The RANDOM EDGE Rule on Three-Dimensional Linear Programs[1] (extended abstract)

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Abstract

The worst-case expected length of the path taken by the simplex algorithm with the RANDOM EDGE pivot rule on a 3-dimensional linear program with n constraints is shown to be bounded by

$$1.3445 \cdot n \;\leq\; f(n) \;\leq\; 1.4943 \cdot n$$

for large enough n.

1 Introduction

The RANDOM EDGE pivot rule is undoubtedly the most natural, and simplest (randomized) pivot rule for the simplex algorithm: "At each iteration, proceed from the current vertex of the polyhedron P of feasible solutions to an improving neighbor, chosen uniformly at random in the one-dimensional skeleton (i.e., the graph) of P."

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Despite its simplicity, this algorithm until now has resisted almost all attempts to analyze its worst-case behavior, with a few exceptions for special cases, among them the linear assignment problem (Tovey [6]), certain linear programs on cubes, including the Klee-Minty cubes (Kelly [4], Gärtner, Henk & Ziegler [2]), and d-dimensional linear programs (i.e., $\dim(P) = d$) with at most $d+2$ constraints (Gärtner et al. [3]). All known results leave open the possibility that the RANDOM EDGE pivot rule yields a strongly polynomial time algorithm – it might be even quadratic. In particular, it is not fooled by the deformed products (defined by Amenta and Ziegler [1]), which yield the well-known exponential examples for all the classical deterministic pivot rules.

Here, we only treat the case of 3-dimensional linear programs, which, of course, is solved in linear time by every (even deterministic) finite variant of the simplex algorithm. Nevertheless, due to the remarks made above, it seems interesting to analyze the RANDOM EDGE pivot rule for this case – and here too, it seems that accurate analysis of RANDOM EDGE is quite hard.

With the usual reductions (see, e.g., Ziegler [7, Lect. 3]), we may assume that our linear program is $\min\{x_3 \;:\; x \in P\}$, where P is a 3-dimensional simple polytope (its graph is 3-regular) with exactly n facets, and hence $2n-4$ vertices and $3n-6$ edges, and no two vertices have the same objective function value x_3. Thus we have an ordering of the vertices $v_{2n-5}, v_{2n-6}, \ldots, v_1, v_0$ by decreasing objective function (i.e., by height). Here $v_0 = v_{\min}$ is the unique minimal (lowest) vertex of the linear program, while $v_{2n-5} = v_{\max}$ is the unique maximal (highest) vertex.

The expected length of the path (i.e., the number of pivot steps) taken by the simplex algorithm on the linear program, starting at vertex v of P and using the RANDOM EDGE rule, is then given by $E(v_0) = 0$ and

$$E(v) \;=\; 1 + \frac{1}{d_v}\sum_{j=1}^{d_v} E(w_j) \qquad (v \neq v_0)\;,$$

where $w_1, \ldots, w_{d_v}$ are the lower neighbors of v. It is easy to see that (in addition to the unique maximal vertex and the unique minimal vertex) there are $n-3$ vertices v with $d_v = 1$ (*1-vertices*) and $n-3$ vertices v with $d_v = 2$ (*2-vertices*); this is the 3-dimensional case of the Dehn-Sommerville equations [7, Thm. 8.21].

Define $f(n)$ to be the maximum expected number of pivot steps taken by the RANDOM EDGE algorithm on any 3-dimensional linear program with n constraints. While it is quite straightforward to construct a sequence of examples with $E(v) \geq \frac{4}{3}\cdot n - \text{const}$, our results in Section 2 (Theorem 2.2) show

$$f(n) \;\geq\; \frac{1721}{1280}\cdot n - \frac{4722}{1280}$$

for infinitely many n in arithmetic progression ($\frac{1721}{1280} = \frac{4}{3} + \frac{43}{3840}$). In Section 3 we prove

$$f(n) \;\leq\; \frac{130}{87}\cdot n - \frac{115}{29}$$

(Theorem 3.2). Both results taken together, this yields that

$$1.3445 \cdot n \;\leq\; f(n) \;\leq\; 1.4943 \cdot n$$

holds for all large enough n. In particular, asymptotically $f(n)$ lies between $(\frac{4}{3}+\varepsilon)\cdot n$ and $(\frac{3}{2}-\varepsilon)\cdot n$ for some $\varepsilon > 0$. Determining "the right coefficient" seems, however, to be very hard.

2 Lower Bounds

The expected number of pivot steps required by the simplex algorithm using Random Edge only depends on the graph of the polytope, directed via the objective function. Therefore, we will describe our examples yielding lower bounds on $f(n)$ by the corresponding directed graphs. The following result provides a nice certificate for a directed graph to come from a 3-dimensional linear program.

Theorem 2.1 (Mihalisin and Klee [5]). *A directed graph D (without loops and parallel arcs) is induced by a 3-dimensional linear program if and only if*

- *it is planar and 3-connected (as an undirected graph),*
- *it is acyclic with a unique source and a unique sink,*
- *it has a unique local sink in every face cycle (these are the non-separating induced cycles), and*
- *it admits three directed paths from its source to its sink that have disjoint sets of interior nodes.*

2.1 Duals of Cyclic Polytopes

Example 1. Our first sequence of examples are wedges, i.e., they are combinatorially equivalent to duals of cyclic polytopes. Figure 1 depicts the orientations of the edges. Here, as well as in the sequel, our convention is that the ordering of the vertices from left to right in the figure defines the (decreasing) ordering of the vertices according to the objective function. It is easy to see that the conditions of the Mihalisin–Klee theorem are satisfied.

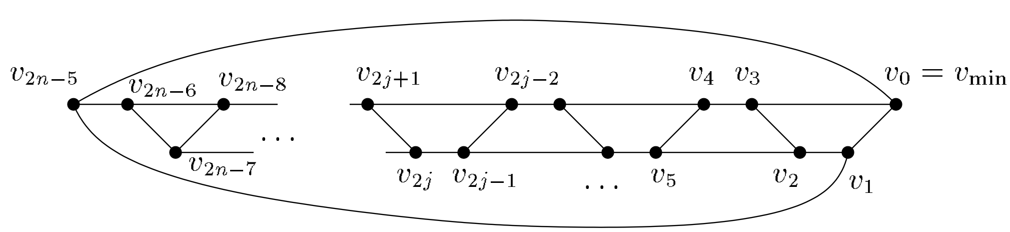


Figure 1: The example on the dual cyclic polytope.

For the expected number of pivot steps $E(v_i)$, we then have the starting values $E(v_0) = 0$ and $E(v_1) = 1$, and the recurrences

$$E(v_{2j}) \;=\; E(v_{2j-1}) + 1, \qquad E(v_{2j+1}) = \tfrac{1}{2}\big(E(v_{2j}) + E(v_{2j-2})\big) + 1$$

for $0 < j \leq n-4$. Thus, using induction, we obtain

$$E(v_{2j}) + 2E(v_{2j+1}) \;=\; 4j + 2$$

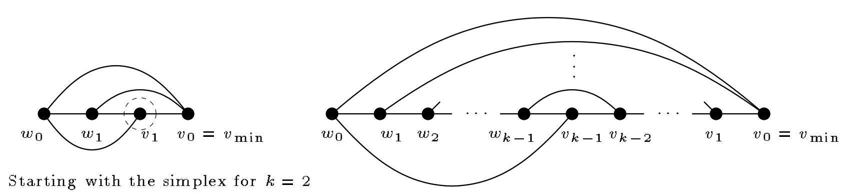
for $0 \leq j \leq n-4$. In particular, for $j = n-4$ this yields

$$\max\Big\{E(v_{2n-8}), E(v_{2n-7})\Big\} \;\geq\; \frac{4n}{3} - \frac{14}{3}.$$

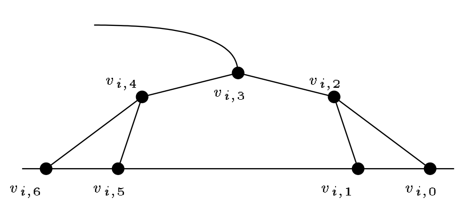
2.2 Improved Lower Bounds

The next examples are based on the construction of a "backbone polytope": This will be a simple 3-polytope P_k with $k+2$ facets and $2k$ vertices, of which k vertices $v_{k-1}, \ldots, v_0$ form a decreasing chain, such that v_0 is the minimal vertex, and v_{i-1} is the only lower neighbor of v_i, for $i > 0$.

Constructing the backbone. We start with the simplex, obtained for $k = 2$ with vertices w_0, w_1, v_1, v_0, as shown on the left. We then inductively cut off the vertex v_{k-1} by a plane, replacing it by a small triangle, as shown on the right.

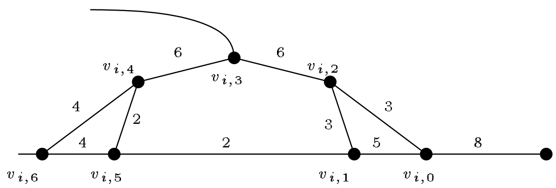


Example 2. Our second sequence of examples is obtained from the backbone polytopes P_k by performing three specific vertex cuts at each vertex v_i, for $i = 0, \ldots, k-1$. Before cutting (in P_k), each vertex v_i ($i > 0$) has indegree 2, while v_0 has indegree 3. The two vertex cuts are supposed to create at each vertex v_i the following configuration (where again all edges are directed "to the right"):



This creates a simple polytope P'_k with $n = k + 2 + 3k = 4k + 2$ facets.

Our starting vertex for RANDOM EDGE on this example will be $v_{k-1,6}$; the expected number of steps taken by RANDOM EDGE is the sum of the probabilities p_e that the edge e is traversed. We think of these probabilities as a "flow" from $v_{k-1,6}$ to $v_{0,0}$. Our figure indicates the flow values on the edges, for a flow of total value 8; equivalently, these are the transversal probabilities in units of $\frac{1}{8}$.

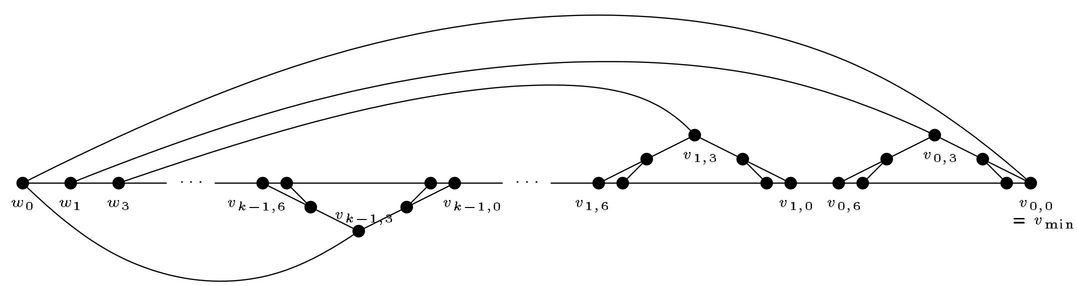


We get the same values for each of the triple-vertex-cut-off configurations, except for the last one, which has no edge leaving the global sink $v_{0,0}$. Thus the expected number of RANDOM EDGE steps, starting from $v_{k-1,6}$, is

$$E(v_{k-1,6}) \quad = \quad k \cdot \frac{43}{8} - 1 = \frac{43}{32}n - \frac{59}{16} \ ,$$

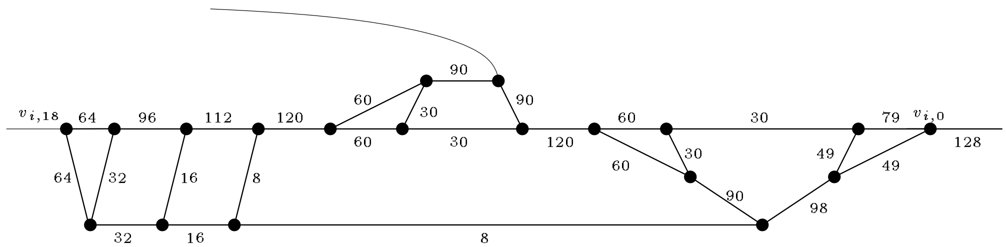
with $k = \frac{1}{4}(n-2)$. Asymptotically, this yields a better lower bound, due to $\frac{43}{32} > \frac{4}{3}$.

The graph of P'_k looks like this:



Example 3. The last examples produced $k = \frac{1}{4}(n-2)$ vertices which are not used. We will further improve the lower bound by using more facets in each local configuration and thus reducing the number of unused vertices (though our final example will still have linearly many unused vertices).

We use the same backbone polytope as before, but we replace each vertex v_i, for $i = 0, \ldots, k-1$, with the following graph. (We do not give an explicit polytopal construction for this example, but it can be constructed by cutting off vertices and edges of the backbone polytope. Alternatively, such a construction is provided by the Mihalisin-Klee Theorem.) To analyze the random path length through this graph we send from $v_{i,18}$ 128 units of flow through the network.



The total flow through all edges is 1721. For each configuration 9 facets are required. Together with the facets from the backbone construction, this yields $n = 10k + 2$. Hence we get

$$E(v_{k-1,18}) \;=\; k \cdot \frac{1721}{128} - 1 \;=\; \frac{1721}{1280}n - \frac{4722}{1280} \;,$$

where $\frac{1721}{1280} = \frac{43}{32} + \frac{1}{1280} = \frac{4}{3} + \frac{1}{96} + \frac{1}{1280} > 1.3445$.

In contrast to the preceeding examples, this example does not contain a directed Hamiltonian path.

Summarizing, we have proved the following bound.

Theorem 2.2. *For $n = 10k + 2 \geq 12$,*

$$f(n) \geq \frac{1721}{1280}n - \frac{4722}{1280}$$

holds.

Starting from the source. By splitting the maximal vertex, one can also construct examples where the expected number of steps *starting at the maximal vertex* is at least $(\frac{1721}{1280} - \varepsilon)n$. This observation is due to Günter Rote.

3 Upper Bounds

Consider any linear program on a simple 3-polytope with the notations as described in the introduction. For a vertex v, let $N_1(v)$ (resp., $N_2(v)$) denote the number of 1-vertices (resp., 2-vertices) that are not higher than v (including v itself). Put $N(v) = N_1(v) + N_2(v)$. This is the number of vertices lower than v. The core of our upper bound on $f(n)$ is the following result.

Theorem 3.1. *For each vertex v, other than the maximal vertex v_{2n-5}, we have*

$$E(v) \leq \frac{46}{87}N_1(v) + \frac{42}{87}N(v) \;.$$

Theorem 3.1 implies

$$E(v) \leq \frac{130(n-3)}{87} + \frac{15}{29} = \frac{130}{87} \cdot n - \frac{115}{29}$$

for all v. Here, the "$\frac{15}{29}$" comes from the fact that Theorem 3.1 is proved only for $v \neq v_{2n-5}$; therefore, we bound $E(v_{2n-5})$ by $1 + \frac{1}{3}\sum_{i=1}^{3} E(w_i)$, where w_1, w_2, w_3 are the neighbors of v_{2n-5}, and we exploit $N(w_1) + N(w_2) + N(w_3) \leq 6n - 21$.

Theorem 3.2. *For every 3-dimensional linear program with n constraints, the expected number of pivot steps taken by* Random Edge *is not more than*

$$\frac{130}{87} \cdot n - \frac{115}{29} .$$

In the remaining part of this section, we briefly sketch the proof of Theorem 3.1. It proceeds by deriving the generic inequality

$$E(v) \leq \alpha N_1(v) + \beta N(v) , \tag{1}$$

where, for most of the proof, α and β are treated as indeterminates. Each step of the proof yields a linear inequality on α and β that needs to be satisfied in order to imply (1). The proof is then completed once it is shown that $(\alpha, \beta) = (\frac{46}{87}, \frac{42}{87})$ satisfies all inequalities; in fact, it is optimal with respect to the objective function $\alpha + 2\beta$ (see below for more details).

Inequality (1) is proved by induction on $N(v)$. The base case $N(v) = 0$ is obvious, since v is the optimum in this case, and $E(v) = 0$. Suppose the theorem holds for all vertices lower than some vertex v.

We express $E(v)$ in terms of the expected costs $E(w)$ of certain vertices w that are reachable from v via a few downward edges. The general form of such a recursive expression will be

$$E(v) = c + \sum_{i=1}^{k} \lambda_i E(w_i) ,$$

where $\lambda_i > 0$ for each $i = 1, \ldots, k$, and $\sum_i \lambda_i = 1$.

Since we assume by induction that $E(w_i) \leq \alpha N_1(w_i) + \beta N(w_i)$, for each i, it suffices to show that

$$\sum_{i=1}^{k} \alpha\lambda_i \left(N_1(v) - N_1(w_i)\right) + \sum_{i=1}^{k} \beta\lambda_i \left(N(v) - N(w_i)\right) \geq c .$$

Write

$$\Delta_1(w_i) = N_1(v) - N_1(w_i) ,$$
$$\Delta(w_i) = N(v) - N(w_i) ,$$

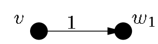
for $i = 1, \ldots, k$. (Of course, these terms are defined with respect to the currently considered vertex v.) Note that $\Delta(w_i)$ is the *distance* between v and w_i, that is, one plus the number of vertices between v and w_i.

We thus need to show that

$$\sum_{i=1}^{k} \alpha\lambda_i\Delta_1(w_i) + \sum_{i=1}^{k} \beta\lambda_i\Delta(w_i) \geq c . \tag{2}$$

This requires a quite extensive case analysis, of which we present only the beginning in this extended abstract in order to indicate the kinds of arguments used. The complete case analysis is given in the appendix.

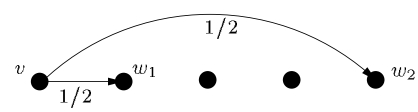
Case 1: v is a 1-vertex.
Let w_1 denote the target of the unique downward edge emanating from v as in the following figure, where (here and in all subsequent figures) each edge is labelled by the probability of reaching it from v.



In this case, $E(v) = 1 + E(w_1)$ holds. In the setup presented above, we have $c = 1$, $\Delta_1(w_1) \geq 1$, and $\Delta(w_1) \geq 1$, thus (2) is implied by

$$\alpha + \beta \geq 1 \ . \tag{3}$$

Case 2: v is a 2-vertex.
Let w_1 and w_2 denote the targets of the two downward edges emanating from v, where w_2 is lower than w_1.



We have
$$E(v) = 1 + \frac{1}{2}E(w_1) + \frac{1}{2}E(w_2) \ ,$$
hence we need to require that
$$\frac{\alpha}{2}\Delta_1(w_1) + \frac{\alpha}{2}\Delta_1(w_2) + \frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1 \ .$$
Note that $\Delta(w_1) \geq 1$.

Case 2.a: $\Delta(w_2) \geq 4$.
Ignoring the effect of the $\Delta_1(w_j)$'s, it suffices to require that
$$\frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1 \ ,$$
which will follow if
$$\beta \geq \frac{2}{5} \ . \tag{4}$$

Case 2.b.i: $\Delta(w_2) = 3$ and one of the two vertices above w_2 and below v is a 1-vertex.
In this case $\Delta_1(w_2) \geq 1$ and $\Delta(w_1) + \Delta(w_2) \geq 4$, so (2) is implied by

$$\frac{1}{2}\alpha + 2\beta \geq 1 \ . \tag{5}$$

We skip the remaining cases 2.b.ii and 2.c ($\Delta(w_2) = 2$) in this extended abstract; the latter one splits into quite a large number of subcases, which become slightly more involved. One ends up with roughly 24 linear inequalities in addition to (3), (4), (5).

Assuming we have α and β that satisfy all these inequalities, each of the induction steps is justified, and the inequality (1) follows. Since we always have $N_1(v), N_2(v) \leq n - 3$, we obtain

$$E(v) \leq \alpha N_1(v) + \beta(N_1(v) + N_2(v)) = (\alpha + \beta)N_1(v) + \beta N_2(v) \leq (\alpha + 2\beta)(n - 3) \ .$$

Hence we choose (α, β) to minimize $\alpha + 2\beta$, subject to all the derived inequalities. This is indeed the choice appearing in the statement of Theorem 3.1.

4 Discussion

The improved lower bounds of Section 2.2 arose from complete enumerations for small n. In particular, the lower bounds provided by examples 2 and 3 are tight for $n = 10, 12$, respectively. Thus, we have $f(10) = \frac{39}{4} = 9.75$ and $f(12) = \frac{1593}{128} \approx 12.45$.

We are convinced that the bound in Theorem 3.2 is not tight. In fact, precisely two of the inequalities of the proof of Theorem 3.1 are tight for $(\alpha, \beta) = (\frac{46}{87}, \frac{42}{87})$. The two corresponding subcases thus constitute the bottleneck for the current upper bound. In order to improve the bound, one should expand these two subcases further, aiming at replacing those two inequalities by weaker ones (at the cost of a longer proof). As a matter of fact, in an earlier (unpublished) version of this manuscript we had obtained an upper bound of $1.5 \cdot n$, using a somewhat more compact enumeration scheme. The current scheme is a refinement, based on further expansion of the preceding one.

At this point, we have no real sense of what the exact bound should be. The refinement of the approach in this section, as just outlined, is not likely to yield substantial improvements in the upper bound, so a radically different approach is probably called for. Such an improvement might be based on the observation that certain local structures involve 1-vertices with one of its upward neighbors lying above v. In fact, if the portion below v contains k such vertices, there must exist at least k vertices higher than v, so the upper bound $\alpha N_1(v) + \beta N(v)$ is much smaller than $(\alpha + 2\beta)(n - 3)$. As a matter of fact, the lower bounds derived in Section 2 do take this constraint into consideration.

Another observation is that the proof of Theorem 3.1 (as detailed in the appendix) uses (twice) the 3-connectivity of the edge graph of P, but it does not use its planarity at all, although it does occassionally run into nonplanar configurations. It is conceivable that further refinement stages might reach nonplanar configurations, whose exclusion would allow us to further improve the bound.

What if we also drop the 3-connectivity assumption? Then we need to consider additional cases, which cause our upper bound to increase. The best upper bound we have at the moment for this relaxed situation is $13n/8 = 1.625 \cdot n$, but we are convinced that it too can be further improved.

Acknowledgements. We are grateful to Emo Welzl and Günter Rote for inspiring discussions and helpful comments.

References

[1] N. Amenta and G. M. Ziegler, *Deformed products and maximal shadows*, in Advances in Discrete and Computational Geometry, B. Chazelle, J. E. Goodman, and R. Pollack, eds., vol. 223 of Contemporary Mathematics, Amer. Math. Soc., Providence RI, 1998, pp. 57–90.

[2] B. Gärtner, M. Henk, and G. M. Ziegler, *Randomized simplex algorithms on Klee-Minty cubes*, Combinatorica, 18 (1998), pp. 349–372.

[3] B. Gärtner, J. Solymosi, F. Tschirschnitz, P. Valtr, and E. Welzl, *One line and n points*, STOC, (2001), pp. 306–315.

[4] D. G. Kelly, *Some results on random linear programs.*, Methods Oper. Res., 40 (1981), pp. 351–355.

[5] J. Mihalisin and V. L. Klee, *Convex and linear orientations of polytopal graphs*, Discrete Comput. Geometry, 24 (2000), pp. 421–435.

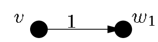
[6] C. A. Tovey, *Low order polynomial bounds on the expected performance of local improvement algorithms*, Math. Programming, 35 (1986), pp. 193–224.

[7] G. M. Ziegler, *Lectures on Polytopes*, vol. 152 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995. Revised edition, 1998; "Updates, corrections, and more" at `www.math.tu-berlin.de/~ziegler`.

Appendix

This appendix contains the complete case analysis of the proof of Theorem 3.1 in the extended abstract (including the cases presented there).

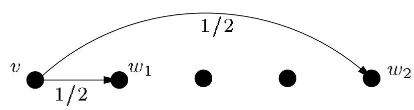
Case 1: v is a 1-vertex.
Let w_1 denote the target of the unique downward edge emanating from v as in the following figure, where each edge is labelled by the probability of reaching it from v.



In this case, $E(v) = 1 + E(w_1)$ holds. In the setup presented above, we have $c = 1$, $\Delta_1(w_1) \geq 1$, and $\Delta(w_1) \geq 1$, thus (2) is implied by

$$\alpha + \beta \geq 1 \ . \tag{3}$$

Case 2: v is a 2-vertex.
Let w_1 and w_2 denote the targets of the two downward edges emanating from v, where w_2 is lower than w_1.



We have

$$E(v) = 1 + \frac{1}{2}E(w_1) + \frac{1}{2}E(w_2) \ ,$$

hence we need to require that

$$\frac{\alpha}{2}\Delta_1(w_1) + \frac{\alpha}{2}\Delta_1(w_2) + \frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1 \ .$$

Note that $\Delta(w_1) \geq 1$.

Case 2.a: $\Delta(w_2) \geq 4$.
Ignoring the effect of the $\Delta_1(w_j)$'s, it suffices to require that

$$\frac{\beta}{2}\Delta(w_1) + \frac{\beta}{2}\Delta(w_2) \geq 1 \ ,$$

which will follow if

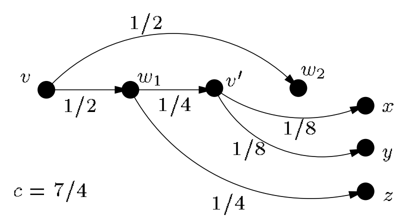
$$\beta \geq \frac{2}{5} \ . \tag{4}$$

Case 2.b.i: $\Delta(w_2) = 3$ and one of the two vertices above w_2 and below v is a 1-vertex.
In this case $\Delta_1(w_2) \geq 1$ and $\Delta(w_1) + \Delta(w_2) \geq 4$, so (2) is implied by

$$\frac{1}{2}\alpha + 2\beta \geq 1 \ . \tag{5}$$

Case 2.b.ii: $\Delta(w_2) = 3$ and the two vertices between v and w_2 are 2-vertices. Denote the second intermediate vertex as v'. We may assume that v' is reachable from v, otherwise we can ignore it and reduce the situation to Case 2.c treated below (be choosing another ordering of the vertices producing the same oriented graph). Three subcases can arise.

First, assume that none of the three edges that emanate from w_1 and v' further down reaches w_2. Denote by x, y the two downward neighbors of v' and by z the downward neighbor of w_1 other than v'. The vertices x, y, z need not be distinct but none of them coinicdes with w_2.



We have here $c = 7/4$.

To make the analysis simpler to follow visually, we present it in a table. Each row denotes one of the target vertices w_2, x, y, z, 'multiplied' by the probability of reaching it from v. The left (resp., right) column denotes a lower bound on the corresponding quantities $\Delta_1(\cdot)$ (resp., $\Delta(\cdot)$). To obtain an inequality that implies (2), one has to multiply each entry in the left (resp., right) column by the row probability times α (resp., times β), and require that the sum of all these terms be $\geq c$.

	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	0	3
$1/8x$	0	4
$1/8y$	0	5
$1/4z$	0	4

Note the following: (a) We do not assume that the rows represent distinct vertices (in fact, $x = z$ is implicit in the table); this does not cause any problem in applying the rule for deriving an inequality from the table. (b) We have to squeeze the vertices so as to make the resulting inequality as sharp (and difficult to satisfy) as possible; thus we made one of x, y the farthest vertex, because making z the farthest vertex would have made the inequality easier to satisfy.

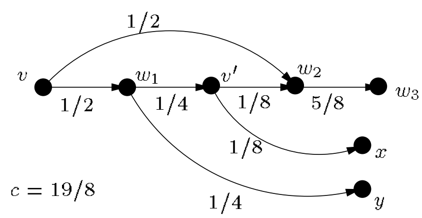
We thus obtain

$$\left(\frac{3}{2}+\frac{4}{8}+\frac{5}{8}+\frac{4}{4}\right)\beta \geq \frac{7}{4},$$

or

$$\beta \geq \frac{14}{29}. \tag{6}$$

Next, assume that w_2 is connected to v'. In this case w_2 is a 1-vertex, and we extend the configuration to include its unique downward neighbor w_3.



Let x denote the other downward neighbor of v' and let y denote the other downward neighbor of w_1. In the following table, the 'worst' case is to make w_3 and y coincide, and make x the farthest vertex.

	$\alpha\Delta_1$	$\beta\Delta$
$5/8 w_3$	1	4
$1/8 x$	1	5
$1/4 y$	1	4

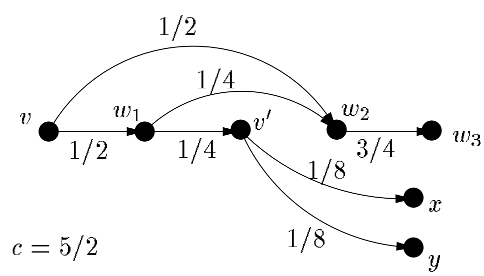
We then obtain

$$\alpha + \left(\frac{20}{8}+\frac{5}{8}+\frac{4}{4}\right)\beta \geq \frac{19}{8},$$

or

$$\alpha + \frac{33}{8}\beta \geq \frac{19}{8}. \tag{7}$$

Finally, assume that w_2 is connected to w_1. Here too w_2 is a 1-vertex, and we extend the configuration to include its unique downward neighbor w_3.



Denoting by x, y the two downward neighbors of v', our table and resulting inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$3/4 w_3$	1	4
$1/8 x$	1	4
$1/8 y$	1	5

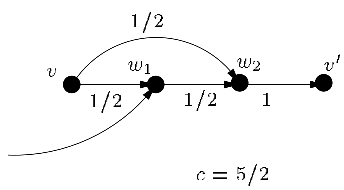
$$\alpha + \frac{33}{8}\beta \geq \frac{5}{2}, \tag{8}$$

which, by the way, is stronger than (7).

Case 2.c: $\Delta(w_2) = 2$. Hence, the only remaining case is that w_1 and w_2 are the two vertices immediately following v.

Case 2.c.i: w_1 is a 1-vertex (whose other upward neighbor lies above v). Its unique downward edge ends at some vertex which is either w_2 or lies below w_2.

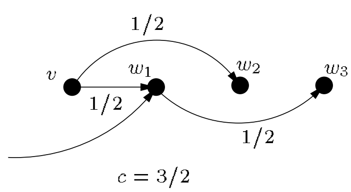
Assume first that this vertex coincides with w_2, which makes w_2 a 1-vertex, whose unique downward neighbor is denoted as v'. The local structure, table, and inequality are



	$\alpha\Delta_1$	$\beta\Delta$
v'	2	3

$$2\alpha + 3\beta \geq \frac{5}{2}. \tag{9}$$

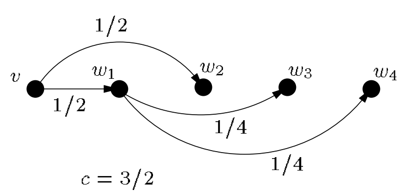
Suppose next that the downward neighbor w_3 of w_1 lies below w_2. We get



	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	1	2
$1/2w_3$	1	3

$$\alpha + \frac{5}{2}\beta \geq \frac{3}{2}. \tag{10}$$

Case 2.c.ii: w_1 is a 2-vertex, both of whose downward neighbors lie strictly below w_2. Denote these neighbors as w_3, w_4, with w_3 lying above w_4.



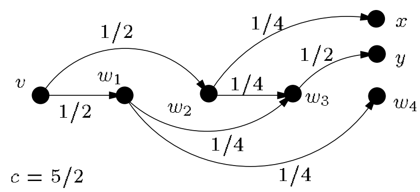
We may assume $\Delta(w_3) = 3$ (i.e., there is no vertex between w_2 and w_3), since $\Delta(w_3) \geq 4$ requires $\beta \geq \frac{6}{13}$ as the sharpest inequalitity, which is already implied by (6).

Case 2.c.ii.1: w_2 is a 1-vertex. Then the table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/2w_2$	0	2
$1/4w_3$	1	3
$1/4w_4$	1	4

$$\frac{1}{2}\alpha + \frac{11}{4}\beta \geq \frac{3}{2}. \tag{11}$$

Case 2.c.ii.2: w_2 is a 2-vertex but w_3 is a 1-vertex. Then w_3 (which satisfies $\Delta(w_3) = 3$) is connected either to w_2 or to a vertex above v. In the former case, let x denote the other downward neighbor of w_2, and let y denote the unique downward neighbor of w_3. The local structure looks like this (with x, y, w_4 not necessarily distinct):



The table depends on whether x precedes or succeeds w_3. In the former case the (worst) table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	0	3
$1/2y$	1	5
$1/4w_4$	1	5

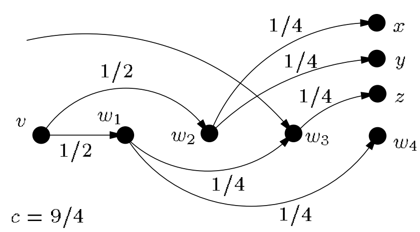
$$\frac{3}{4}\alpha + \frac{9}{2}\beta \geq \frac{5}{2}. \tag{12}$$

In the latter case the (worst) table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	4
$1/2y$	1	4
$1/4w_4$	1	5

$$\alpha + \frac{17}{4}\beta \geq \frac{5}{2}. \tag{13}$$

The next case is where the other upward neighbor of w_3 lies above v. Let x, y denote the two downward neighbors of w_2, and let z denote the unique downward neighbor of w_3. The local structure is:



The table depends on how many of x, y precede w_3. If both precede w_3, the table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	0	3
$1/4y$	0	4
$1/4z$	1	6
$1/4w_4$	1	6

$$\frac{1}{2}\alpha + \frac{19}{4}\beta \geq \frac{9}{4}. \tag{14}$$

If only one of x, y precedes w_3, say x, the table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	0	3
$1/4y$	1	5
$1/4z$	1	5
$1/4w_4$	1	6

$$\frac{3}{4}\alpha + \frac{19}{4}\beta \geq \frac{9}{4}, \tag{15}$$

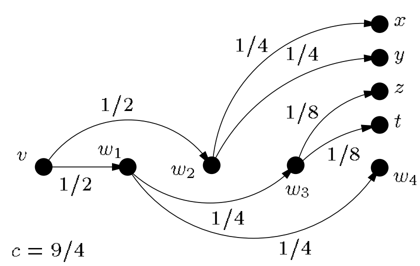
which is weaker than (14).

Finally, if none of x, y precedes w_3, the table and inequality become

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	4
$1/4y$	1	4
$1/4z$	1	5
$1/4w_4$	1	5

$$\alpha + \frac{9}{2}\beta \geq \frac{9}{4}. \tag{16}$$

Case 2.c.ii.3: Both w_2 and w_3 are 2-vertices. We have to consider the following type of configuration (where x, y, z, t, w_4 need not all be distinct, but $x \neq y$ and $z \neq t$, and we may assume $x \neq t$, $y \neq z$; also, because $\Delta(w_3) = 3$, both x and y are lower than w_3):



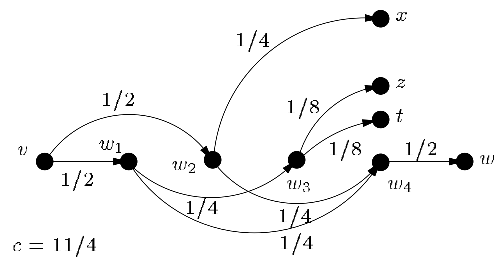
Intuitively, a worst table is obtained by 'squeezing' x, y, z, t, and w_4 as much to the left as possible, placing two of them at distance 4 from v, two at distance 5, and one at distance 6. However, squeezing them this way will make some pairs of them coincide and form 1-vertices, which will affect the resulting tables and inequalities.

Suppose first that among the three 'heavier' targets x, y, w_4, at most one lies at distance 4 from v. The worst table and the associated inequality are (recall that $x \neq y$):

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	0	4
$1/4y$	0	5
$1/8z$	0	4
$1/8t$	0	6
$1/4w_4$	0	5

$$\frac{19}{4}\beta \geq \frac{9}{4}. \tag{17}$$

Suppose then that among $\{w_4, x, y\}$, two are at distance 4 from v, say w_4 and y. Then $w_4 = y$ is a 1-vertex, and we denote by w its unique downward neighbor. The local structure is:



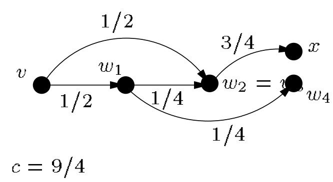
Two equally worst tables, and the resulting common inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	5
$1/8z$	1	6
$1/8t$	1	7
$1/2w$	1	5

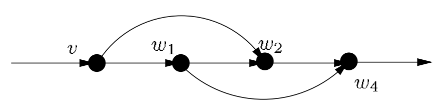
	$\alpha\Delta_1$	$\beta\Delta$
$1/4x$	1	6
$1/8z$	1	6
$1/8t$	1	5
$1/2w$	1	5

$$\alpha + \frac{43}{8}\beta \geq \frac{11}{4}\ . \qquad (18)$$

Case 2.c.iii: w_1 is a 2-vertex that reaches w_2. Then w_2 is a 1-vertex, and we denote by x its unique downward neighbor.



A crucial observation is that x cannot be equal to w_4. Indeed, if they were equal, then w_4 would be a 1-vertex.



In this case, cutting the edge graph G of P at the downward edge emanating from x and at the edge entering v would have disconnected G, contradicting the fact that G is 3-connected.

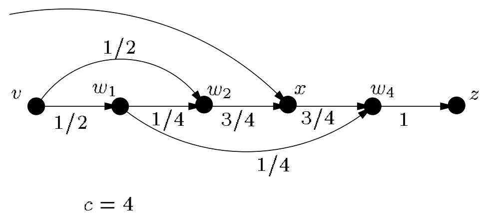
We first dispose of the case where x lies lower than w_4. The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/4x$	1	4
$1/4w_4$	1	3

$$\alpha + \frac{15}{4}\beta \geq \frac{9}{4}. \qquad (19)$$

In what follows we thus assume that x lies above w_4.

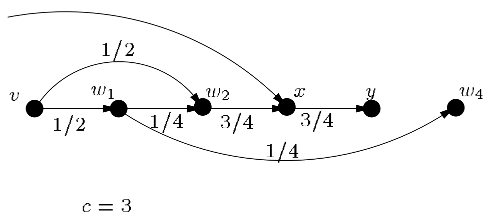
Case 2.c.iii.1: x is a 1-vertex that precedes w_4. Suppose first that w_4 is the unique downward neighbor of x. Then w_4 is a 1-vertex, and we denote its unique downward neighbor by z. The local structure, table and inequality are:



	$\alpha\Delta_1$	$\beta\Delta$
z	3	5

$$3\alpha + 5\beta \geq 4. \qquad (20)$$

Suppose next that the unique downward neighbor y of x is not w_4. The local structure, table and inequality look like this (y is drawn above w_4 because this yields a sharper inequality):

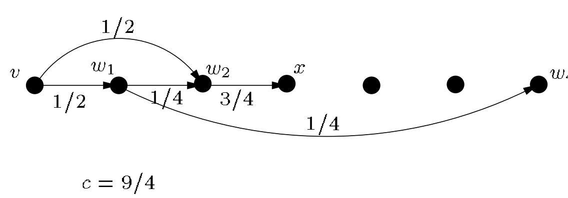


	$\alpha\Delta_1$	$\beta\Delta$
$3/4y$	2	4
$1/4w_4$	2	5

$$2\alpha + \frac{17}{4}\beta \geq 3. \qquad (21)$$

Case 2.c.iii.2: x is a 2-vertex that precedes w_4. This subcase splits into several subcases, where we assume, respectively, that $\Delta(w_4) \geq 6$, $\Delta(w_4) = 4$, and $\Delta(w_4) = 5$.

Case 2.c.iii.2(a). Suppose first that $\Delta(w_4) \geq 6$. The configuration looks like this:

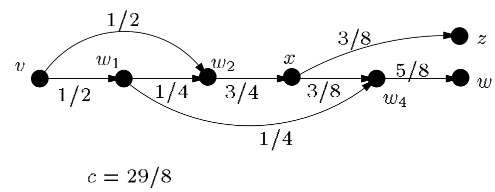


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/4x$	1	3
$1/4w_4$	1	6

$$\alpha + \frac{15}{4}\beta \geq \frac{9}{4}. \qquad (22)$$

Case 2.c.iii.2(b). Suppose next that $\Delta(w_4) = 4$, and that one of the downward neighbors of x is w_4. Let z denote the other downward neighbor. w_4 is a 1-vertex, and we denote by w its unique downward neighbor.

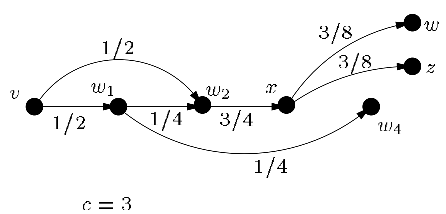


The 3-connectivity of the edge graph of P implies, as above, that $w \neq z$. Since we assume that $\Delta(w_4) = 4$, z also lies below w_4, and the table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$5/8w$	2	5
$3/8z$	2	6

$$2\alpha + \frac{43}{8}\beta \geq \frac{29}{8}. \tag{23}$$

Suppose next that $\Delta(w_4) = 4$ and w_4 is not a downward neighbor of x. Denote those two neighbors as w and z, both of which lie lower than w_4, by assumption, and are clearly distinct. The configuration, table and inequality look like this:

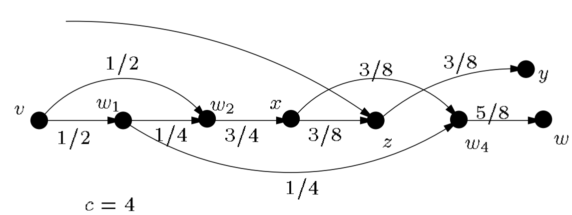


	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	1	4
$3/8w$	1	5
$3/8z$	1	6

$$\alpha + \frac{41}{8}\beta \geq 3. \tag{24}$$

Case 2.c.iii.2(c). It remains to consider the case $\Delta(w_4) = 5$. Let z denote the unique vertex lying between x and w_4. We may assume that z is connected to x, for otherwise z is not reachable from v, and we might as well reduce this case to the case $\Delta(w_4) = 4$ just treated.

Consider first the subcase where the other downward neighbor of x is w_4 itself. Then w_4 is a 1-vertex, and we denote by w its unique downward neighbor. This subcase splits further into two subcases: First, assume that z is a 1-vertex, and let y denote its unique downward neighbor. Clearly, y must lie below w_4 (it may coincide with or precede w). The configuration looks like this:

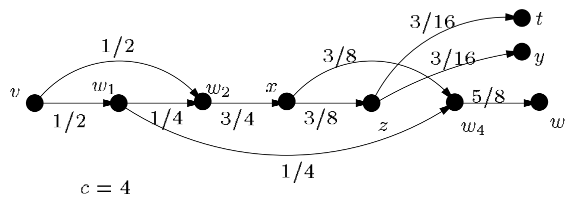


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/8y$	3	6
$5/8w$	3	6

$$3\alpha + 6\beta \geq 4. \tag{25}$$

In the other subcase, z is a 2-vertex; we denote its two downward neighbors as y and t. The vertices w, y, t all lie below w_4 and may appear there in any order. The configuration looks like this:

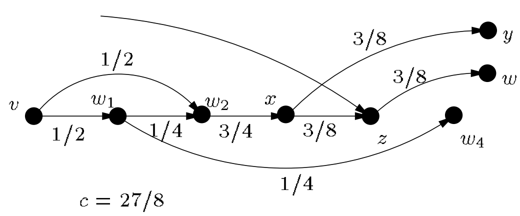


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/16y$	2	6
$3/16t$	2	7
$5/8w$	2	6

$$2\alpha + \frac{99}{16}\beta \geq 4. \tag{26}$$

Consider next the subcase where w_4 is not a downward neighbor of x. Denote the other downward neighbor of x as y, which lies strictly below w_4. This subcase splits into three subcases. First, assume that z is a 1-vertex, and denote its unique downward neighbor as w. The configuration looks like this:

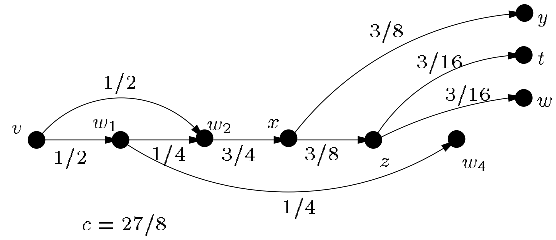


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	2	5
$3/8y$	2	6
$3/8w$	2	5

$$2\alpha + \frac{43}{8}\beta \geq \frac{27}{8}. \tag{27}$$

Second, assume that z is a 2-vertex, so that none of its two downward neighbors is w_4. Denote these neighbors as w and t. All three vertices y, t, w lie strictly below w_4. The configuration looks like this:

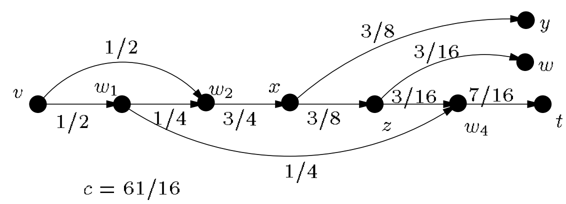


The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$1/4w_4$	1	5
$3/8y$	1	6
$3/16w$	1	6
$3/16t$	1	7

$$\alpha + \frac{95}{16}\beta \geq \frac{27}{8}. \tag{28}$$

Finally, assume that z is a 2-vertex, so that one of its two downward neighbors is w_4. Denote the other neighbor as w. In this case w_4 is a 1-vertex, and we denote its unique downward neighbor as t. All three vertices y, t, w lie strictly below w_4. The configuration looks like this:



The table and inequality are

	$\alpha\Delta_1$	$\beta\Delta$
$3/8y$	2	6
$3/16w$	2	7
$7/16t$	2	6

$$2\alpha + \frac{99}{16}\beta \geq \frac{61}{16}, \tag{29}$$

which, by the way, is weaker than (26).