Mathematical Programming	**Lecture 20**
OR 630 Fall 2005	**November 03, 2005**
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1 Dantzig-Wolfe Decomposition

We want to solve the large-scale LP:

$$\begin{aligned} \min\ & c_1^T x_1 + ... + c_k^T x_k \\ & A_{01}x_1 + ... + A_{0k}x_k = b_0 \\ & A_{11}x_1 \qquad\qquad\quad = b_1 \\ & ... \\ & \qquad\qquad\qquad A_{kk}x_k = b_k \\ & x_1, x_2, ..., x_k \geq 0, \end{aligned} \tag{P}$$

where $x_j \in \mathbb{R}^{n_j}, 1 \leq j \leq k$, $b_0 \in \mathbb{R}^{m_0}$, $b_j \in \mathbb{R}^{m_j}, 1 \leq j \leq k$, and $A_{ij} \in \mathbb{R}^{m_i \times m_j}, i = 0..k, j = 1..k$. Therefore, there are totally $m_0 + \sum m_j$ constraints and $\sum n_j$ variables. This LP is in a Block-angular Form, i.e. in the form of Fig. 1.

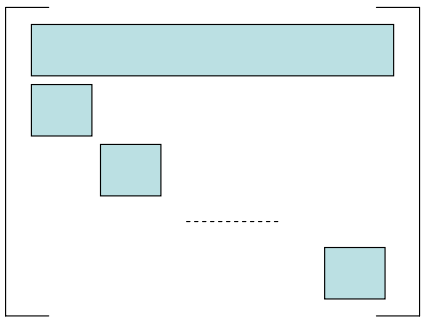

Figure 1: Block-angular Form

Application: a corporation has k divisions:

Division $j\,(1 \leq j \leq k)$ has its own decision variables x_j and its own "local" constraints, $A_{jj}x_j = b_j, x_j \geq 0$. Also, the corporation has its own resources/goals and corresponding linear constraints. The objective is to minimize cost.

Note: we allow $k = 1$, i.e. "only one division." The key is that the part $A_{11}x_1 = b_1, x_1 \geq 0$ of the problem should be easier to deal with (e.g. network flow).

Note: (P) is just

$$\begin{aligned} \min\ & c_1^T x_1 + ... + c_k^T x_k \\ & A_{01}x_1 + ... + A_{0k}x_k = b_0 \\ & x_1 \in Q_1, x_2 \in Q_2, ..., x_k \in Q_k \end{aligned}$$

where Q_j is the polyhedron $\{x_j \in \mathbb{R}^{n_j} : A_{jj}x_j = b_j, x_j \geq 0\}$, which is assumed to be nonempty for all j otherwise the problem is infeasible.

Now, we use the *representation theorem* (Thm 2 in notes 9/6, Thm 1 of 9/8 or recitation notes III of 9/14):

$$Q_j = \{x_j = \sum_{h=1}^{N_j} \lambda_{jh} v_{jh} + \sum_{i=1}^{R_j} \mu_{ji} d_{ji} : \lambda_{jh} \geq 0 \text{ all } h, \sum_h \lambda_{jh} = 1, \mu_{jh} \geq 0, \text{ all } i\}$$

where $v_{jh}, h = 1, .., N_j$, are all the extreme points of Q_j and $d_{ji}, i = 1, .., R_j$, are all the extreme rays of Q_j. Here R_j can be 0, if Q_j is bounded.

So we can substitute for each x_j in (P) to get the following Master Problem:

$$\begin{aligned} \min\ & \sum_{j=1}^{k} \left(\sum_h (c_j^T v_{jh})\lambda_{jh} + \sum_i (c_j^T d_{ji})\mu_{ji} \right) \\ & \sum_{j=1}^{k} \left(\sum_h (A_{0j} v_{jh})\lambda_{jh} + \sum_i (A_{0j} d_{ji})\mu_{ji} \right) = b_0 \qquad (MP) \\ & \sum_h \lambda_{jh} = 1, \ j = 1, 2, ..., k \\ & \lambda_{jh}, \mu_{ji} \geq 0, \text{ all } j, h, i. \end{aligned}$$

(P) has $m_0 + \sum_1^k m_j$ rows and $\sum_1^k n_j$ variables.
(MP) has $m_0 + k$ rows and $\sum_1^k (N_j + R_j)$ variables.

We want to solve (MP) using the revised simplex method and column generation.

Proposition 1 *(P) and (MP) have the same optimal value (possibly $-\infty$ or $+\infty$) and every feasible solution of (P) corresponds to a feasible solution of (MP) with the same objective function value and vice-versa.*

Proof: Immediate from representation theorem. □
Important Note: The correspondence is NOT 1-1.

How can we apply the revised simplex method to (MP)? We need an initial basic feasible solution and a way to generate new columns as needed.

For the initial solution, we can solve, say:

$$\min c_j^T x_j, x_j \in Q_j,$$

for each j. If infeasible, quit; otherwise we generate a vertex, say v_{j1} (either optimal or adjacent to an unbounded ray).

Compute the corresponding column $\begin{pmatrix} A_{0j}v_{j1} \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix}$ in (MP) and introduce artificial variables for the first m_0 constraints and solve the phase I problem, again by column generation.

So, suppose we now have a basic feasible solution to (MP), involving some λ_{jh}'s and μ_{ji}'s. We also have a corresponding dual solution $\bar{y} = \begin{pmatrix} \bar{y}_0 \\ \bar{z} \end{pmatrix}$ where $\bar{y}_0 \in \mathbb{R}^{m_0}, \bar{z} \in \mathbb{R}^k$. We are optimal if all the reduced costs of variables λ_{jh} and μ_{ji} are nonnegative.

Look at the reduced cost of λ_{jh}: it is

$$(c_j^T v_{jh}) - (A_{0j}v_{jh})^T \bar{y}_0 - \bar{z}_j = (c_j - A_{0j}^T \bar{y}_0)^T v_{jh} - \bar{z}_j \geq 0(?)$$

We can check this by solving

$$\begin{aligned} \min\ & (c_j - A_{0j}^T \bar{y}_0)^T x_j \\ & A_{jj}x_j = b_j \qquad\qquad (SP_j) \\ & x_j \geq 0. \end{aligned}$$

(a) If the optimal value is $\geq \bar{z}_j$, then reduced cost of each λ_{jh} is ≥ 0.

(b) If the optimal value is $< \bar{z}_j$, then λ_{jh}, where v_{jh} is an optimal solution, has negative reduced cost in (MP), and we can calculate its column $\begin{pmatrix} A_{0j}v_{jh} \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix}$ with cost $c_j^T v_{jh}$ in (MP).

So we can continue the simplex method.

(c) Suppose (SP_j) is unbounded, then we have found an extreme ray d_{ji} with $(c_j - A_{0j}\bar{y}_0)^T d_{ji} < 0$. So we compute its column and enter it into the basis.

Note: $(c_j - A_{0j}\bar{y}_0)^T d_{ji} = (c_j^T d_{ji}) - (A_{0j} d_{ji})^T \bar{y}_0$ is the reduced cost of μ_{ji}, therefore all reduced costs of μ_{ji}'s in (a) and (b) are ≥ 0 (because the subproblem is bounded).

Note: Each (SP_j) can be interpreted as a divisional problem, where the costs c_j are modified by $A_{0j}^T \bar{y}_0$, which can be thought of division j's contribution to meeting corporate goals.

An example of a problem in $\mathbb{R}^3$ is illustrated in Fig. 2. Q is the 3-dimensional polytope shown, while (P) has two additional equality constraints, defining the line cutting through Q. So the feasible region of (P) is the line segment consisting of the part of the line intersecting the polytope. Here are some comments on this example.

1. (MP) problem could have many optimal solutions, although the corresponding (P) only has one optimal.The optimal solution indicated in the figure can be written as a convex combination of extreme points a, c, e or as a convex combination of a, d, e.

2. For the final $\bar{y}_0$, $c_j - A_{0j}^T \bar{y}_0$ is normal to the top face of Q, so all its vertices are optimal in the final subproblem.

3. The simplex iterations for (P) update its basic feasible solution in the feasible region in the figure, so there is at most one iteration after Phase I is done. However problem (MP) updates its basic feasible solution by changing extreme points which in fact have a convex combination in the feasible region in the figure, e.g. extreme points a, c, f give a convex combination in the feasible region. Hence there are many more basic feasible solutions to (MP) than to (P) here.

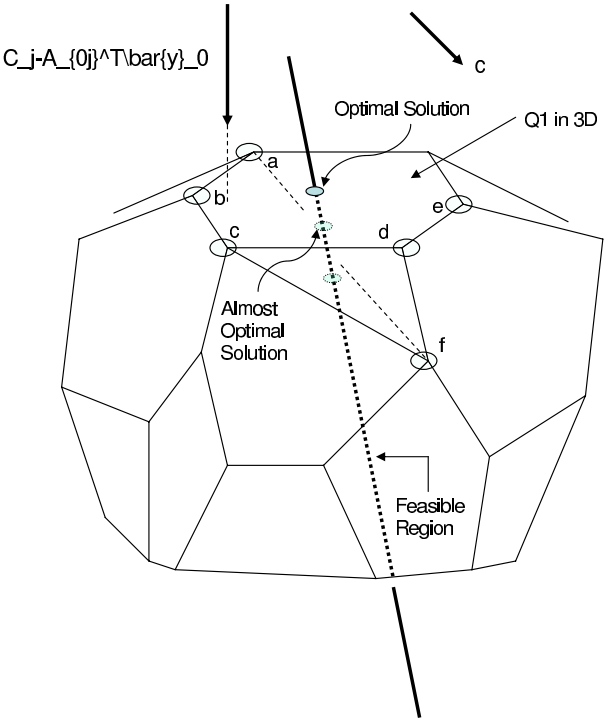

Figure 2: An example