REVIEWS

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Mathematics for Finance: An Introduction to Financial Engineering. By Marek Capiński and Tomasz Zastawniak. Springer, London, 2003, x + 310 pp., ISBN 1-85233-330-8, $34.95.

Reviewed by **Philip Protter**

Mathematical finance (or financial engineering, as it is often known) is a young subject for mathematics, but is highly popular with students. No doubt the allure of being connected to vast sums of money is a part of the attraction. Yet it is a difficult subject, requiring a broad array of knowledge of subjects that are traditionally considered hard to learn.

Forty years ago, options and what are now called "financial derivatives" were little known. Options were traded on the Chicago Board Options Exchange (CBOE), primarily for commodities such as pork bellies, orange juice, coffee, and precious metals. Let us take a minute to describe a situation where an option is useful. Imagine a small Indiana farmer raising pigs. The price is high now, but his pigs are only 80% grown. He can market them now and make a handsome profit, or he can wait until they are fully grown and make a larger profit if the price stays up, but end up making significantly less money if the price falls. He could solve the problem by buying a forward contract, locking in a prearranged price and thus a sure profit, but what if the price rises further? Then he will kick himself for having locked in the price. An option, on the other hand, gives him the right, but not the obligation, to sell his pigs at the prearranged price, thus guaranteeing him the nice profit but not excluding a potentially bigger one.

The prices of options such as the one just described were set by the market: supply and demand. In the United States there is a fervent belief that the market knows best and, if left alone, will arrive at a fair and just price. There are many unspoken hypotheses involved with this belief, and in the case of commodities, several were violated. It will suffice to point out that small farmers were buying options sold by large banks and companies. In the early 1970s, Black, Scholes, and Merton showed that by using the Itô stochastic calculus and a simple model describing the dynamics of the price of a risky asset, one could arrive at a fair price for an option. They did this using a key idea: if one sells the option for $\$x$, there is a hedging strategy by which one can use that $\$x$ to trade in the commodity over time until the option is due and end up with exactly what is owed to the option purchaser at the settlement time. There is no risk at all, except the implicit risk that the model for the dynamic price of the commodity is wrong. Therefore, if the market price is larger than $\$x$, one can charge the market price and match the option and have money left over. If the market price is less than $\$x$, one can buy the options and make money in reverse. It turned out that the market is often wrong, but the breakthrough of Black and Scholes went largely ignored by the financial players. This gradually changed, largely through the efforts of a few visionary people at Wells Fargo Bank, who worked not so much with commodities as with the (then) new concepts of portfolio insurance and index funds (see [**2**]).

The option I described has the result of removing the risk for the pig farmer. For a (usually rather small) fee, he can buy what amounts to an insurance policy on the price of pork bellies. This is known as a *transfer of risk*: the option seller assumes the risk the farmer is not willing to assume, just as an insurance company assumes (for a fee) the financial risk of one's house burning down. This example also shows the utility of such insurance, since now the farmer will not slaughter the pigs before they are fully grown, and society as a whole will benefit (assuming that people eat pork). Once the methodology for pricing this transfer of risk became widely known, the concept spread widely. It has transformed modern business and arguably helped to create the financial boom years of the 1990s. One can now insure against currency fluctuations, dangerous drops in stock prices in one's portfolio, and all manner of (often fairly esoteric) business operations by this form of risk transfer. Options are also widely used for less virtuous goals, such as helping companies and executives avoid paying taxes, and of course for what amounts, simply, to gambling.

The ideas behind the model of Black and Scholes go back to the turn of the last century. In 1900 a French Ph.D. student of Poincaré named Louis Bachelier proposed what we now know as Brownian motion as a model for the French stock market (the Paris *bourse*). He linked Brownian motion to the fundamental solution of the heat equation, a connection admired by Poincaré. Nevertheless, since economics was not a subject considered worthy of study by the Parisian mathematics community at the time, Bachelier's thesis was not appreciated and fell into obscurity, even though he had essentially constructed Brownian motion five years earlier than Einstein, who in a famous 1905 paper used a model of Brownian motion to argue his case for the existence of molecules. Bachelier's work was referenced by researchers from time to time (including A. N. Kolmogorov), but its significance was not appreciated until the statistician L. J. Savage showed it to the (eventual) Nobel Prize-winning economist Paul Samuelson. Samuelson modified it and proposed geometric Brownian motion as a model for stock prices instead. In a fundamental paper with an appendix by H. P. McKean, Jr., he further connected the pricing of options to solving partial differential equations. In the case of American options (a more complicated situation than my previous example), these PDEs become free boundary problems, which of course are notoriously difficult. All this preceded by decades the work of Black, Scholes, and Merton.

The mathematics involved in the Black-Scholes paradigm is measure-theoretic probability theory, Brownian motion, stochastic processes including Markov processes and martingale theory, Itô's stochastic calculus, stochastic differential equations, and partial differential equations. Those prerequisites give one entry to the subject, which is why it is best taught to advanced Ph.D. students. One might expect an American undergraduate to know calculus-based probability theory and to have had some exposure to PDEs and perhaps, if one is lucky, an economics course or two, but not much more. Therefore, any attempt to teach such a subject to undergraduates is fraught with compromise and almost inevitably involves "dumbing down." Indeed, it becomes analogous to teaching integration to freshmen: the students learn the mechanics, and those who continue on to take a junior level analysis course begin to understand the subject, but they understand it fully only when they study Lebesgue measure theory. In this spirit, an undergraduate course in financial mathematics, seen as a first exposure only, might indeed be useful. One could even teach Brownian motion heuristically using the physics analogy, one could teach the mechanics of Itô calculus in a fashion analogous to the way we teach the mechanics of calculus to freshmen, and one could explain the connection to some PDEs they might know. (Indeed, a recent lovely article by Rob Almgren [**1**] could be used to inspire a PDE-based pedagogic approach to the subject. Personally I do not favor such an approach, since I think it gives a biased view of the subject, one favored primarily by mathematicians who know and like PDEs but are rather innocent of the more significant results coming from a probabilistic approach. This leads to a misleading perception that the subject has a lot to do with PDEs, which—while important—are ultimately only a tangential element of the theory and its applications. Nevertheless, a lot of mathematicians know and like PDEs, and perhaps for that reason such an approach is popular.)

The book under review does none of these things. Instead, it uses discrete time theory. One can do this by basing the material on the Cox-Ross-Rubinstein theorem and the like, approximating the Black-Scholes model with discrete-time binomial trees. But while one can verify the truth of the Black-Scholes formula this way, and while the main mathematical tool is the central limit theorem, in my opinion little light is actually shed on what is going on. This is especially true because if one varies the method only a little (for example, by using a nonrecombining tree or a trinomial model), it then breaks down, destroying what little intuition might otherwise have been gained. This puts the professor in the untenable position of hoping that no student asks such a question, which I suppose is likely—more's the pity. Continuing my analogy, this approach is like trying to make calculus easy by teaching it using only sums because sums are considered more elementary than integrals. The book becomes merely some economics explained with fairly simple and uninteresting mathematics.

It need not have been so. Had the authors been willing to require a little more background of the student, a stochastic calculus approach could have been used in the spirit described earlier. Such an approach would stand a better chance of revealing the inner workings of the subject. Alas, the authors' approach is all too common. This is not to condemn discrete time models out of hand; indeed, the recent book by H. Föllmer and A. Scheid [**3**] is one of the deepest introductions to the subject, and the now classic book by S. Pliska [**4**] is rough going but ultimately rewarding. These good books require a sophisticated reader, however, and are clearly at the graduate-student level.

Let me give some idea of what the student (and the professor) will be missing. A fundamental idea, and one that is unique to the mathematics of economics, is the hypothesis of the absence of arbitrage opportunities. In words, an arbitrage opportunity is the possibility to make a profit, when starting with nothing, by assuming no risk. (An example is a sweepstakes where it costs nothing to enter, but where there is a possibility, however remote, of winning something.) It is assumed by economists that such opportunities do not exist in financial markets, since if they were to exist, the person or firm offering the opportunity would go bankrupt almost instantaneously. Suppose then that we are given a financial model (Ω, P, S), where P is a complete probability measure on the measurable space Ω and $S = (S_t)_{0 \le t \le T}$ is a nonnegative stochastic process on Ω that represents the price of a risky asset such as a stock. We call a probability measure Q on Ω *risk-neutral* if it is equivalent[1] to P and transforms S into a local martingale, and we denote the collection of risk-neutral measures by $\mathcal{Q}$. The set $\mathcal{Q}$ is convex (and possibly empty), and one can prove by stochastic calculus that if it consists of only one element, call it P^*, then P^* has an integral representation property known as martingale representation; namely, if X is a random variable with mean zero in (for example) $L^2(dP^*)$, then there exists a stochastic process H such that $X = \int_0^T H_s dS_s$. The "First Fundamental Theorem of Finance" states that there is no arbitrage if and only if $\mathcal{Q}$ is nonempty. The "Second Fundamental Theorem of Finance" states that, assuming there is no arbitrage, $\mathcal{Q}$ consists of exactly one element P^* if and only if the

[1]Equivalence here means that Q and P have the same sets of probability zero; that is, they are mutually absolutely continuous.

market is *complete*, that is, every contingent claim in the market (i.e., every $L^2(dP^*)$ random variable) can be perfectly hedged (i.e., has an integral representation with respect to S). Of equal importance, a market is complete if and only if every contingent claim X has a unique fair price; moreover, that price is given by $E^*\{X\}$, where E^* denotes expectation under the unique risk-neutral measure P^*. Not only is this theory elegant and simple to state (but not to prove), but much of the subject is devoted to the calculation of the hedging strategy H and the unique price $E^*\{X\}$.

This simple and clarifying way of viewing the subject is obscured to the point of being opaque in the book under review and in similar books aimed at (in the reviewer's opinion) too unsophisticated an audience. To be fair, this book does indeed state in some manner the first fundamental theorem in chapter four, but not in a fashion that elucidates its meaning or enhances its appeal. Indeed, the authors begin the treatment with the dreaded phrase, "In this section, which can be omitted on first reading," and the only clue the student may have that the theorem is important is that its title contains the word "fundamental."

I have been told that one of the subjects that is often the core of the undergraduate mathematics major, abstract algebra (groups, rings, modules, fields, etc.), was taught only at the graduate level in (for example) the 1930s, and over the years it has worked its way down to younger students. Some subjects can do that. I await the day when C^*-algebras, K-theory, and homotopy theory are standard in the undergraduate curriculum. Perhaps it is the same with mathematical finance: it simply is not (yet?) meant to be an undergraduate subject. In a way that is too bad, because the subject is beautiful and powerful, and expertise in it is much needed in industry.

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