OR 6327: Semidefinite Programming. Spring 2012.

Homework Set 4. Due: Thursday April 19.

1. Assume $X \succ 0$, $S \succ 0$, and P is invertible. Suppose that the scaled matrices PXP^T and $P^{-T}SP^{-1}$ commute. Show that MXS, where $M = P^TP$, is symmetric and positive definite, and that the MZ search directions given by this P at the iterate (X, y, S) are well-defined.

2. In this question, all operators are restricted to $\mathbb{M}^n$.

a) If G and H are $n \times n$ matrices, show that the adjoint of $G \odot H$ is $G^T \odot H^T$, i.e., $U \bullet (G \odot H)V = (G^T \odot H^T)U \bullet V$ for all $U, V \in \mathbb{M}^n$.

b) If G and H are $n \times n$ matrices, show that $(G \odot H)(J \odot J) = (GJ \odot HJ)$ and $(J \odot J)(G \odot H) = (JG \odot JH)$.

c) Suppose $G, H \in \mathbb{M}^n$, with eigenvalues $(\lambda_i)_1^n$ and $(\mu_j)_1^n$ respectively. Show that, if G and H commute, then $G \odot H$ has eigenvalues $\frac{1}{2}(\lambda_i\mu_j + \lambda_j\mu_i)$, $1 \leq i \leq j \leq n$. (You may want to start first with the case that G and H are diagonal.)

3. It appears that computing the scaling point W for X and S requires two eigenvalue decomposions, one of S to get $S^{\pm 1/2}$ and then a second of $S^{1/2}XS^{1/2}$ to get its square root. Show that it suffices to compute a Cholesky factorization of S and then a single eigenvalue decomposition.

4. Suppose you know that all feasible solutions of a standard primal form SDP problem (P) satisfy $I \bullet X \leq \gamma$ for some positive γ. Show that (P) can be written in an equivalent way so that its dual is the problem of minimizing the maximum eigenvalue of a matrix depending linearly on some parameters.

5. a) Suppose q is a unit eigenvector corresponding to the largest eigenvalue of $C - \mathcal{A}^*y$. Let $f(y) := \lambda_{\max}(C - \mathcal{A}^*y)$. Show that $-\mathcal{A}(qq^T)$ is a subgradient of f at y.

b) More generally, suppose that q is a unit vector with $q^T(C - \mathcal{A}^*y)q \geq \lambda_{\max}(C - \mathcal{A}^*y) - \epsilon$ for some nonnegative ϵ. Show that $-\mathcal{A}(qq^T)$ is an ϵ-subgradient of f at y, i.e., that $f(z) \geq f(y) + (-\mathcal{A}(qq^T))^T(z - y) - \epsilon$ for all z.