

FORMULATION AND INTERPRETATION OF DUAL:

1. The product mix problem:

$$\begin{aligned} \text{Max} \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1, \dots, m \\ & x_j \geq 0 \quad \text{for } j=1, \dots, n \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Max} \\ \text{s.t.} \end{aligned}} \right\} (P)$$

P is a product-mix problem in which n products are being produced using some m resources.

- (D.V.) x_j : # of units (per year for ex.) produced for product j .
- b_i : # of units available for resource i .
- a_{ij} : # of units of resource i consumed by resource j in order to produce 1 unit of product j .
- c_j : profit obtained from 1 unit of product j .

Consider the dual problem:

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^m b_i w_i \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} w_i \geq c_j \quad \text{for } j=1, \dots, n \\ & w_i \geq 0 \quad \text{for } i=1, \dots, m \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Min} \\ \text{s.t.} \end{aligned}} \right\} (D)$$

Let w_i denote the shadow price (or in economics the fair market price/rent) of resource i .

We can interpret the dual constraints as follows:

Assume that the manufacturer decided not to produce anything but considers to rent out the resources he has in his hand. He wants to price resource i with w_i . Then every unit of product J not manufactured would result in a loss of c_J but earn a rent of

$\sum_{j=1}^m a_{ij} w_i$ (since it releases a_{ij} unit of resource i).

So dual constraints ensure that he does not lose money.

On the other hand, in order to survive in the competitive market, the manufacturer must provide fair prices so the dual objective seeks to minimize the total rent.

Note that, complementary slackness conditions says that if not all available units of resource i are utilized then the marginal worth of an extra unit of resource i (w_i^*) would be naturally zero. $(w_i^* \cdot (b_i - \sum_{j=1}^n a_{ij} x_j^*)) = 0$

And, if the net value $\sum_{i=1}^m a_{ij} w_i^*$ of the resources consumed by a unit level of production of product J exceeds the realizable profit c_J , then product J should not be produced.

2. The diet problem:

$$\begin{array}{ll}
 \text{Min} & \sum_{j=1}^n c_j x_j \\
 \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \geq b_i \quad i=1, \dots, m \\
 & x_j \geq 0 \quad j=1, \dots, n
 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{Min} \\ \text{s.t.} \end{array}} \right\} (P)$$

P is a diet problem where there are n different foods and m different nutrients.

x_j = Amount of food type J consumed

b_i = Minimal requirement of nutrient type i .

a_{ij} = Amount of nutrient i obtained from one unit of food J

c_j = ^{unit} Cost of food type J .

"Interpretation" of primal: mixing non-negative quantities of available foods, to synthesize ideal menu (food) at minimal cost.

Consider the dual:

$$\begin{array}{ll}
 \text{Max} & \sum_{i=1}^m b_i p_i \\
 \text{s.t.} & \sum_{i=1}^m a_{ij} p_i \leq c_j \quad j=1, \dots, n \\
 & p_i \geq 0 \quad i=1, \dots, m
 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{Max} \\ \text{s.t.} \end{array}} \right\} (D)$$

p_i = fair price of nutrient i in "nutritional market".

c_j = value of food j in "food market"

b_i = nutrient i found in "ideal" menu (food).

"Interpretation" of dual: We want to sell a menu in nutritional market at maximum profit. In order to survive in competitive market, this menu should cost less than available ~~prices~~ ^{prices for each item} in the food market.

$x_j \cdot (c_j - \sum_{i=1}^m a_{ij} p_i^*) = 0 \Rightarrow$ If the nutritional value is less than the cost for food j , we don't use food j .

$p_i \cdot (\sum_{j=1}^n a_{ij} x_j^* - b_i) = 0 \Rightarrow$ If there is an ideal menu providing excess amount of nutrient i , then an extra unit of nutrient i does not sell in nutritional market (price = 0).

3. The transportation problem:

Let there be m suppliers and n consumers.

s_i = amount of goods supplied by supplier i

d_j = " " " demanded by consumer j .

c_{ij} = cost of transporting one unit of good from sup. i to cons. j .

The transportation problem is:

$$\begin{array}{l} \min \quad \sum c_{ij} x_{ij} \\ \text{s.t.} \quad \sum_{i=1}^m x_{ij} = d_j \quad j=1, \dots, n \\ \quad \quad \sum_{j=1}^n x_{ij} = s_i \quad i=1, \dots, m \\ \quad \quad x_{ij} \geq 0 \quad \forall i, j \end{array} \quad (P)$$

where x_{ij} denotes number of units transported from i to j .

The dual problem is

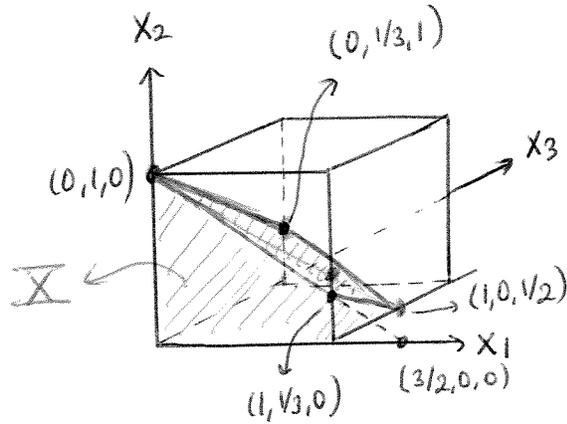
$$\begin{aligned} \text{Max } & \sum_{j=1}^n d_j w_j + \sum_{i=1}^m s_i u_i \\ \text{s.t. } & w_j + u_i \leq c_{ij} \\ & w_j, u_i \text{ unrestricted} \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Max } \\ \text{s.t. } \end{aligned}} \right\} (D)$$

Let w_j, u_i denote the price consumer j or supplier i wants to pay in order to have transportation. Obviously, they don't want to pay more than its value (or cost c_{ij}) so dual constraints rise. We want to maximize the total value of the system for customers so obj. funct. follows.

$$x_{ij}^* (c_{ij} - (w_j^* + u_i^*)) = 0 \quad \text{(if cust } j \text{ and supplier } i \text{ does not} \\ \text{in turn we don't provide service} \\ \text{will to pay for the transportation} \\ \text{always holds! (no interpretation)})$$

$$w_j^* (d_j - \sum_{i=1}^m x_{ij}^*) = 0$$

Example 2:



$$\underline{X} = \begin{array}{r} X_1 + 3/2 X_2 + X_3 \leq 3/2 \\ X_1 \leq 1 \\ X_2 \leq 1 \\ X_3 \leq 1 \\ X_i \geq 0 \end{array}$$

Consider $(3/2, 0, 0)$

$$X_1 = 3/2, X_2 = 0, X_3 = 0$$

$$s_1 = 0, s_2 = -1/2, s_3 = 1, s_4 = 1$$

→ A basic solution
but NOT basic feasible
since $s_2 = -1/2 < 0!$

Example AMPL Model for a diet problem:

Minimize

$$3.19 X_{BEEF} + 2.59 X_{CHK} + 2.29 X_{FISH} + 2.89 X_{HAM} + 1.89 X_{MCH} + 1.99 X_{MTL} + 1.99 X_{SPG} + 2.49 X_{TUR}$$

Subject to

$$60 X_{BEEF} + 8 X_{CHK} + 8 X_{FISH} + 40 X_{HAM} + 15 X_{MCH} + 70 X_{MTL} + 25 X_{SPG} + 60 X_{TUR} \geq 700$$

$$20 X_{BEEF} + 0 X_{CHK} + 10 X_{FISH} + 40 X_{HAM} + 35 X_{MCH} + 30 X_{MTL} + 50 X_{SPG} + 20 X_{TUR} \geq 700$$

$$10 X_{BEEF} + 20 X_{CHK} + 15 X_{FISH} + 35 X_{HAM} + 15 X_{MCH} + 15 X_{MTL} + 25 X_{SPG} + 15 X_{TUR} \geq 700$$

$$15 X_{BEEF} + 20 X_{CHK} + 10 X_{FISH} + 10 X_{HAM} + 15 X_{MCH} + 15 X_{MTL} + 15 X_{SPG} + 10 X_{TUR} \geq 700$$

$$X_{BEEF} \geq 0, X_{CHK} \geq 0, X_{FISH} \geq 0, X_{HAM} \geq 0, \\ X_{MCH} \geq 0, X_{MTL} \geq 0, X_{SPG} \geq 0, X_{TUR} \geq 0$$

FOOD LIST:

BEEF, CHK, ..., TUR
 ↓ ↓
 chicken turkey.

NUTRITION LIST:

A, B1, B2, C vitamin

```

set NUTR;
set FOOD;

param cost {FOOD} > 0;
param f_min {FOOD} >= 0;
param f_max {j in FOOD} >= f_min[j];

param n_min {NUTR} >= 0;
param n_max {i in NUTR} >= n_min[i];

param amt {NUTR,FOOD} >= 0;

var Buy {j in FOOD} >= f_min[j], <= f_max[j];

minimize Total_Cost: sum {j in FOOD} cost[j] * Buy[j];

subject to Diet {i in NUTR}:
    n_min[i] <= sum {j in FOOD} amt[i,j] * Buy[j] <= n_max[i];
    
```

Figure 2-1: Diet model in AMPL (diet.mod).

```

set NUTR := A B1 B2 C ;
set FOOD := BEEF CHK FISH HAM MCH MTL SPG TUR ;

param: cost f_min f_max :=
  BEEF 3.19 0 100
  CHK 2.59 0 100
  FISH 2.29 0 100
  HAM 2.89 0 100
  MCH 1.89 0 100
  MTL 1.99 0 100
  SPG 1.99 0 100
  TUR 2.49 0 100 ;

param: n_min n_max :=
  A 700 10000
  C 700 10000
  B1 700 10000
  B2 700 10000 ;

param amt (tr):
  A C B1 B2 :=
  BEEF 60 20 10 15
  CHK 8 0 20 20
  FISH 8 10 15 10
  HAM 40 40 35 10
  MCH 15 35 15 15
  MTL 70 30 15 15
  SPG 25 50 25 15
  TUR 60 20 15 10 ;

```

Figure 2-2: Data for diet model (diet.dat).

```

ampl: display Buy;
Buy [*] :=
  BEEF 0
  CHK 0
  FISH 0
  HAM 0
  MCH 46.6667
  MTL -1.07823e-16 → 0
  SPG -1.32893e-16 → 0
  TUR 0
;

```

PRODUCTION/TRANSPORTATION EXAMPLE:

```

set ORIG;      # origins (steel mills)
set DEST;     # destinations (factories)
set PROD;     # products

param rate {ORIG,PROD} > 0;      # tons per hour at origins
param avail {ORIG} >= 0;        # hours available at origins
param demand {DEST,PROD} >= 0;  # tons required at destinations

param make_cost {ORIG,PROD} >= 0;      # manufacturing cost/ton
param trans_cost {ORIG,DEST,PROD} >= 0; # shipping cost/ton

var Make {ORIG,PROD} >= 0;      # tons produced at origins
var Trans {ORIG,DEST,PROD} >= 0; # tons shipped

minimize Total_Cost:
    sum {i in ORIG, p in PROD} make_cost[i,p] * Make[i,p] +
    sum {i in ORIG, j in DEST, p in PROD}
        trans_cost[i,j,p] * Trans[i,j,p];

subject to Time {i in ORIG}:
    sum {p in PROD} (1/rate[i,p]) * Make[i,p] <= avail[i];

subject to Supply {i in ORIG, p in PROD}:
    sum {j in DEST} Trans[i,j,p] = Make[i,p];

subject to Demand {j in DEST, p in PROD}:
    sum {i in ORIG} Trans[i,j,p] = demand[j,p];

```

Figure 4-6: Production/transportation model, 3rd version (steelp.mod).

```

set ORIG := GARY CLEV PITT ;
set DEST := FRA DET LAN WIN STL FRE LAF ;
set PROD := bands coils plate ;

param avail := GARY 20 CLEV 15 PITT 20 ;

param demand (tr):
    bands  FRA  DET  LAN  WIN  STL  FRE  LAF :=
    coils  300  300  100  75  650  225  250
    plate  500  750  400  250  950  850  500
    plate  100  100  0  50  200  100  250 ;

param rate (tr):
    bands  GARY  CLEV  PITT :=
    coils  200  190  230
    plate  140  130  160
    plate  160  160  170 ;

param make_cost (tr):
    bands  GARY  CLEV  PITT :=
    coils  180  190  190
    plate  170  170  180
    plate  180  185  185 ;

param trans_cost :=
    [*,* ,bands]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
    GARY  30  10  8  10  11  71  6
    CLEV  22  7  10  7  21  82  13
    PITT  19  11  12  10  25  83  15

    [*,* ,coils]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
    GARY  39  14  11  14  16  82  8
    CLEV  27  9  12  9  26  95  17
    PITT  24  14  17  13  28  99  20

    [*,* ,plate]:  FRA  DET  LAN  WIN  STL  FRE  LAF :=
    GARY  41  15  12  16  17  86  8
    CLEV  29  9  13  9  28  99  18
    PITT  26  14  17  13  31  104  20 ;

```

Figure 4-7: Data for production/transportation model (steelp.dat).