Oct. 26, 2005

Recitation 8:

Dual Simplex:

Ex:

	Z	x_1	x_2 ↓	x_3	x_4	x_5	RHS
	1	2	6	10	0	0	0
x_4	0	-2	4	1	1	0	2
← x_5	0	4	-2	-3	0	1	-1 → pivot row

candidates: -2, -3

ratio rule: $\frac{6}{2}$, $\frac{10}{3}$

$\frac{6}{2}$ → smallest

	Z	x_1	x_2	x_3	x_4	x_5	RHS
	1	14	0	1	0	3	-3
x_4	0	6	0	-5	1	2	0
x_2	0	-2	1	3/2	0	-1/2	+1/2

→ optimal

Note: Dual Simplex is useful when RHS changes or a new constraint is added.

NETWORK FLOW PROBLEMS:

Given a network $G = (N, A)$ and information:

b_i: supply of node i (>0 source / <0 sink)

u_{ij}: cap. of arc (i,j)

c_{ij}: cost of unit flow on (i,j)

D.V. x_{ij}: amount of flow from i to j

$$\min \sum_{(i,j)\in A} c_{ij} \cdot x_{ij}$$

s.t.

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i \quad \forall i$$

flow conservation

$$0 \le x_{ij} \le u_{ij} \quad \forall i,j \quad (\text{or } l_{ij} \le x_{ij} \le u_{ij})$$

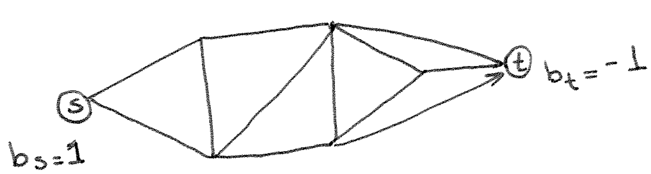
capacity

$$x_{ij} \geq 0 \quad \forall i,j$$

non-neg.

Note: $\sum b_i = 0$ for feasibility.

The Shortest Path Problem:



edge cap $= \infty$

$b_i = 0 \quad \forall i$

$b_s = 1$

$b_t = -1$

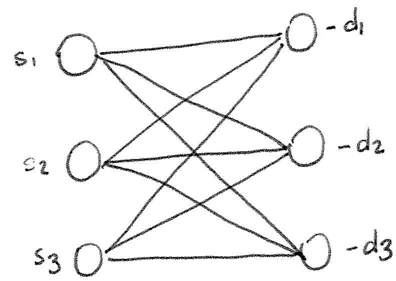
c_{ij}: cost of edge (i,j).

Transportation Problem:

m supp. / n cust.

s_i: supply of supplier i / d_J = dem. of cust J.

c_{ij}: cost of transportation



$$\min \sum\sum c_{ij} x_{ij}$$

s.t.

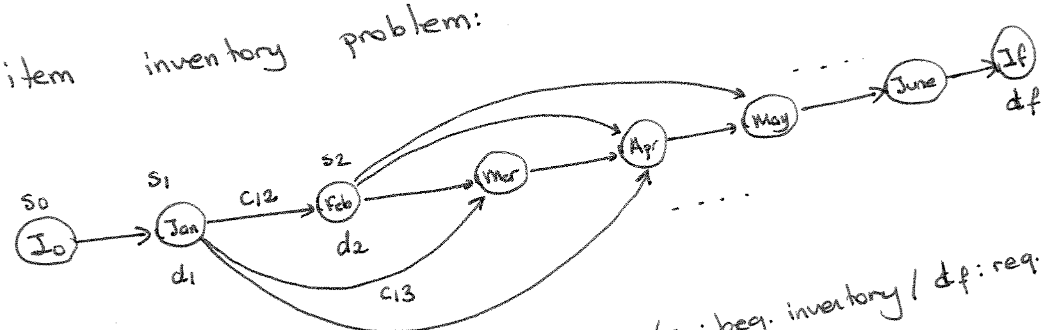
$$0 - \sum_i x_{ij} = -d_J \quad \forall J$$

$$\sum_J x_{ij} - 0 = b_i \quad \forall i$$

$$x_{ij} \geq 0 \quad \forall i, J.$$

Assignment Problem: Special case of transportation w. $d_j = 1$, $b_i = 1$ $(m=n)$

Single-item inventory problem:



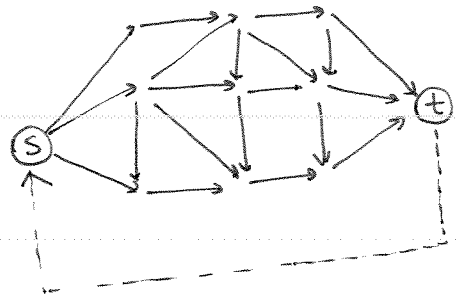
(s_0: beg. inventory / d_f: req. inv. at the end)

s_i: amount produced in i^{th} month

d_J: demand in i^{th} month

c_{iJ}: cost of producing 1 unit in month i to use in month J (prod + inventory)

The Maximum Flow Problem:



$c_{ij} = 0 \quad \forall\, i,j \quad (i,j) \neq (t,s)\,, \qquad c_{ts} = -1$

$u_{ij} =$ cap. of arc

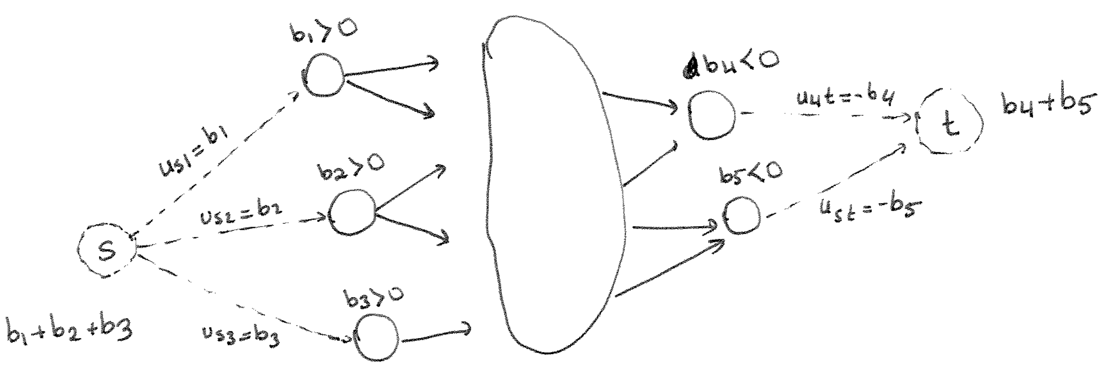
$b_i = 0 \quad \forall\, i,j.$

Special Algorithm: Ford-Fulkerson

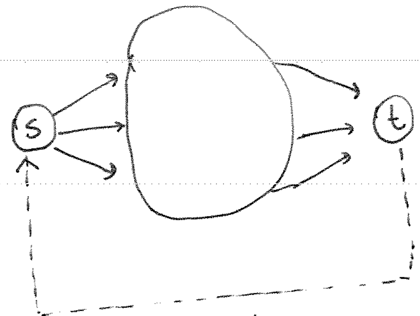
Theorem: Max-Flow = Min-Cut

<u>Notes</u>:

(1) Every network flow problem can be reduced to a problem with "exactly one" source and "exactly one" sink:



② Every network can be reduced to a circulation problem (without nodes & sinks)

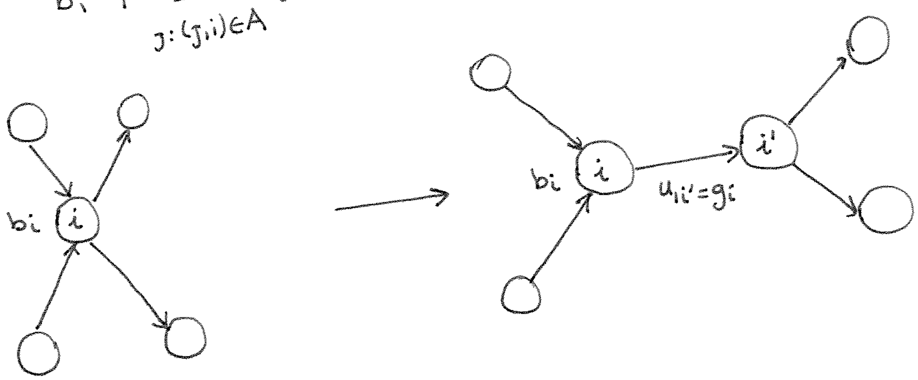


$u_{ts} = b_s$

$c_{ts} = -M$ → since cost is large, opt. sol will try to set $X_{ts} = b_s$ if it does not then the prob is infeasible.

③ Capacity restrictions on nodes can be handled:

$$b_i + \sum_{j:(j,i)\in A} X_{ji} \leq g_i$$



④ Lower bounds on flows: $X_{ij} \geq l_{ij}$ ⇒ $X_{ij}' = X_{ij} - l_{ij}$ (update b_i's)

Problem 1 from midterm: Add slack vars u_t and v_t:

$$
\begin{aligned}
w_1 \qquad\qquad\qquad\qquad + u_1 \qquad\qquad &= I \\
-w_1 + x_1 \qquad\qquad\qquad\qquad + v_1 &= C - I \\
&\vdots \\
\sum_1^t w_i - \sum_1^{t-1} x_i \qquad + u_t \qquad\qquad &= I \\
-\sum_1^t w_i + \sum_1^t x_i \qquad\qquad + v_t &= C - I
\end{aligned}
$$

$$\ldots \; w,\ x,\ u,\ v \geq 0$$

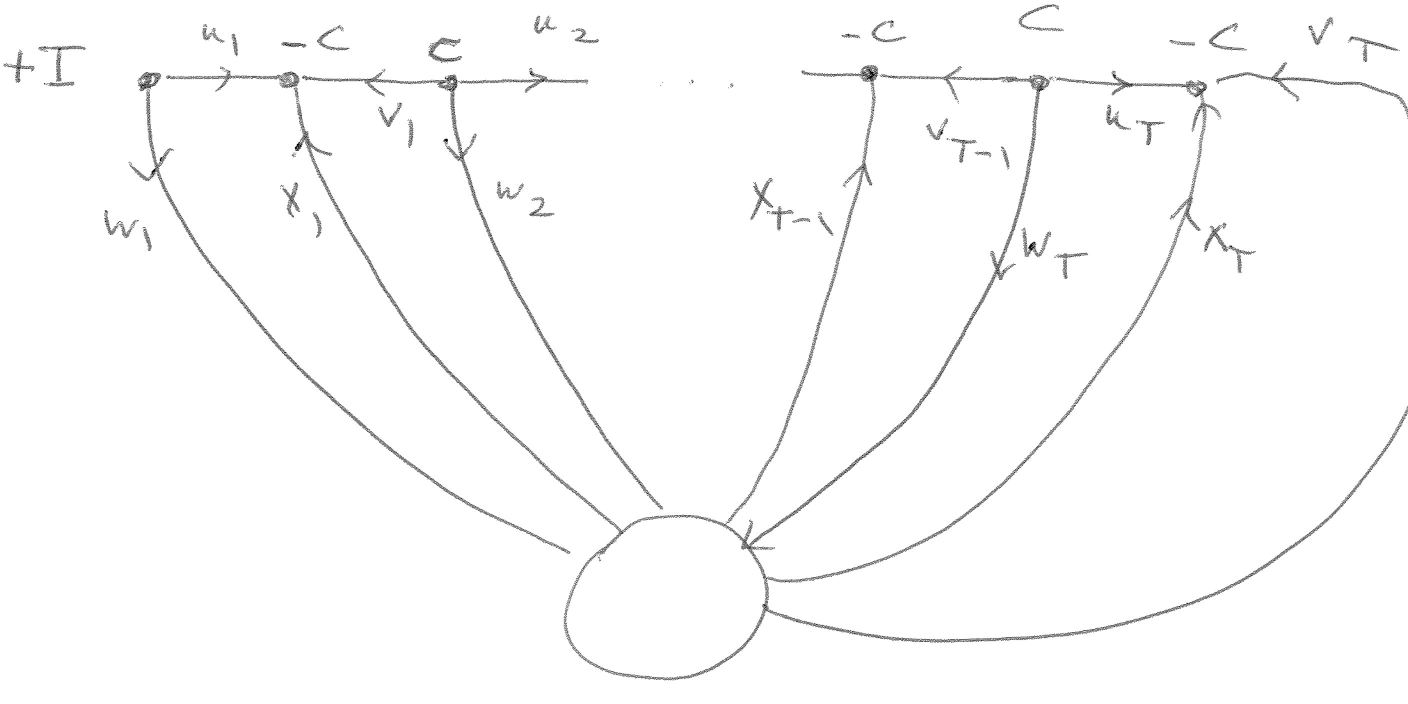
Now keep the first eq'n + take sums of consec. pairs:

$$
\begin{aligned}
w_1 \qquad\qquad + u_1 \qquad\qquad &= I \\
x_1 \qquad\qquad + u_1 + v_1 &= C \\
&\vdots \\
w_t \qquad + v_{t-1} + u_t &= C \\
x_t \qquad\qquad + u_t + v_t &= C
\end{aligned}
$$

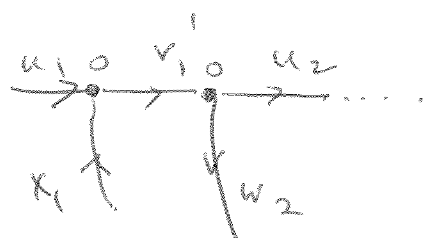
Finally, reverse the sign of every x constraint + then append $-$ the sum of all constraints:

$$
\begin{aligned}
w_1 \qquad\qquad\qquad & + u_1 && = I \\
-x_1 \qquad\qquad & - u_1 - v_1 && = -C \\
w_2 \qquad\qquad & + v_1 + u_2 && = C \\
& \vdots && \\
-w_1 + x_1 - w_2 \quad \cdots \quad & -w_T + x_T + v_T && = C - I
\end{aligned}
$$

$$\cdots \geq 0 .$$



If allow capacities,



$v_t' \leq C$.

OR 630: Mathematical Programming I. Fall 2005.

Take-home Midterm Exam. Available at 3 pm Monday, October 17th, from Donna Moore in Rhodes 206. Due: 4 pm on Tuesday, October 18th.

This must be all your own work. No discussions with any other students or faculty, or email to discussion groups or friends at other universities ... are allowed. If you are stuck on a problem I can provide some hints to get you going at a slight cost. You can consult any of the course materials, lecture or recitation notes, or Bertsimas-Tsitsiklis or Chvatal.

1. (30 points: 8, 7, 6, 5, 4) A partnership trades in the purchase and sale of agricultural commodities. It owns a large storage tank, which can hold C gallons of soybean oil. At the beginning of period 1, it holds I gallons ($0 < I < C$). In period t, $t = 1, \ldots, T$, the partnership can sell any oil it has, at selling price p_t per gallon. This must be delivered at the beginning of the period. It can also order more oil, at a purchase cost of c_t per gallon, which will be delivered at the end of the period. Any oil left at the end of period T is worthless. There is no holding cost, and the partnership knows all the prices and costs at the beginning of the planning horizon, periods 1 through T. It wants a plan for selling and buying to maximize its net profits.

Let w_t denote the number of gallons sold in period t and x_t the number of gallons bought. Do not use any other variables, such as storage amounts. In each period t, there are two constraints: first, the amount sold cannot exceed the amount on hand. In the first period, this constraint is just $w_1 \leq I$; in later periods, it bounds w_t by I together with the net amount bought over the first $t-1$ periods. The second constraint limits the amount purchased to what will fit in the tank. In the first period, this constraint is just $x_1 \leq C - I + w_1$; in later periods, it bounds x_t by $C - I$ together with the net amount sold in the first $t-1$ periods and the amount sold in period t. Also, all variables are nonnegative.

a) Formulate the problem of finding a profit-maximizing plan over the planning horizon as a linear programming problem.

b) Let y_t be a dual variable corresponding to the bound on w_t, and z_t a dual variable corresponding to the bound on x_t, $t = 1, \ldots, T$. State the dual problem to that in (a).

c) Obtain an optimal solution to the problem in (b) by inspection, by assigning values to the dual variables in the order $z_T, y_T, \ldots, z_1, y_1$. (A very heuristic argument of optimality suffices.)

d) Show how you could obtain an optimal solution to the problem in (a) from your solution in (c).

e) How could you show rigorously that your solutions constructed above are indeed optimal?

2. (30 points: 8, 12, 10) Consider the two related linear programming problems

$$(P) \quad \begin{array}{rl} \min_x & c^T x \\ Ax &= b, \\ x &\geq 0, \end{array} \qquad (\hat{P}) \quad \begin{array}{rl} \min_x & c^T x \\ \hat{A}x &= b, \\ x &\geq 0, \end{array}$$

where $\hat{A} = A + \mu b c^T$, $\mu \geq 0$, and $c \geq 0$.