Semidefinite Programming	**Lecture 6**
OR 6327 Spring 2012	**February 9, 2012**
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Today, we will talk about control theory. For more information, please see Carsten Scherer's notes (see the link on the home page). Consider the linear system

$$\dot{x} = Rx, \quad x = x(t) \in \mathbb{R}^n, R \in \mathbb{R}^{n \times n}.$$

Is $x(t)$ bounded for all t, whatever $x_0 = x(0)$? If we can find $v(x) = x^T Y x$ with $Y \succ 0$ such that v is nonincreasing, then we're done. Let's compute $\dot{v}$, which is

$$\dot{v} = \dot{x}^T Y x + x^T Y \dot{x} = x^T R^T Y x + x^T Y R x.$$

Hence, if we have $Y \succ 0$ with $R^T Y + Y R \preceq 0$, then x remains bounded.

This condition is also necessary if R can be diagonalized. Let $R = P \Lambda P^{-1}$ where $P \in \mathbb{C}^{n \times n}$ is the matrix of eigenvectors and $\Lambda = \mathrm{Diag}(\lambda)$ for $\lambda \in \mathbb{C}^n$, the vector of eigenvalues. Since x remains bounded, each $\mathrm{Re}(\lambda_i) \leq 0$. But

$$\begin{aligned} & P^{-1} R P = \Lambda \\ \Rightarrow \quad & P^{-1} R P + P^H R^T P^{-H} = \Lambda + \Lambda^H = 2\mathrm{Diag}\left(Re(\lambda)\right) \preceq 0 \\ \Rightarrow \quad & (P^{-H} P^{-1}) R + R^T (P^{-H} P^{-1}) = 2 P^{-H} \mathrm{Diag}\left(Re(\lambda)\right) P^{-1} \preceq 0. \end{aligned}$$

Now $P^{-H} P^{-1}$ is Hermitian and positive definite, and in fact it is real. Let $Y = P^{-H} P^{-1}$ and we're done. Note that, since the conditions are homogeneous, we can replace $Y \succ 0$ by $Y \succeq I$, so we can find such a Y by solving, say,

$$\begin{array}{lrcl} \max_Y & -I \bullet Y & & \\ & R^T Y + Y R & \preceq & 0 \\ & Y & \succeq & I. \end{array}$$

What if x evolves according to $\dot{x} \in \mathrm{conv}\{R_1, \cdots, R_k\} x$ (a differential inclusion)? Certainly for x to remain bounded it is sufficient if there is a nonincreasing Lyapounov function

$$v(x) := x^T Y x, \qquad Y \succ 0.$$

Hence, $R_i^T Y + Y R_i \preceq 0$, $i = 1, \cdots, k$, and $I - Y \preceq 0$ is a sufficient condition.

Suppose instead we have the system $\dot{x} = Rx + Bu$ with $B \in \mathbb{R}^{n \times k}$ and $u \in \mathbb{R}^k$. Can we choose u to control x? In particular, can we choose $u = Px, P \in \mathbb{R}^{r \times n}$? If we choose such a feedback control, $\dot{x} = (R + BP)x$, so we want P and Y such that

$$\begin{aligned} & (R + BP)^T Y + Y(R + BP) \preceq 0 \text{ and } Y \succ 0 \\ \Leftrightarrow \quad & R^T Y + Y R + P^T B^T Y + Y B P \preceq 0 \text{ and } Y \succ 0 \\ \Leftrightarrow \quad & Y^{-1} R^T + R Y^{-1} + Y^{-1} P^T B^T + B P Y^{-1} \preceq 0 \text{ and } Y \succ 0. \end{aligned}$$

Note that, by using Y^{-1} instead of Y, we have put the two variables together, so we can change variables and get linear constraints. Indeed, in terms of $Z = Y^{-1}$ and $Q = PY^{-1}$, we have the following constraints

$$\begin{aligned} ZR^T + RZ + Q^T B^T + BQ &\preceq 0 \\ Z &\succeq I. \end{aligned}$$

(Note, the last constraint is not equivalent to $Y \succeq I$, but ensures that Z and $Y = Z^{-1}$ are positive definite.) Having solved this system, we can set $Y = Z^{-1}$ and $P = QY$. There is an obvious extension of this to get sufficient conditions for a linear feedback control to keep x bounded when $\dot{x} \in \text{conv}\{R_1, \cdots, R_k\}x + Bu$.

Next, we will talk about Euclidean distance matrices.

Definition 1 *(Euclidean Distance Matrix) We say $D \in \mathbb{M}^n$ is a Euclidean distance matrix (EDM) (of dimension $\leq r$) iff there are points $x_1, \cdots, x_n$ (in $\mathbb{R}^r$) with $||x_i - x_j||^2 = d_{ij}$.*

We want to characterize EDMs. We use $P \in \mathbb{R}^{n\times(n-1)}$ with $[\frac{e}{\sqrt{n}}, P]$ orthogonal where $e = [1; \cdots; 1] \in \mathbb{R}^n$.

Theorem 1 *The matrix $D \in \mathbb{M}^n$ is an EDM (of dimension $\leq r$) iff there is $U \in \mathbb{M}_+^{n-1}$ (of rank $\leq r$) with*

$$D = \text{diag}\,(PUP^T)e^T + e(\text{diag}\,(PUP^T))^T - 2PUP^T.$$

Proof: Suppose there is such a U (of rank r). Then there is Q ($\in \mathbb{R}^{(n-1)\times r}$) with $U = QQ^T$. Let x_i^T be the ith row of PQ (in $\mathbb{R}^r$). Then

$$d_{ij} = (x_i^T x_i).1 + 1.(x_j^T x_j) - 2x_i^T x_j = ||x_i - x_j||^2.$$

Conversely, suppose D is the EDM corresponding to $x_1, \cdots, x_n$ (in $\mathbb{R}^r$). Then, since $[\frac{e}{\sqrt{n}}, P]$ is orthogonal, $PP^T = I - \frac{ee^T}{n}$. Let $X = [x_1, \cdots, x_n]$ and $Y = XPP^T = X - \frac{Xee^T}{n} = [x_1 - \frac{\sum x_i}{n}, \cdots, x_n - \frac{\sum x_i}{n}]$. Hence, D is also the EDM corresponding to Y, so

$$D = \text{diag}\,(Y^T Y)e^T + e(\text{diag}\,(Y^T Y))^T - 2Y^T Y.$$

Also, $Y^T Y = PP^T X^T XPP^T$, so taking $U = P^T X^T XP$ does the trick (and has rank at most r). $\square$

This motivates the "SDPs" (in primal form):

$$\begin{array}{ll} \min_{U\in\mathbb{M}^{n-1}} & ||D - \text{diag}\,(PUP^T)e^T + e(\text{diag}\,(PUP^T))^T - 2PUP^T||_F^2 \\ \text{s.t.} & U \succeq 0, \end{array}$$

if there is noise in the distance measurements D, or

$$\begin{array}{ll} \min_{U\in\mathbb{M}^{n-1}} & I \bullet U \\ \text{s.t.} & \text{diag}\,(PUP^T)e^T + e(\text{diag}\,(PUP^T))^T - 2PUP^T = D \\ & U \succeq 0. \end{array}$$

(The latter tends to give a low rank solution.)

A related problem is the sensor localization problem (see e.g. the paper by So and Ye, with a link on the home page): there are some known locations called "anchors" and n unknown locations called "sensors". And we also know $||x_i - x_j||^2$ for some pairs $ij \in E_x$ and $||x_j - a_k||^2$ for some pairs $jk \in E_a$.

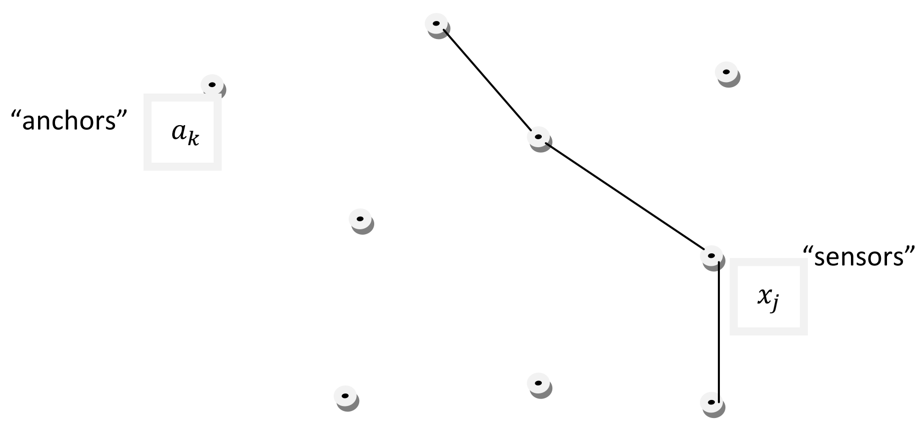

Figure 1: Sensor localization problem

Then, with $X := [x_1, \cdots, x_n]$,

$$||x_i - x_j||^2 = (e_i - e_j)^T X^T X (e_i - e_j), \qquad ||x_j - a_k||^2 = \begin{pmatrix} a_k \\ -e_j \end{pmatrix}^T \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \begin{pmatrix} a_k \\ -e_j \end{pmatrix}.$$

Then we have

$$(e_i - e_j)^T Y (e_i - e_j) = d_{ij},\ ij \in E_x$$

$$\begin{pmatrix} a_k \\ -e_j \end{pmatrix}^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \begin{pmatrix} a_k \\ -e_j \end{pmatrix} = f_{kj},\ kj \in E_a$$

$$Y = X^T X.$$

The last constraint is nonlinear, but we can relax it to $Y \succeq X^T X$, or equivalently

$$\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0.$$

This will give an SDP formulation for the problem.