Mathematical Programming — **Lecture 22**
OR 630 Fall 2005 — **November 10, 2005**
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First, let's talk about the complexity of algorithm. How could we show that *any* simplex method is exponential? The only hope is by studying the graph (vertices, edges) of the feasible regions of LPs.

Definition 1 *Given a pointed polyhedron Q and two vertices v and w of Q, $d_Q(v,w)$ is the smallest k such that there are vertices $v_0 = v, v_1, \ldots, v_k = w$ of Q with $[v_{j-1}, v_j]$ an edge of Q for $1 \leq j \leq k$.*

Definition 2 *The diameter of Q is $\delta(Q) := \max\{d_Q(v,w) : v$ and w are vertices of $Q\}$.*

Finally:

Definition 3 $\Delta_u(d,n) := \max\{\delta(Q) : Q$ *is a pointed polyhedron in* $\Re^d$ *with* n *facets*$\}$;
$\Delta(d,n) := \max\{\delta(Q) : Q$ *is a bounded polyhedron in* $\Re^d$ *with* n *facets*$\}$.

In 1957, W. M. Hirsch conjectured $\Delta_u(d,n) \leq n - d$.

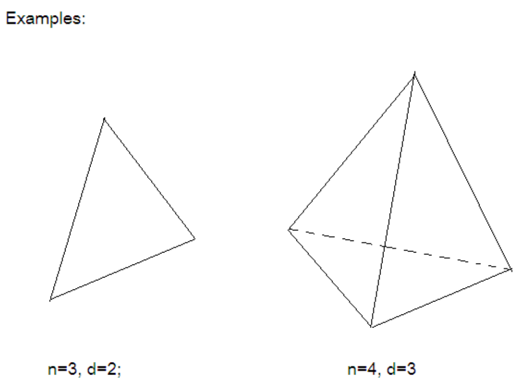
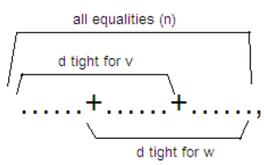


Figure 1: Examples which satisfy $\Delta_u(d,n) \leq n - d$

If $n \leq 2d$, $\Delta_u(d,n) = n - d$ seems best possible: since at least $n - d$ inequalities must be



exchanged. Klee and Walkup ('67) showed that it is false (they constructed a particular counter example):

$$\Delta_u(d,n) \geq n - d + \min\{[\frac{d}{4}], [\frac{n-d}{4}]\}.$$

But the bounded case is still open (right or wrong). It is called the *Hirsch conjecture*:

$$?\Delta(d,n) \leq n - d.???$$

Is it polynomial or exponential? The answer (Kalai and Kleitman, 1992): It is subexponential:

$$\Delta_u(d,n) \leq (4d)^{\log_2 N} = n^{\log_2 d + 2}.$$

Log (this bound) is $\mathrm{O}(\log_2 n \cdot \log_2 d)$; Log (polynomial) is linear in $\log_2 n, \log_2 d$; Log (exponential) is linear in n, d. (We can see that the first one is between the other two, so it is superpolynomial, but subexponential.)

Is there a "simple" simplex method that is subexponential? Kalai gave a randomized simplex method whose expected number of steps is:

$$\exp(K\sqrt{n \log_2 d}).$$

Here, K is an absolute constant. (For the TSP, n can be gigantic, so it may still look like an exponential, but it is just an upper bound.)

Now, let's talk about the complexity of *problems*.

Introduction to the ellipsoid method. Let's be more precise about "what is a polynomial-time algorithm"?

Definition 4 *An **instance** of an optimization problem, is a feasible set F and a cost function $c : F \to \Re$. The objective is to $\min c$ over F. An optimization problem is just a set of such instances.*

The LP problem is the set of all instances where F is a polyhedron and c is a linear function.

An algorithm applies to a problem, and generates a solution for all its instances. Question: How long does the algorithm take? For LP, an instance is defined by (A, b, c) (the data set). We'll require the data to be integer-valued (or equivalently rational). We can write down an integer

$$Z = \pm(Z_k 2^k + Z_{k-1} 2^{k-1} + \cdots + Z_0 2^0),$$

with each $Z_j = 0$ or 1, its binary representation in k bits, where k is about $\lceil \log_2 |Z| \rceil$ (rounded up).

Definition 5 *size*$(Z) := \lceil \log(|Z|+1) \rceil + 1$, *and size*$(A,b,c) := \sum_i \sum_j$ *size*$(a_{ij}) + \sum_i$ *size*$(b_i) + \sum_j$ *size*$(c_j) = L$.

This is $\mathrm{O}(\min \log_2 U)$, when U bounds all $|a_{ij}|'s$, $|b_i|$'s and $|c_j|$'s.

Definition 6 *A **polynomial-time algorithm** for a problem is one that, applied to any instance of that problem, gives a solution in a number of bit operations ($+, -, \times$, comparisons) that is bounded by a polynomial in its size.*

Note, if an algorithm takes a polynomial number of arithmetical operations $(+, -, \times, \div)$ on integers whose size remains polynomial in the size of the instance, this is a polynomial-time algorithm.

Is there a polynomial-time algorithm for LP? Yes. Khachiyan (1979,1980) showed a polynomial-time algorithm: $O(n^2L)$ iterations, each requiring $O(n^2)$ arithmetical operations on integers of length $O(L)$. He used the ellipsoid algorithm, which was developed by Yudin and Nemirovski (1976) and Shor (1977) for general convex programming.

The ellipsoid algorithm : This algorithm is applied to the feasibility form of LP : $A^Tx \leq b$. Assume A is an $m \times n$ matrix, so there are n inequalities in m unknowns.
A problem

$$\begin{array}{rrcl} \min_{\bar{x}} & \bar{c}^T\bar{x} & & \\ & \bar{A}\bar{x} & = & \bar{b}, \\ & \bar{x} & \geq & 0. \end{array}$$

where $\bar{A}$ is an $\bar{m} \times \bar{n}$ matrix, can be transferred into

$$\begin{array}{rcrcrl} \bar{A}\bar{x} & & & \leq & \bar{b}, & \\ -\bar{A}\bar{x} & & & \leq & -\bar{b}, & \\ -1\bar{x} & & & \leq & 0, & \\ & & \bar{A}^T\bar{y} & \leq & \bar{c}, & \text{(dual)} \\ \bar{c}^T & - & \bar{b}^T\bar{y} & \leq & 0. & \text{(optimal)} \end{array}$$

So it is enough to be able to "solve" $A^Tx \leq b$.
Suppose we know,

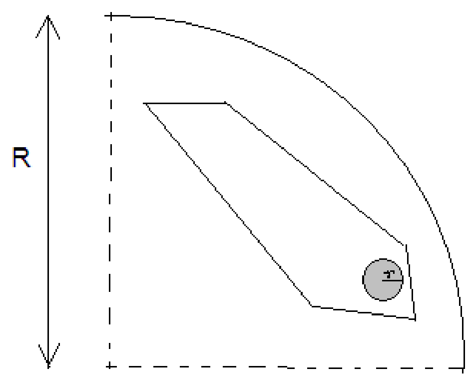


Figure 2: Q and $B(\hat{x}, r)$.

$$Q := \{x \in \Re^m, A^Tx \leq b\} \subseteq B(0, R) = \{x \in \Re^m : \| x \| \leq R\},$$

and, *if* Q is nonempty,

$$B(\hat{x}, r) \subseteq Q, \text{where } B(\hat{x}, r) = \{x \in \Re^m : \parallel x - \hat{x} \parallel \leq r\}.$$

for some (unknown!!) $\hat{x}$ and some $0 < r < R$.

What does the algorithm do? This algorithm generates a sequence of ellipsoids:

$$E_k = \{x \in \Re^m : (x - x_k)^T B_k^{-1}(x - x_k) \leq 1\},$$

where $B_k \in \Re^{m \times m}$ is symmetric and positive definite, i.e., $v^T B_k v > 0$ for all $v \neq 0$. So $E_0 = B(0, R) \supseteq Q$. At iteration k, either $x_k \in Q$ (great — stop! for we only need to find a feasible solution), or find a constraint, say $a_j^T x \leq b_j$, that is violated by x_k. Then find the

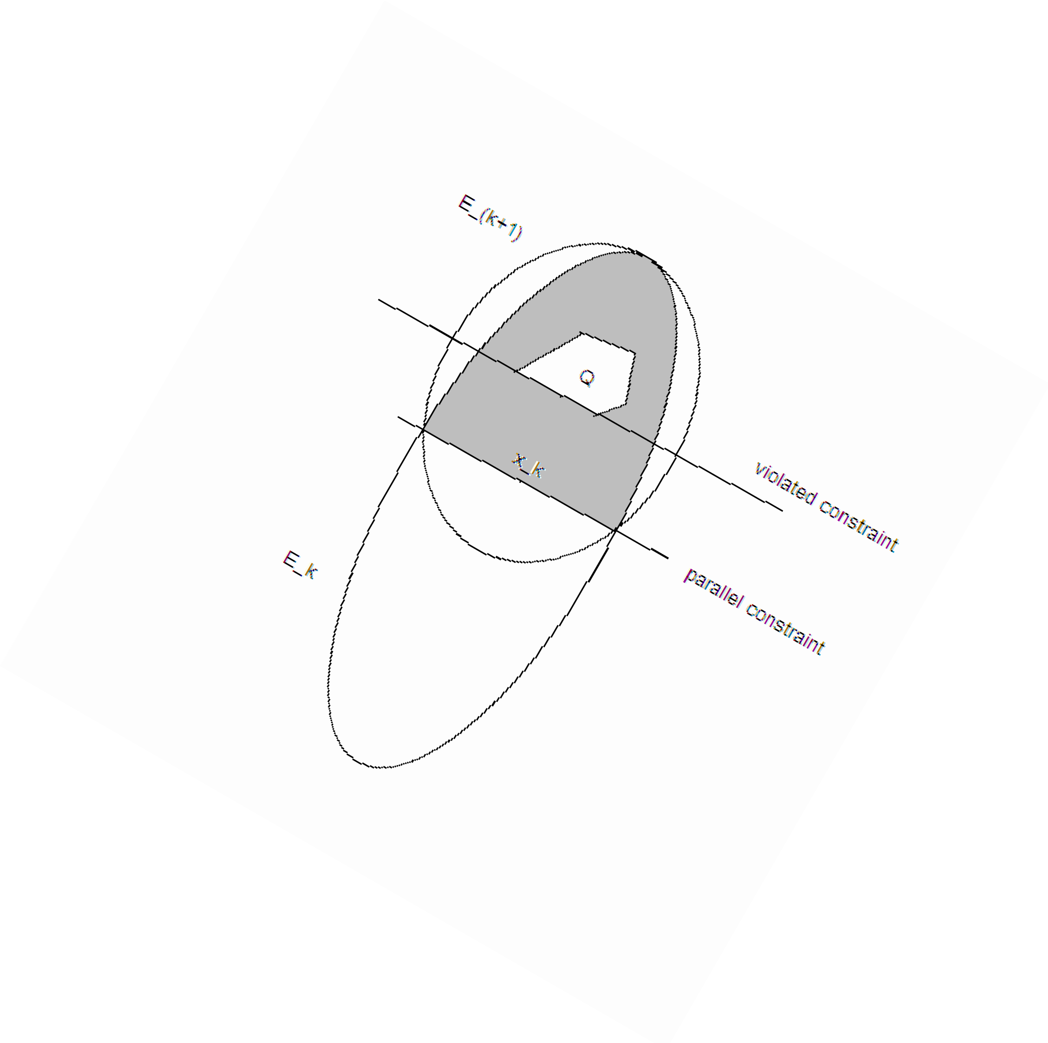


Figure 3: The iteration steps.

minimum volume ellipsoid E_{k+1} containing $\{x \in E_k : a_j^T x \leq a_j^T x_k\}$.

Key:

$$\mathrm{vol}(E_{k+1}) < \exp(-\frac{1}{2(m+1)}) \times \mathrm{vol}(E_k).$$

This is the magic thing that makes everything work.