Semidefinite Programming	**Lecture 5**
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Convex Quadratically-Constrained Quadratic Programming .

Consider the problem

$$\begin{array}{lll} \min_y & f_0(y) & \\ & f_i(y) & \leq 0, \quad i = 1, \ldots, n, \end{array}$$

where each f_i is a convex quadratic function of $y \in \mathbb{R}^m$.

We can assume the objective is linear, so we have

$$\begin{array}{lll} \max_y & b^T y & \\ & f_i(y) & \leq 0, \quad i = 1, \ldots, n, \end{array}$$

where each $f_i(y) = y^T C_i y - d_i^T y - \epsilon_i$ with C_i psd. We can write $C_i = G_i^T G_i$ where $G_i \in \mathbb{R}^{r_i \times m}$. Then

$$\begin{aligned} f_i(y) \leq 0 &\iff d_i^T y + \epsilon_i \geq (G_i y)^T (G_i y) \\ &\iff M_i = \begin{bmatrix} d_i^T y + \epsilon & (G_i y)^T \\ G_i y & I \end{bmatrix} \succeq 0 \end{aligned}$$

(this follows from Schur complements, but we have to consider the two cases when $d_i^T y + \epsilon$ is zero and when it is positive). So our problem can be formulated as

$$\begin{array}{ll} \max_y & b^T y \\ & \mathrm{Diag}\,(M_1, \ldots, M_n) \succeq 0. \end{array}$$

Alternatively

$$\begin{aligned} f_i(y) \leq 0 &\iff (d_i^T y + \epsilon_i + 1)^2 \geq (d_i^T + \epsilon_i - 1)^2 + (2G_i y)^T (2G_i y) \\ &\iff (d_i^T y + \epsilon_i + 1) \geq \left\| \begin{matrix} d_i^T y + \epsilon_i - 1 \\ 2G_i y \end{matrix} \right\|_2 \\ &\iff d_i^T y + \epsilon_i + 1 \geq \frac{1}{d_i^T y + \epsilon_i + 1} \left\| \begin{matrix} d_i^T y + \epsilon_i - 1 \\ 2G_i y \end{matrix} \right\|_2^2 \\ &\iff W_i = \begin{bmatrix} d_i^T y + \epsilon_i + 1 & d_i^T y + \epsilon_i - 1 & (2G_i y)^T \\ d_i^T y + \epsilon_i - 1 & d_i^T y + \epsilon_i + 1 & 0 \\ 2G_i y & 0 & (d_i^T y + \epsilon_i + 1) I \end{bmatrix} \succeq 0 \end{aligned}$$

(several steps of this argument need to ensure that $d_i^T y + \epsilon + 1$ is positive, whichever the direction of the implications). So we could replace all the M_is with W_is in the formulation above.

However, this analysis also shows another formulation of the problem in dual conic programming form as

$$\begin{array}{ll} \max_y & b^T y \\ & d_i^T y + \epsilon_i + 1 \geq \left\| \begin{matrix} d_i^T y + \epsilon_i - 1 \\ 2G_i y \end{matrix} \right\|_2, \quad \text{all } i, \end{array}$$

a second-order cone programming problem, which will typically be much more efficient to solve.

In fact, we have used

Proposition 1

$$\begin{bmatrix} \gamma & v^T \\ v & \gamma I \end{bmatrix} \succeq 0 \quad \textit{iff } \gamma \geq \|v\|_2.$$

Truss topology design is the problem of choosing where and what size rods to use in a framework to support one or more loads.

For more details in what follows, see Ben-Tal & Nemirovskii, Lectures on Modern Convex Optimization.

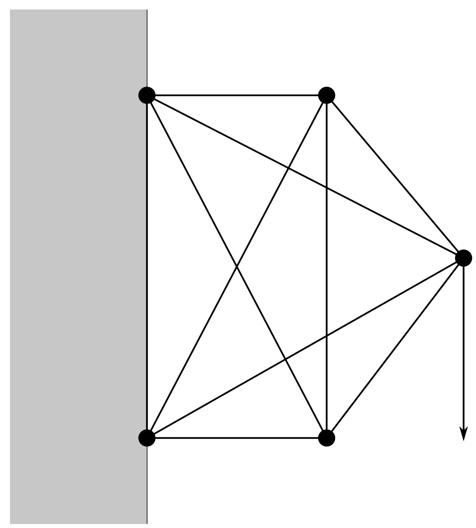
Suppose we put a rod in potential link j with cross-sectional area y_j. This gives a stiffness matrix

$$A(y) = \sum y_j b_j b_j^T \succeq 0$$

and "determines" displacements d (indexed by 2 or 3 components for each free node) that can support a load vector f by

$$A(y)d = f \quad \text{(Hooke's Law)}.$$

We may also have constraints $a \leq y \leq b$ and $l^T y \leq w$ and we also want the "maximum stiffness" or "minimum compliance" (proportional to the work done)



$$\min f^T d.$$

So the problem is

$$\begin{array}{lll} \min & f^T d & \\ & A(y)d & = f \\ & l^T y & \leq w, \quad a \leq y \leq b. \end{array}$$

This is nonlinear in the variables d and y. If $A(y) \succ 0$, we could eliminate d and the first set of constraints and minimize $f^T A(y)^{-1} f$ to get

$$\begin{array}{ll} \min & \eta \\ & \begin{bmatrix} \eta & f^T \\ f & A(y) \end{bmatrix} \succeq 0, \quad a \leq y \leq b, \quad l^T y \leq w. \end{array}$$

In fact, this works even if $A(y)$ might be singular at the solution.

Proposition 2 *Suppose $A \succeq 0$. Then*

(i) $f^T d$ is the same for all solutions to $Ad = f$;

(ii) $$\begin{array}{ll} \min & f^T d \\ & Ad = f \end{array} \quad = \quad \begin{array}{c} \min \quad \eta \\ \left[\begin{smallmatrix} \eta & f^T \\ f & A \end{smallmatrix}\right] \succeq 0 \end{array}.$$

Proof:

(i) Suppose $Ad_1 = Ad_2 = f$. Then $f^T d_1 = d_2^T A d_1 = d_1^T A d_2 = f^T d_2$.

(ii) Suppose first $f \notin \text{Range}(A)$, so the left-hand minimum is ∞. Since there is no d with $Ad = f$, so there exists v with $Av = 0$ and $f^T v < 0$. Then

$$\begin{bmatrix} \beta \\ v \end{bmatrix}^T \begin{bmatrix} \eta & f^T \\ f & A \end{bmatrix} \begin{bmatrix} \beta \\ v \end{bmatrix} = \eta\beta^2 + 2f^T v\beta < 0$$

for any η by choosing $\beta > 0$ small enough. Thus the right-hand minimum is also ∞.

Now assume there is some d with $Ad = f$. Then the left-hand minimum is $f^T d$ for such a d. Also,

$$\begin{bmatrix} d^T A d & d^T A \\ Ad & A \end{bmatrix} = \begin{bmatrix} d^T \\ I \end{bmatrix} A \begin{bmatrix} d & I \end{bmatrix} \succeq 0.$$

So we can choose $\eta = d^T A d = f^T d$, and the right-hand minimum is at most the left-hand minimum.

Also, if $\begin{bmatrix} \eta & f^T \\ f & A \end{bmatrix} \succeq 0$, then

$$\begin{aligned} 0 \leq \begin{bmatrix} 1 \\ -d \end{bmatrix}^T \begin{bmatrix} \eta & f^T \\ f & A \end{bmatrix} \begin{bmatrix} 1 \\ -d \end{bmatrix} &= \eta - 2f^T d + d^T A d \\ &= \eta - f^T d. \end{aligned}$$

So the right-hand minimum is at least the left-hand minimum. □

So the TTD problem can be formulated as

$$\begin{array}{ll} \min & \eta \\ & \begin{bmatrix} \eta & f^T \\ f & A(y) \end{bmatrix} \succeq 0 \\ & a \leq y \leq b, \quad l^T y \leq w. \end{array}$$

In fact, it can be formulated as a linear programming problem!! (See Ben-Tal & Nemirovskii.)

But in practice, the truss has to withstand different load vectors and then we got the robust TTD problem with constraints

$$\begin{bmatrix} \eta & f_i^T \\ f_i & A(y) \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n,$$

not known to be reducible to linear programming (although it can in fact be reduced to a second-order cone programming problem).

Robust Mathematical Programming .

Consider the problem

$$\begin{aligned} \max \quad & b^T y \\ & a_j^T y \le c_j, \quad j = 1, \ldots, n, \end{aligned}$$

where some or all of the $(a_j; c_j)$s are not known exactly. Change variables to $(y; -1)$ to get the constraints

$$\begin{aligned} & a_j^T y \le 0, \quad \text{for all } a_j \in \mathcal{E}_j, \quad j = 1, \ldots, k, \\ & a_j^T y \le c_j, \quad j = k+1, \ldots, n, \end{aligned}$$

where the last "certain" set of constraints forces the final component of y to be -1. This is a semi-infinite set of linear constraints (infinite number of constraints in a finite number of variables).

Assume each $\mathcal{E}_j$ is "ellipsoid-like", i.e., of the form $\{\bar{a}_j + G_j u_j : \|u_j\|_2 \le 1\}$. Consider some $j = 1, \ldots, k$. Then

$$\begin{aligned} & a_j^T y \le 0 \quad \text{all } a_j \in \mathcal{E}_j \\ \Longleftrightarrow & \max_{a_j \in \mathcal{E}_j} a_j^T y \le 0 \\ \Longleftrightarrow & \max_{\|u_j\| \le 1} (\bar{a}_j + G_j u_j)^T y \le 0 \\ \Longleftrightarrow & \bar{a}_j^T y + \max_{\|u_j\| \le 1} u_j^T (G_j^T y) \le 0 \\ \Longleftrightarrow & \bar{a}_j^T y + \|G_j^T y\|_2 \le 0. \end{aligned}$$

So the robust linear programming problem is equivalent to the dual form conic programming problem

$$\begin{aligned} \max \quad & b^T y \\ & -\bar{a}_j^T y \ge \|G_j^T y\|_2, \quad j = 1, \ldots, k, \\ & a_j^T y \le c_j, \qquad j = k+1, \ldots, n. \end{aligned}$$

This is a SOCP (second order conic programming) problem.