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| **Mathematical Programming** | **Lecture 10** |
| **OR 630 Fall 2005** | **September 27, 2005** |
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Last lecture, we were looking at the simplex algorithm to solve the linear programming problem in its standard form:

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b, \\ & x & \geq & 0. \end{array} \tag{1}$$

We found a criterion for optimality of a basic feasible solution.

Before stating the theorem, let's recall some notation:

- $\overline{c} = c - A^T \overline{y} = c - A^T B^{-T} c_B$;
- $\overline{y} = B^{-T} c_B$; and
- $\overline{x} = \begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix}$, the current b.f. solution.

**Theorem 1** *(Optimality Criterion) If for every $j \in \nu$ the non-basic set of indices, $c_j - a_j^T \overline{y}$ is non-negative, then $\overline{x}$ is optimal for (P).*

$\square$

We can first note that the reduced cost for a basic variable is $\overline{c}_j = c_j - a_j^T \overline{y} = 0$ , and $\overline{c}_j$ (probably nonzero) for a non-basic one (i.e.: $\overline{c}_B = 0$, $\overline{c}_N = c_N - N^T B^{-T} c_B$).

We also have an equation relating $\zeta$ (the objective function value) to the values of the non-basic variables:

$$\zeta - \overline{c}^T x = c_N^T B^{-1} b \text{ or } \zeta - \overline{c}_N^T x_N = \overline{\zeta}.$$

Also if we denote

- $\overline{A} = B^{-1} A$;
- $\overline{b} = B^{-1} b$;
- $\overline{A}_B = B^{-1} B = I$; and
- $\overline{A}_N = B^{-1} N$,

we have that $Ax = b$ is equivalent to $\overline{A} x = \overline{b}$ or $x_B + B^{-1} N x_N = \overline{b}$ or $x_B + \overline{A}_N x_N = \overline{b}$.
We call the columns of $\overline{A}$: $\overline{a}_1, ..., \overline{a}_n$ so that $\overline{a}_j = B^{-1} a_j$.

We have the basic feasible solution:

$$\overline{x} = \begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix} = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix}$$

corresponding to the basis matrix $B$. We can compute $\overline{y} = B^{-T}c_B$ and hence compute $\overline{c}$ and finally check if $\overline{c}_N$ is non-negative. In the case where there exists some negative component related to index $q$: $\overline{c}_q < 0$, we want to increase $x_q$, and at the same time, hold all the other non-basic variables equal to zero. We get the objective function value: $\zeta = \overline{\zeta} + \overline{c}_q x_q$

To minimize the objective function we want $x_q$ as large as possible but we have to know what happens to the basic variables when we choose a particular value for $x_q$.

Next, we compute $\overline{a}_q = B^{-1}a_q$ and we note that again if we hold all other non-basic variables fixed: $x_B = \overline{b} - \overline{a}_q x_q$, so as long as $x_B = \overline{b} - \overline{a}_q x_q$ remains true, our solution stay feasible. So if the updated column $\overline{a}_q$ is nonpositive: $\overline{a}_q \leq 0$, we can make $x_q$ as large as we want.

Let's now consider the following ray:

$$x(\alpha) = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix}$$

where $\alpha \geq 0$ and $e_q$ denotes a vector whose components are indexed by $j \in \nu$ with all entries equal to 0 except for a 1 in the row corresponding to index $q$.

Remark that because $\overline{a}_q \leq 0$, we have for every $\alpha \geq 0$, $\begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} \geq 0$ and

$$Ax(\alpha) = (B\ N) \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha (B\ N) \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} = b + \alpha(-BB^{-1}a_q + a_q) = b,$$

so the ray consists of feasible points. The objective function value is:

$$c^T x(\alpha) = (c_B^T\ c_N^T) \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha (c_B^T\ c_B^T) \begin{pmatrix} -\overline{a}_q \\ e_q \end{pmatrix} = c_B^T B^{-1} b + \alpha(-c_B^T B^{-1} a_q + c_q)$$

and because $c_B^T B^{-1} = \overline{y}^T$ we get: $c^T x(\alpha) = \overline{\zeta} + \alpha(c_q - a_q^T \overline{y})$.

Therefore if we choose $q$ such that $c_q - c_B^T B^{-1} a_q < 0$, we have $\lim_{\alpha \to \infty} c^T x(\alpha) = -\infty$.

We therefore have an unboundedness criterion:

**Theorem 2** *(Unboundedness Criterion) Let $\overline{x}$ be the basic feasible solution corresponding to the basis matrix $B$ and let $c_B^T B^{-1} = \overline{y}^T$.*
*If there is some $q \in \nu$ such that $\overline{c}_q = c_q - a_q^T \overline{y} < 0$ and $\overline{a}_q = B^{-1} a_q \leq 0$ then (P) is unbounded.*
*Indeed, the objective function is unbounded below on the feasible ray* $\begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix} + \alpha \begin{pmatrix} -B^{-1}a_q \\ e_q \end{pmatrix}$.

□

**Remark 1** *In Theorem 1 we found an optimal solution for the dual too.*
*Similarly here we can find a certificate that the dual is infeasible (cf a future HW)*

The remaining case is when we choose $q$ with $\overline{c}_q < 0$, compute $\overline{a}_q$ and find that at least one component $\overline{a}_{iq}$ is positive.

Let's come back to the previous example with $c_1 = 4$:

$$
(E) \quad
\begin{array}{lrcrcrcrcrcrcl}
\min & 4x_1 & +7x_2 & +5x_3 & & & & & & & & & & \\
& 2x_1 & + & x_2 & + & x_3 & - & x_4 & & & & & = & 5, \\
& x_1 & + & 3x_2 & + & x_3 & & & - & x_5 & & & = & 5, \\
& x_1 & + & x_2 & + & 4x_3 & & & & & - & x_6 & = & 2, \\
& & & & & & & & & & & x & \geq & 0,
\end{array}
$$

With basic indices $\beta = \{4, 1, 6\}$ and non-basic indices $\nu = \{2, 3, 5\}$ and the basis:

$$
B = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & 4 & 0 \end{pmatrix}, \quad c_B = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.
$$

So we find: $\overline{y} = B^{-T} c_B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$,

and then $\overline{c}_N = c_N - N^T \overline{y} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$.

So we choose $q = 2$ with $\overline{c}_2 = -5$.

We want to increase $x_2$ (the number of bananas) and compute:

$$
\overline{a}_2 = B^{-1} a_2 = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.
$$

We see that $\overline{a}_2$ is positive.

Interpretation of $\overline{a}_2$ and $\overline{c}_2$

$\overline{a}_2 = B^{-1} a_2$ therefore $B\overline{a}_2 = a_2 =$ amount of all the nutrients in each banana $= \sum_{i \in \beta} \overline{a}_{i2} B_i$ where $B_i$ is the ith column of $B$.

We can think of $\overline{a}_{i2}$ as a recipe of a banana. In our case: $\overline{a}_2 = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$, so we can make a "synthetic banana" out of:

- 5 units of food 4,

- 3 units of food 1, and
- 2 units of food 6.

In another way, we can make a banana out of 3 apples throwing away 5 unit of nutrient 1 (corresponding to food 4) and throwing away 2 unit of nutrient 3 (corresponding to food 6). We obtain then a recipe of making a nonbasic food out of the basic foods.

Now let's look at $\overline{c}_2 = c_2 - a_2^T \overline{y} = c_2 - c_B^T (B^{-1} a_2)$:

- $c_2$ is the cost of a real banana.
- $B^{-1} a_2$ is the recipe for a synthetic banana.
- $c_B^T (B^{-1} a_2)$ is the cost of a synthetic banana.

If $\overline{c}_2 < 0$, the real cost of a banana is less than the cost of a synthetic one.
Therefore we should eat real bananas and adjust the level of the basic foods by decreasing according to the recipe for a banana.
So if we increase $x_2$ to $\alpha$, we find: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} - \alpha \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$
The largest possible value for $\alpha$ is 1 (i.e., $x_4$ then hits 0).
The new solution is $x_+ = (5; 0; 0; 5; 0; 3) + 1(-3; 1; 0; -5; 0; -2) = (2; 1; 0; 0; 0; 1)$

We have a new basic feasible solution corresponding to $\beta_+ = \{2, 1, 6\}$ and basis

$$B_+ = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

This is effectively a basic feasible solution because the basis $B_+$ is non singular.

Also the cost has become 15 ($\overline{\zeta} = c_B^T \overline{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} . \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} = 20$, $\overline{c}_q = -5$, $\alpha = 1$).

Let's go back to the general case of a simplex iteration:

First we choose $q \in \nu$ with $\overline{c}_q < 0$ and compute $\overline{a}_q = B^{-1} a_q$.
Because we are assuming that $\overline{a}_q$ is not nonpositive (otherwise we are in the unbounded case), there exists a positive component $\overline{a}_{iq}$.
If we set $x_q$ to $\alpha \geq 0$ and keep all other non-basic variables at 0, the basic variables change according to $x_B = \overline{b} - \alpha \overline{a}_q$.

We want $\alpha$ as large as possible (because the updated objective function is $\zeta = \overline{\zeta} + \alpha \overline{c}_q$), so we choose $\max\{\alpha \text{ s.t. } x_B \geq 0\}$

The maximum $\alpha$ is $\overline{\alpha} = \min\{\frac{\overline{b}_i}{\overline{a}_{iq}} : \text{s.t. } \overline{a}_{iq} > 0\} = \frac{\overline{b}_p}{\overline{a}_{pq}}$ for some $p$ with $\overline{a}_{pq} > 0$.
Note that $\overline{b}_p \geq 0$ (because it's a basic feasible solution) and therefore $\overline{\alpha} \geq 0$.

Note that if $\overline{b}_p \neq 0$ then $\overline{\alpha} > 0$ and the new solution will be different from the current one with strictly lower objective function value. Note also that if there is a $p' \neq p$ such that $\frac{\overline{b}_{p'}}{\overline{a}_{p'q}} = \frac{\overline{b}_p}{\overline{a}_{pq}}$ then the new solution will have some basic variable equal to zero.

We can now "increase" $x_q$ to level $\overline{\alpha}$ and make it a basic variable replacing the old $p^{th}$ basic variable and update the set of new basic variables indices to $\beta_+$ with $q$ replacing the $p$th index. The new "basis matrix" is obtained by replacing the $p$th column of $B$ by $a_q$:
$B_+ = (B_1, B_2, ..., B_{p-1}, a_q, B_{p+1}, ..., B_m)$.

We put quotes because we don't know yet that $B_+$ is a basis!

**Claim 1** *$B_+$ is a basis matrix.*

proof:

Because $a_q = B\overline{a}_q$ we can build the new basic matrix by multiplying $B$ by the identity matrix with $\overline{a}_q$ instead of the $p^{th}$ column:

$$B_+ = BE_p = B \begin{pmatrix} 1 & 0 & \dots & 0 & \overline{a}_{1q} & 0 & \dots & & 0 \\ 0 & 1 & & & \vdots & & & & \vdots \\ \vdots & & \ddots & 0 & & & & & \\ & & & 1 & \vdots & & & & \\ & & & 0 & \overline{a}_{pq} & & & & \\ & & & & \vdots & & \ddots & & \vdots \\ \vdots & & & & \vdots & & & 1 & 0 \\ 0 & \dots & & 0 & \overline{a}_{nq} & 0 & \dots & 0 & 1 \end{pmatrix};$$

these two matrices $B$ and $E_p$ are non-singular: the determinant of the second one is $\overline{a}_{pq} > 0$

**Remark 2** *This decomposition of $B_+$ gives a easy way to find $B_+^{-1}$ and plays a key role in the efficiency of the algorithm.*

We have now a new non-negative solution $x_+ = \begin{pmatrix} \overline{b} \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\overline{a}_q \\ \overline{e}_q \end{pmatrix}$, satisfying $Ax = b$ and having non-zeros exactly in positions corresponding to the indices of $\beta_+$. Therefore $x_+$ is the basic feasible solution corresponding to $\{B_+, \beta_+\}$

We've just finished one simplex iteration.