \geq 0$, one has*

$$E\left(\frac{1}{X_u} - K\right)^+ < E\left(\frac{1}{X_t} - K\right)^+. \tag{32}$$

Proof. Fix $u > t$. Recall that BES(3), being the norm of a three dimensional Brownian motion, has the Brownian scaling property when starting from zero. That is to say, for any $c > 0$,

$$\left(\frac{1}{\sqrt{c}} X_{cs},\ s \geq 0\right) \stackrel{\mathcal{L}}{=} (\ X_s,\ s \geq 0\ ),$$

where the above equality is equality in law.

Take $c = u/t$, and apply the above equality for X_s when $s = t$, to infer that $c^{-1/2} X_u$ has the same law as X_t, and thus

$$E\left(\frac{1}{X_u} - K\right)^+ = E\left(\frac{c^{-1/2}}{X_t} - K\right)^+ = c^{-1/2} E\left(\frac{1}{X_t} - \sqrt{c}K\right)^+. \tag{33}$$

Note that for any $\sigma > 1$, we have

$$\frac{1}{\sigma}(x - \sigma K)^+ < (x - K)^+, \quad \forall\, x > 0.$$

Since $c > 1$, taking $\sigma = \sqrt{c}$, one deduces from (33)

$$E\left(\frac{1}{X_u} - K\right)^+ < E\left(\frac{1}{X_t} - K\right)^+,$$

which proves the result. □

We conclude this section with an example of a strict local martingale S where $E(S_t - K)^+$ is not asymptotically decreasing for any K. This, coupled with the earlier Bessel result, establishes the fact that functions which are not sublinear at infinity can display a variety of characteristics when applied to strict local martingales.

We inductively construct a process in successive intervals $[i, i+1)$ by the following recipe. The process starts at zero. The process in the odd interval $[2i, 2i+1)$ is an exponential Brownian motion $\exp(B_t - t/2)$ starting from S_{2i} and independent of the past. On the even intervals $[2i+1, 2i+2)$ the process S is an inverse Bessel process starting from S_{2i+1} and again independent of the past. The constructed process is always a positive local martingale. The value of the function $E(S_t - K)^+$ is increasing in the odd intervals due to the martingale component, and decreasing (at least when $K \leq 1/2$) in the Bessel component by Proposition 7.

One might object to the fact that this process is not strict local throughout. But, one can mix the two components, by a sequence of coin tosses which decides whether to use Brownian or the Bessel component in the corresponding interval. By choosing the probability of heads in these coins in a suitably predictable manner, we can generate a local martingale which is strict throughout but $E(S_t - K)^+$ does not decrease anywhere.

4.2 Application to financial bubbles

A natural question is: what happens to a financial market when the no arbitrage condition yields a strict local martingale (rather than a true martingale) under a risk neutral measure? Several authors have looked at this problem and offered solutions to anomalies which might result from the lack of the martingale property. One interesting perspective offered in this direction is the theory of price bubbles as argued in 2000 by Loewenstein and Willard [27]. They propose that to identify a bubble one needs to look at the difference between the market price of an asset and its fundamental price. Their argument is later complemented and further developed by Cox and Hobson [5] and the two articles by Jarrow, Protter, and Shimbo [21], [22]. Please see the latter articles for the definitions of the market and the fundamental prices of an asset and any of the other financial terms that follow. In particular, the authors in [21] and [22] classify bubbles into three types in an arbitrage-free market satisfying Merton's *No Dominance* condition (see [21] or [29]). One, in which the difference between the two price processes under an equivalent local martingale measure is a uniformly integrable martingale; two, when it is a martingale but non-uniformly integrable; and last, when it is a strict local martingale. In a static market with infinite horizon, for a stock which pays no dividends, Example 5.4 in [22] shows that the difference between the two prices is actually the current market price of the stock. Thus a stock price which behaves as a strict local martingale under an equivalent local martingale measure is an example of a price bubble of the third kind. Cox and Hobson [5], too, use this definition of stock price bubbles. They further furnish several interesting examples of bubbles both where volatility increases with price levels, and where the bubble is the result of a feedback mechanism. They go on to exhibit (among other things) how in the presence of bubbles put-call parity might not hold and call prices do not tend to zero as strike tends to infinity.

We mention in passing that in the Economics literature the study of bubbles is older. For example, the possibility that bubbles can exist in discrete-time infinite-horizon economies has been explored in Diba and Grossman [9], Kocherlakota [26], and Tirole [34]. In fact, it has only recently been studied using the tools of mathematical finance. In this vein, other than the ones already mentioned, see, for example, Jarrow and Madan [20], Gilles [15], Gilles and Leroy [16], and Huang and Werner [18].

We consider a market with a single risky asset (stock) and zero spot interest rate. Let $\{S_t\}$, $t \in (0, \infty)$, be a positive continuous strict local martingale which models the discounted price of the (non-dividend paying) stock under an equivalent local martingale measure. We have the following result which follows immediately from Proposition 3 and the subsequent Bessel example.

Proposition 10. *Suppose for a European option, the discounted pay-off at time T is given by a convex function $h(S_T)$ which is sublinear at infinity, i.e., $\lim_{x\to\infty} h(x)/x = 0$. Then the price of the option is increasing with the time to maturity, T, whether or not a bubble is present in the market. In other words, $E(h(S_T))$ is an increasing function of T. For example, consider the put option with a pay-off $(K - x)^+$.*

However, for a European call option, the price of the option $E(S_T - K)^+$ with strike K might decrease as the maturity increases.

One should note the distinction from the case of true martingales for which the price curve strictly increases by Jensen's inequality. This, in turn, has interesting implications for American options. A standard truism for financial markets is Merton's *no early exercise* theorem which states that the prices of American calls and European calls are the same (given identical strike prices, maturity times, and the absence of dividends), which in turn implies that the optimal exercise time of an American call is at maturity (hence "no early exercise"). This famous theorem can be found in many textbooks, for example in that of Shreve [33, p. 363]. The proof of this result however uses that the risky asset price is a (true) martingale under the risk neutral measure. If it is in fact a strict local martingale, then "no early exercise" need not hold, as shown in both [5] and [21], [22].

However, once we establish that the prices of European options decrease with maturity, it readily follows that "no early exercise" cannot hold. This feature may seem strange at first glance, but if we assume the existence of a financial bubble, the intuition is that it is advantageous to purchase a call with a short expiration time, since at the beginning of a bubble prices rise, sometimes dramatically. However in the long run it is disadvantageous to have a call, increasingly so as time increases, since the likelihood of a crash in the bubble taking place increases with time.

Of course, pricing a European option by the usual formula when the underlying asset price is a strict local martingale is itself controversial. For example, Heston, Loewenstein, and Willard [17] observe that under the existence of bubbles in the underlying price process, put-call parity might not hold, American calls have no optimal exercise policy, and lookback calls have infinite value. Madan and Yor [28] have recently argued that when the underlying price process is a strict local martingale, the price of a European call option with strike rate K should be modified as

$$\lim_{n\to\infty} E\left[(S_{T\wedge T_n} - K)^+\right],$$

where $T_n = \inf\{t \geq 0 : S_t \geq n\}$, $n \in \mathbb{N}$, is a sequence of hitting times. Please see the forthcoming notes by Bentata and Yor [2], which is very closely related to our work, for a more complete picture. This proposal does however, in effect, try to hide the presence of a bubble and act as if the price process is a true martingale under the risk neutral measure, rather than a strict local martingale.

Let us also mention that a different approach to such market anomalies has been studied extensively in Fernholz and Karatzas [11], Fernholz, Karatzas, and Kardaras [13], and Karatzas and Kardaras [24]. Such an approach also involves strict local martingales and a fine analysis using Bessel processes for specific examples. In [11] the authors investigate the case when the *candidate Radon-Nikodým derivative* for the risk-neutral measure turns out to be a strict local martingale. See Proposition 3.4 (also Remark 4.2) for the details. This is intimately connected with what the authors call a *weakly diverse market* which results in a number of anomalies similar to the case of bubbles. For example, put-call parity fails to hold in such markets. See, Remark 9.1 and 9.3 in [13]. Also see Example 9.2 for anomalies in the price of European call option. In a very interesting example [24, Example 4.6], the authors explicitly compute an arbitrage portfolio in the case of a one stock market modeled by a three dimensional Bessel process. This makes explicit the proposed arbitrage possibilities by Delbaen and Schachermayer

[6]. Please see the survey [12] by Fernholz and Karatzas for an exposition of all these results and how they fit together in stochastic portfolio theory.

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