OR 6327: Semidefinite Programming. Spring 2012.

Homework Set 2. Due: Thursday March 8.

1. Consider the dynamical system $\dot{x} = Ax$, where $x \in \Re^2$ and

$$A = \begin{bmatrix} -4 & 0 \\ 6 & -1 \end{bmatrix}.$$

a) Show that $x^T x$ may not be decreasing, by choosing a suitable $x(0)$.

b) Give a certificate that x converges to zero for any starting point using a linear matrix inequality. For extra credit, show that x converges exponentially to zero in some sense.

2. Suppose the graph $G = (V, E)$ has $n = 2k$ nodes.

a) Find an SDP relaxation for the problem of finding a minimum cardinality bisection, a cut that has k nodes on either side.

b) Find an SDP relaxation for the problem like (a), except that the discrepancy between the numbers of nodes on the two sides of the cut can be at most p.

3. Show directly that $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$, where $\mathrm{STAB}(G) :=$ $\mathrm{conv}\{\chi^S : S \text{ is a stable set of } G\}$, $\mathrm{QSTAB}(G) := \{x \in \Re^n_+ : (\chi^C)^T x \leq 1 \text{ for every clique } C \text{ of } G\}$, and $\mathrm{TH}(G)$ is defined as
$\{x \in \Re^n_+ : \sum_i (c^T u_i)^2 x_i \leq 1 \text{ for every orthonormal representation } (c, u_1, \ldots, u_n) \text{ of } G\}$.

4. State primal and dual SDP formulations to approximate the problem of globally minimizing $p(z) = z^4 + z^2 + 6z$ over $\Re^1$. Show that the solutions $(y_0; \ldots; y_4) = (1; -1; 1; -1; 1)$ and

$$X = \begin{bmatrix} 4 & 3 & -1 \\ 3 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

are optimal. Use these to find an optimal solution z^* and express $p(z) - p(z^*)$ as a sum of squares.