Mathematical Programming	**Lecture 24**
OR 630 Fall 2005	**November 22, 2005**
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Let's continue with the proof of Theorem 2.

Theorem 2 *The minimum volume ellipsoid containing* $\{x \in E : a^T x \leq a^T z\}$ *is* $E(z_+, B_+)$ *where*

$$\begin{aligned} z_+ &= z - \tau \tfrac{Ba}{\sqrt{a^T Ba}}, \\ B_+ &= \delta \left(B - \sigma \tfrac{Ba^T aB}{a^T Ba} \right). \end{aligned}$$

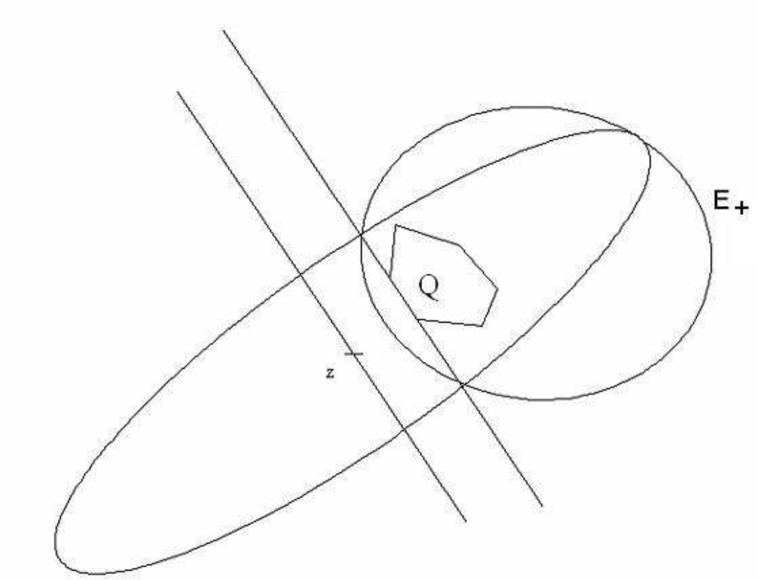

Figure 1: Ellipsoid method

Proof: Last time we have derived the inequalities

$$\begin{aligned} (x-z)^T B^{-1}(x-z) &\leq 1, && \text{(E1)} \\ ((x-z)^T \bar{a} + \alpha)(\bar{a}^T(x-z)+1) &\leq 0. && \text{(E2)} \end{aligned}$$

where $\bar{a} = \frac{a}{\sqrt{a^T Ba}}$. Let's continue by deriving the sum $(1-\sigma)(\text{E1}) + \sigma(\text{E2})$:

$$(x-z)^T \left((1-\sigma)B^{-1} + \sigma \bar{a}\bar{a}^T\right)(x-z) + \sigma(1+\alpha)\bar{a}^T(x-z) \leq (1-\sigma) - \sigma\alpha.$$

We would like to arrange the terms so that we have a "complete square" on the left hand side. Note that $\bar{a}^T B \bar{a} = 1$ and this implies

$$\left((1-\sigma)B^{-1} + \sigma\bar{a}\bar{a}^T\right) B\bar{a} = (1-\sigma)\bar{a} + \sigma\bar{a}(\bar{a}^T B\bar{a}) = (1-\sigma)\bar{a} + \sigma\bar{a} = \bar{a}.$$

We can use this fact as well as $\bar{a}^T B\bar{a} = 1$ to derive

$$\left(x - z + \frac{\sigma(1+\alpha)}{2}B\bar{a}\right)^T \left((1-\sigma)B^{-1} + \sigma\bar{a}\bar{a}^T\right)\left(x - z + \frac{\sigma(1+\alpha)}{2}B\bar{a}\right) \le (1-\sigma) - \sigma\alpha + \frac{\sigma^2(1+\alpha)^2}{4}.$$

Now to simplify the right hand side let's denote

$$\delta = \frac{1}{1-\sigma}\left((1-\sigma) - \sigma\alpha + \frac{\sigma^2(1+\alpha)^2}{4}\right),$$

and the RHS becomes $\delta(1-\sigma)$. Then we can use the Sherman-Morrison formula to derive

$$\left(B^{-1} + \frac{\sigma}{1-\sigma}\bar{a}\bar{a}^T\right)^{-1} = B - \sigma B\bar{a}\bar{a}^T B.$$

So our inequality can finally take the form of

$$(x - z_+)^T B_+^{-1}(x - z_+) \le 1,$$

where

$$z_+ = z - \frac{(1+\alpha)\sigma}{2}B\bar{a} \qquad \text{and} \qquad B_+ = \delta(B - \sigma B\bar{a}\bar{a}^T B).$$

Now we are free to choose any σ to make the formulas as nice as possible. So let's choose

$$\sigma = \frac{2(1+m\alpha)}{(1+m)(1+\alpha)},$$

so that

$$(1-\sigma) = \frac{m-1}{m+1}\cdot\frac{1-\alpha}{1+\alpha} \qquad \text{and} \qquad \delta = \frac{(1-\alpha)^2 m^2}{(m^2-1)}.$$

Then σ lies between 0 and 1 as long as $-1/m \le \alpha \le 1$, so our inequalities are valid. Also, z_+ is given as in the statement of the theorem because of the choice of τ.
This defines an ellipsoid $E_+ = E(z_+, B_+)$ with a volume $\mathrm{Vol}(E_+) = \sqrt{\det B_+}$ so that

$$\frac{\mathrm{Vol}(E_+)}{\mathrm{Vol}(E)} = \sqrt{\frac{\det B_+}{\det B}} = \sqrt{\delta^m(1-\sigma)} = \sqrt{\left(\frac{(1-\alpha^2)m^2}{(m^2-1)}\right)^m \frac{(m-1)}{(m+1)}\frac{(1-\alpha)}{(1+\alpha)}} =$$

$$= (1-\alpha^2)^{\frac{m-1}{2}}(1-\alpha)\left(\frac{m^2}{m^2-1}\right)^{\frac{m-1}{2}}\left(\frac{m}{m+1}\right).$$

There is still one parameter, α, left in the model. Note that if $\alpha = -1/m$ then $\sigma = 0$ and $\delta = 1$, and that gives us exactly the same ellipsoid as the one we started with. On the other hand, for $\alpha = 0$ we find that the ratio of the volumes is

$$\frac{\mathrm{Vol}(E_+)}{\mathrm{Vol}(E)} = \left(\frac{m^2}{m^2-1}\right)^{\frac{m-1}{2}}\left(\frac{m}{m+1}\right) = \left(1 + \frac{1}{m^2-1}\right)^{\frac{m-1}{2}}\left(1 - \frac{1}{m+1}\right) <$$

$$< \left(\exp\frac{1}{m^2-1}\right)^{\frac{m-1}{2}} \exp\frac{-1}{m+1} = \exp\frac{-1}{2(m+1)}.$$

□

Thus we have proved Theorem 2 except for the "minimum volume" part. But even without this we know that in $O(m^2 \ln \frac{R}{r})$ iterations we either find an $x \in Q$ or we know that Q is empty. (For the full gory details, see M. J. Todd, "On minimum volume ellipsoids containing part of a given ellipsoid," *Mathematics of Operations Research* 7 (1982), 253-261.)

However, all this holds only under the assumption that if Q is nonempty, it contains a ball $B(\hat{x}, r)$. This is not always the case in problems we are interested in, but if we restrict ourselves to integer input, we can modify the problem to ensure that this assumption holds.

Recall that we are searching for an feasible solution to the problem $A^T x \leq b$ where $A \in \mathbb{Z}^{m\times n}$. We have already defined, for an integer z, $\text{size}(z) = \lceil \log_2(|z|+1)\rceil + 1$. We can generalize this to say that for a vector $v \in \mathbb{Z}^p$, $\text{size}(v) = \sum \text{size}(v_j)$ and for a matrix $M \in \mathbb{Z}^{p\times q}$ we have that $\text{size}(M) = \sum \text{size}(m_{ij})$. Let's denote $L := \text{size}(A)+ \text{size}(b)$.

Lemma 1 The following is true:

(a) If $v \in \mathbb{Z}^p$ then $||\,v||_\infty \leq ||\,v||_2 \leq ||\,v||_1 \leq 2^{\text{ size }(v)}$.

(b) If $M \in \mathbb{Z}^{p\times q}$, then $|det(M)| \leq 2^{\text{ size }(M)}$.

(c) Any square submatrix of $[A^T, b]$ has an absolute value of determinant bounded by 2^L.

Proof:

(a) Let's denote

$$\beta_j = \begin{cases} 2 & \text{if } |v_j| = 1, \\ |v_j| & \text{otherwise.} \end{cases}$$

Then

$$||\,v||_1 = \sum |v_j| \leq \sum \beta_j \leq \prod_{v_j\neq 0} \beta_j \leq \prod_{v_j \neq 0} 2^{\text{size}(v_j)} = 2^{\sum \text{size}|v_j|} \leq 2^{\text{size}|v|}.$$

The other two inequalities are well known.

(b) We can use Hadamard's inequality (which represents the determinant as the volume of a solid, which is biggest if its edges are orthogonal) to get

$$|\det(M)| \leq \prod_{j=1}^{p} ||\,m_j||_2 \leq \prod_{j=1}^{p} 2^{\text{size}(m_j)} = 2^{\text{size(M)}}.$$

(c) As the size of the submatrix is less than or equal to the size of the matrix itself, we can again use the Hadamard's inequality to prove this part.

□

Lemma 2 If $A^T x \le b$ is feasible, it has a solution x with $||\,x||_\infty \le 2^L$.

Proof: We want to find a basic feasible solution to $A^T x \le b$. There is a possibility that this describes a not-pointed polyhedron, but we can transform it to be pointed. If it contains a line $\{x + \alpha d, \alpha \in \mathbb{R}\}$, we can choose an index j such that $d_j \ne 0$, set $x_j = 0$ and continue. Note that setting $x_j = 0$ in this case keeps feasibility. After a sufficient number of steps we either obtain a pointed polyhedron (that has an extreme point – a BFS we are looking for) or we set all x's to zero and then $x = 0$ is a feasible solution.

Anyhow, we have a basic feasible solution – so it satisfies the system of equations $A^T_{JK} x_J = b_K$ where we have possibly removed some redundant equations to get a square system. Now we know the system can be solved using Cramer's rule, so that $x_j = \frac{\det(M_j)}{\det(M_0)}$ where both matrices M_j and M_0 are submatrices of $[A^T, b]$. Hence for the determinant of M_j we know that $|\det(M_j)| \le 2^L$ while $|\det(M_0)| \ge 1$ because the matrix M_0 is nonsingular and integer-valued. So, together $|x_j| \le 2^L$ and hence $||x||_\infty \le 2^L$. $\square$

So we can replace the system $A^T x \le b$ with

$$\begin{aligned} A^T x &\le b\,, \\ x &\le 2^L e\,, \\ -x &\le 2^L e. \end{aligned}$$

and assume this system is our new $A^T x \le b$. Then all feasible solutions lie in $B(0, \sqrt{m}2^L)$, while the size of the new system is polynomial in L.

Lemma 3 *Suppose the system $A^T x \le b$ has size L. Then:*

(a) If $A^T x \le b$ is infeasible, so is $A^T x \le b + \frac{2^{-L}}{m+2} e$.

(b) If this system is feasible, there is some $\hat{x} \in \mathbb{R}^m$ for which

$$B\left(\hat{x}, \frac{2^{-2L}}{m+2}\right) \subseteq \{x : A^T x \le b + \frac{2^{-L}}{m+2} e\}.$$

Proof:

(a) If $A^T x \le b$ is infeasible, then by the Farkas lemma, the system

$$Ay = 0, \qquad b^T y = -1, \qquad y \ge 0.$$

is feasible. Throwing out the linearly dependent rows of A we can find a BFS $\hat{y}$, that has (by the Cramer's rule argument used in the previous proof) every component bounded by 2^L.

So we have $||\,\hat{y}||_\infty \le 2^L$ and at most $(m+1)$ components of $\hat{y}$ are nonzero. Let's check

$$\left(b + \frac{2^{-L}}{m+2} e\right)^T \hat{y} = b^T \hat{y} + \frac{2^{-L}}{m+2} \sum \hat{y}_i < b^T \hat{y} + 1 = 0,$$

and therefore $\hat{y}$ satisfies the Farkas lemma system

$$A\hat{y} = 0, \qquad \left(b + \frac{2^{-L}}{m+2}e\right)^T \hat{y} < 0, \qquad \hat{y} \geq 0.$$

So the "dual" system $A^T x \leq b + \frac{2^{-L}}{m+2}e$ is infeasible.

(b) Let $\hat{x}$ be feasible in $A^T x \leq b$ and $||\,\hat{x}||_\infty \leq 2^L$. Look at $x \in B\left(\hat{x}, \frac{2^{-2L}}{m+2}\right)$. Let's compute

$$a_i^T x = a_i^T \hat{x} + a_i^T (x - \hat{x}) \leq b_i + ||\,a_i||_2||\,x - \hat{x}||_2 \leq b_i + 2^L \frac{2^{-2L}}{m+2} = b_i + \frac{2^{-L}}{m+2}.$$

Therefore we know that

$$B\left(\hat{x}, \frac{2^{-2L}}{m+2}\right) \subseteq \{x : A^T x \leq b + \frac{2^{-L}}{m+2}e\}.$$

□

So we see that if we add the bounding constraints and perturb the RHS, we obtain a system that satisfies the conditions for the ellipsoid method for

$$r = \frac{2^{-2L}}{m+2} \qquad \text{and} \qquad R = \sqrt{m}2^L + \frac{2^{-2L}}{m+2}.$$

For this r and R we have $\log(R/r)$ polynomial in L.

The last problem that could arise is what if the solution obtained by the ellipsoid method is not feasible in the original problem? In that case we need to adjust the obtained solution - we can start by investigating whether setting $x_1 = 0$ gives us a feasible solution. If so, we add that constraint (or eliminate x_1) and we can continue with the other variables. Eventually we get a feasible system with as many variables set to zero as possible. Then we can look at the inequality constraints and see if there is a feasible solution if we force the first to hold as an equality. If so, we make that an equality and move to the next inequality and so on. Eventually we have set as many variables to zero and as many constraints to be equalities as possible. We take the resulting system of equations and solve it (in polynomial time) to get a feasible solution to the original inequalities. We're finally done!