Mathematical Programming	**Lecture 10**
OR 630 Fall 2005	**September 27, 2005**
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Last lecture, we were looking at the simplex algorithm to solve the linear programming problem in its standard form:

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b, \\ & x & \geq & 0. \end{array} \tag{1}$$

We found a criterion for optimality of a basic feasible solution.

Before stating the theorem, let's recall some notation:

- $\overline{c} = c - A^T \overline{y} = c - A^T B^{-T} c_B$;
- $\overline{y} = B^{-T} c_B$; and
- $\overline{x} = \begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix}$, the current b.f. solution.

Theorem 1 *(Optimality Criterion) If for every $j \in \nu$ the non-basic set of indices, $c_j - a_j^T \overline{y}$ is non-negative, then $\overline{x}$ is optimal for (P).*

□

We can first note that the reduced cost for a basic variable is $\overline{c}_j = c_j - a_j^T \overline{y} = 0$, and $\overline{c}_j$ (probably nonzero) for a non-basic one (i.e.: $\overline{c}_B = 0$, $\overline{c}_N = c_N - N^T B^{-T} c_B$).

We also have an equation relating ζ (the objective function value) to the values of the non-basic variables:

$$\zeta - \overline{c}^T x = c_N^T B^{-1} b \text{ or } \zeta - \overline{c}_N^T x_N = \overline{\zeta}.$$

Also if we denote

- $\overline{A} = B^{-1} A$;
- $\overline{b} = B^{-1} b$;
- $\overline{A}_B = B^{-1} B = I$; and
- $\overline{A}_N = B^{-1} N$,

we have that $Ax = b$ is equivalent to $\overline{A}x = \overline{b}$ or $x_B + B^{-1} N x_N = \overline{b}$ or $x_B + \overline{A}_N x_N = \overline{b}$.
We call the columns of $\overline{A}$: $\overline{a}_1, ..., \overline{a}_n$ so that $\overline{a}_j = B^{-1} a_j$.

We have the basic feasible solution:

$$\overline{x} = \begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix} = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix}$$

corresponding to the basis matrix B. We can compute $\overline{y} = B^{-T}c_B$ and hence compute $\overline{c}$ and finally check if $\overline{c}_N$ is non-negative. In the case where there exists some negative component related to index q: $\overline{c}_q < 0$, we want to increase x_q, and at the same time, hold all the other non-basic variables equal to zero. We get the objective function value: $\zeta = \overline{\zeta} + \overline{c}_q x_q$

To minimize the objective function we want x_q as large as possible but we have to know what happens to the basic variables when we choose a particular value for x_q.

Next, we compute $\overline{a}_q = B^{-1}a_q$ and we note that again if we hold all other non-basic variables fixed: $x_B = \overline{b} - \overline{a}_q x_q$, so as long as $x_B = \overline{b} - \overline{a}_q x_q$ remains true, our solution stay feasible. So if the updated column $\overline{a}_q$ is nonpositive: $\overline{a}_q \leq 0$, we can make x_q as large as we want.

Let's now consider the following ray:

$$x(\alpha) = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix}$$

where $\alpha \geq 0$ and e_q denotes a vector whose components are indexed by $j \in \nu$ with all entries equal to 0 except for a 1 in the row corresponding to index q.

Remark that because $\overline{a}_q \leq 0$, we have for every $\alpha \geq 0$, $\begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} \geq 0$ and

$$Ax(\alpha) = (B\ N) \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha (B\ N) \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} = b + \alpha(-BB^{-1}a_q + a_q) = b,$$

so the ray consists of feasible points. The objective function value is:

$$c^T x(\alpha) = (c_B^T\ c_N^T) \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha (c_B^T\ c_B^T) \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} = c_B^T B^{-1} b + \alpha(-c_B^T B^{-1} a_q + c_q)$$

and because $c_B^T B^{-1} = \overline{y}^T$ we get: $c^T x(\alpha) = \overline{\zeta} + \alpha(c_q - a_q^T \overline{y})$.

Therefore if we choose q such that $c_q - c_B^T B^{-1} a_q < 0$, we have $\lim_{\alpha \to \infty} c^T x(\alpha) = -\infty$.

We therefore have an unboundedness criterion:

Theorem 2 *(Unboundedness Criterion) Let $\overline{x}$ be the basic feasible solution corresponding to the basis matrix B and let $c_B^T B^{-1} = \overline{y}^T$.*
If there is some $q \in \nu$ such that $\overline{c}_q = c_q - a_q^T \overline{y} < 0$ and $\overline{a}_q = B^{-1}a_q \leq 0$ then (P) is unbounded.
Indeed, the objective function is unbounded below on the feasible ray $\begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix} + \alpha \begin{pmatrix} -B^{-1}a_q \\ e_q \end{pmatrix}$.
□

Remark 1 *In Theorem 1 we found an optimal solution for the dual too.*
Similarly here we can find a certificate that the dual is infeasible (cf a future HW)

The remaining case is when we choose q with $\overline{c}_q < 0$, compute $\overline{a}_q$ and find that at least one component $\overline{a}_{iq}$ is positive.

Let's come back to the previous example with $c_1 = 4$:

$$(E) \quad \begin{array}{rrrrrrrrrrrrr} \min & 4x_1 & +7x_2 & +5x_3 & & & & & & & & & \\ & 2x_1 & + & x_2 & + & x_3 & - & x_4 & & & & = & 5, \\ & x_1 & + & 3x_2 & + & x_3 & & & - & x_5 & & = & 5, \\ & x_1 & + & x_2 & + & 4x_3 & & & & & - x_6 & = & 2, \\ & & & & & & & & & & x & \geq & 0, \end{array}$$

With basic indices $\beta = \{4, 1, 6\}$ and non-basic indices $\nu = \{2, 3, 5\}$ and the basis:

$$B = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & 4 & 0 \end{pmatrix}, \quad c_B = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.$$

So we find: $\overline{y} = B^{-T} c_B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$,

and then $\overline{c}_N = c_N - N^T \overline{y} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$.

So we choose $q = 2$ with $\overline{c}_2 = -5$.

We want to increase x_2 (the number of bananas) and compute:

$$\overline{a}_2 = B^{-1} a_2 = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.$$

We see that $\overline{a}_2$ is positive.

Interpretation of $\overline{a}_2$ and $\overline{c}_2$

$\overline{a}_2 = B^{-1} a_2$ therefore $B\overline{a}_2 = a_2 =$ amount of all the nutrients in each banana $= \sum_{i \in \beta} \overline{a}_{i2} B_i$ where B_i is the ith column of B.

We can think of $\overline{a}_{i2}$ as a recipe of a banana. In our case: $\overline{a}_2 = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$, so we can make a "synthetic banana" out of:

- 5 units of food 4,

- 3 units of food 1, and
- 2 units of food 6.

In another way, we can make a banana out of 3 apples throwing away 5 unit of nutrient 1 (corresponding to food 4) and throwing away 2 unit of nutrient 3 (corresponding to food 6). We obtain then a recipe of making a nonbasic food out of the basic foods.

Now let's look at $\overline{c}_2 = c_2 - a_2^T \overline{y} = c_2 - c_B^T(B^{-1}a_2)$:

- c_2 is the cost of a real banana.
- $B^{-1}a_2$ is the recipe for a synthetic banana.
- $c_B^T(B^{-1}a_2)$ is the cost of a synthetic banana.

If $\overline{c}_2 < 0$, the real cost of a banana is less than the cost of a synthetic one. Therefore we should eat real bananas and adjust the level of the basic foods by decreasing according to the recipe for a banana.

So if we increase x_2 to α, we find: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} - \alpha \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

The largest possible value for α is 1 (i.e., x_4 then hits 0).

The new solution is $x_+ = (5; 0; 0; 5; 0; 3) + 1(-3; 1; 0; -5; 0; -2) = (2; 1; 0; 0; 0; 1)$

We have a new basic feasible solution corresponding to $\beta_+ = \{2, 1, 6\}$ and basis

$$B_+ = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

This is effectively a basic feasible solution because the basis B_+ is non singular.

Also the cost has become 15 ($\overline{\zeta} = c_B^T \overline{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} = 20$, $\overline{c}_q = -5$, $\alpha = 1$).

Let's go back to the general case of a simplex iteration:

First we choose $q \in \nu$ with $\overline{c}_q < 0$ and compute $\overline{a}_q = B^{-1}a_q$.

Because we are assuming that $\overline{a}_q$ is not nonpositive (otherwise we are in the unbounded case), there exists a positive component $\overline{a}_{iq}$.

If we set x_q to $\alpha \geq 0$ and keep all other non-basic variables at 0, the basic variables change according to $x_B = \overline{b} - \alpha \overline{a}_q$.

We want α as large as possible (because the updated objective function is $\zeta = \overline{\zeta} + \alpha \overline{c}_q$), so we choose $\max\{\alpha \text{ s.t. } x_B \geq 0\}$

The maximum α is $\overline{\alpha} = \min\{\frac{\overline{b}_i}{\overline{a}_{iq}} : \text{s.t. } \overline{a}_{iq} > 0\} = \frac{\overline{b}_p}{\overline{a}_{pq}}$ for some p with $\overline{a}_{pq} > 0$. Note that $\overline{b}_p \geq 0$ (because it's a basic feasible solution) and therefore $\overline{\alpha} \geq 0$.

Note that if $\overline{b}_p \neq 0$ then $\overline{\alpha} > 0$ and the new solution will be different from the current one with strictly lower objective function value. Note also that if there is a $p' \neq p$ such that $\frac{\overline{b}_{p'}}{\overline{a}_{p'q}} = \frac{\overline{b}_p}{\overline{a}_{pq}}$ then the new solution will have some basic variable equal to zero.

We can now "increase" x_q to level $\overline{\alpha}$ and make it a basic variable replacing the old p^{th} basic variable and update the set of new basic variables indices to β_+ with q replacing the pth index. The new "basis matrix" is obtained by replacing the pth column of B by a_q:
$B_+ = (B_1, B_2, ..., B_{p-1}, a_q, B_{p+1}, ..., B_m)$.

We put quotes because we don't know yet that B_+ is a basis!

Claim 1 B_+ *is a basis matrix.*

proof:

Because $a_q = B\overline{a}_q$ we can build the new basic matrix by multiplying B by the identity matrix with $\overline{a}_q$ instead of the p^{th} column:

$$B_+ = BE_p = B \begin{pmatrix} 1 & 0 & \dots & 0 & \overline{a}_{1q} & 0 & \dots & & 0 \\ 0 & 1 & & & \vdots & & & & \vdots \\ \vdots & & \ddots & 0 & & & & & \\ & & & 1 & \vdots & & & & \\ & & & 0 & \overline{a}_{pq} & & & & \\ & & & & \vdots & & \ddots & & \vdots \\ \vdots & & & & \vdots & & & 1 & 0 \\ 0 & \dots & & 0 & \overline{a}_{nq} & 0 & \dots & 0 & 1 \end{pmatrix};$$

these two matrices B and E_p are non-singular: the determinant of the second one is $\overline{a}_{pq} > 0$

Remark 2 *This decomposition of B_+ gives a easy way to find B_+^{-1} and plays a key role in the efficiency of the algorithm.*

We have now a new non-negative solution $x_+ = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ \overline{e}_q \end{pmatrix}$, satisfying $Ax = b$ and having non-zeros exactly in positions corresponding to the indices of β_+. Therefore x_+ is the basic feasible solution corresponding to $\{B_+, \beta_+\}$

We've just finished one simplex iteration.

Mathematical Programming	**Homework 5**
OR 630 Fall 2005	**Due October 7, 2005**

1. The indication for unboundedness in the simplex method shows the existence of feasible solutions to the primal problem with objective function values unbounded below. This implies via weak or strong duality that the dual problem is infeasible. Show how to obtain a short certificate of the infeasibility of (D) from the quantities already computed.

2. Consider an LP problem is in the form we considered for the simplex interpretation of the simplex method:

$$\min\{c^T x : \tilde{A}x = \tilde{b},\, e^T x = 1,\, x \geq 0\},$$

where $e \in \mathbf{R}^n$ is a vector of ones. Suppose you have a basic feasible solution $\bar{x}$ for this problem, and you compute all the reduced costs $\bar{c}_N$. Show how you can obtain a lower bound on the optimal value of the problem and hence a bound on how far $\bar{x}$ is from optimality.

3. Suppose you are solving a standard form LP problem with n variables from a given basic feasible solution, and you know that every basic solution has at most one basic variable zero. Show that the simplex method will either terminate or improve the objective function value within n iterations from any basic feasible solution, and deduce that it will terminate in a finite number of iterations.