Mathematical Programming	**Lecture 15**
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1 Sensitivity Analysis (cont'd)

1.1 Case (a): Changes in c (cont'd)

Last time we considered changes in c. We considered two cases: changes in a basic c_j and changes in a non-basic c_j. In addition to remarks 1 and 2 at the end of the last lecture we have the following remark:

Remark 3 *As long as $\bar{y}$ remains feasible, the corresponding change in the optimal $\bar{\zeta}$ is 0 for case (i) (the non-basic case) and $\delta\bar{x}_j$ in case (ii) where $c_j \leftarrow c_j + \delta$.*

1.2 Case (b): Changes in $\bar{b}$

Since B and c_B remain unchanged, so does $\bar{y}$. However $\bar{x}_B = B^{-1}b$ does change. The case where $\bar{x}_B$ remains non-negative can be easily handled: say b_i changes to $b_i + \delta$:

$$b \leftarrow b + \delta e_i.$$

Then

$$\bar{x}_B \leftarrow B^{-1}b + \delta B^{-1}e_i.$$

The basis B remains optimal basis (with adjusted $\bar{x}$) as long as

$$B^{-1}b + \delta B^{-1}e_i \geq 0.$$

When $(B^{-1}e_i)_p \geq 0$, we can derive from this condition a lower bound on δ, and when $(B^{-1}e_i)_p < 0$, this condition yields an upper on δ. Overall, we get an allowed range for δ which includes 0.

Remark 4

1. *Note that this is not symmetric with case (a) because (P) and (D) are not completely symmetric. But, if the i'th constraint in (P) was initially an inequality, we could add a slack or a surplus variable, so we have $\pm e_i$ as a column of A. If this column is also in B, then $B^{-1}e_i$ is some $\pm e_p$, and then we get a single lower or upper bound on δ, symmetric to case (a).*

2. *Suppose that the change in b_i is such that $\bar{x}_B$ does remain feasible and hence optimal (recall that $\bar{y}$ does not change). Then $\bar{\zeta}$ becomes:*

$$\begin{aligned}\bar{\zeta}_{new} &= c_B^T\left(B^{-1}b + \delta B^{-1}e_i\right)\\ &= \bar{\zeta}_{old} + \delta\bar{y}_i.\end{aligned}$$

(Compare with Remark 3 above.) Equivalently,

$$\bar{\zeta}_{new} = b_{new}^T \bar{y} = (b + \delta e_i)\, \bar{y}.$$

Therefore, as long as the change is small enough (so that the new $\bar{x}$ remains feasible), then $\bar{y}_i$ measures the rate of change in the objective function as a function of b_i. This is a kind of shadow price *of the i'th constraint or a* marginal price*. Note that if the current optimal solution $\bar{x}$ is non-degenerate ($B^{-1}b > 0$), then the range of δ includes 0 in its interior, and this marginal value is accurate for all sufficiently small changes in b_i.*

3. *We can perform a similar analysis if b changes to $b + \delta\hat{b}$, $\hat{b} \in \mathcal{R}^m$, $\delta \in \mathcal{R}$.*

4. *Suppose that the changes in b are large enough so that the new $\bar{x}_B$ has at least one negative component: we want to re-optimize but we don't have a basic feasible solution anymore.*

1.3 Case (c): Changes in A

(i) Change in a column a_j:

- (i.1) Change in a non-basic $a_j \longrightarrow$ dealt with in case (a)(i). $\bar{x}$ stays the same and if $\bar{y}$ is not feasible in the changed constraint, continue the algorithm.
- (i.2) Change in a basic a_j: we have a change in a column of B, so both $\bar{x}$ and $\bar{y}$ change. This change is similar to the way we change B in the simplex iteration: this is a rank-one change in B. So,
  - (i.2.1) Check if the new B is non-singular (the entry of $B^{-1}a_{j,\text{new}}$ corresponding to the position of $a_{j,\text{old}}$ in the basis has to be non-zero).
  - (i.2.2) If so, compute the new B^{-1} (using the rank-one formula). Hence, $\bar{x}$ and $\bar{y}$ can be computed.

(ii) Change in a row of A:
This is also a rank-one change - change of a row of B. This can be analyzed as before: if the new B is non-singular, we can calculate $\bar{x}$ and $\bar{y}$, and re-optimize if we get a feasible $\bar{x}$ but infeasible $\bar{y}$ as before.

2 Reoptimization

However, what if we get a basic but not feasible primal solution $\bar{x}$ (due to changes in A or b)?

A) Default Method:
Suppose for example we have changed a_j to $\tilde{a}_j$, $j \in \beta$ and changing a_j to $\tilde{a}_j$ in B makes it singular. Then, consider

$$\begin{aligned} &\min x_j \\ \sum_{i \neq j} a_i x_i + a_j x_j + \tilde{a}_j \tilde{x}_j &= b \end{aligned}$$

$$\begin{aligned} x &\geq 0 \\ \tilde{x}_j &\geq 0 \end{aligned}$$

Now, solve this as a phase I problem (where the old x_j is now an artificial variable), starting with the current basic feasible solution, and when $x_j = 0$ switch to the phase II problem as in the usual phase I - phase II technique.

B) Dual Simplex Algorithm:
Consider a case where the new $\bar{x}$ is not feasible but the new $\bar{y}$ is still feasible (e.g., too big a change in just b, as in part 4 of Remark 4 above).
Recall that in the primal simplex algorithm:

- $\bar{x}$ is feasible.
- $\bar{y}$ is infeasible until termination (since one or more reduced costs remain negative).
- We always have complementary slackness.

Now we have:

- $\bar{x}$ is infeasible.
- $\bar{y}$ is feasible.
- Complementary slackness.

$\Rightarrow$ We should switch to a *Dual Simplex Algorithm.*

In general, we could apply the "usual" simplex method to the dual problem starting with $\bar{y}$. But this problem is not in standard form, and putting it in standard form is awkward for finding an initial basis matrix and its inverse. We will try to simulate this method while keeping "primal" quantities like B^{-1}. We can write the primal problem as

$$(P) \qquad \begin{aligned} &\min_x c_B^T x_B + c_N^T x_N \\ &\quad Bx_B + Nx_N \; = \; b \\ &\quad x_B, x_N \geq 0, \end{aligned}$$

with current solution : $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$. Its dual can be written

$$(D) \qquad \begin{aligned} &\max_y b^T y \\ &\quad B^T y \; \leq \; c_B \\ &\quad N^T y \; \leq \; c_N, \end{aligned}$$

with current solution $\bar{y} = B^{-T} c_B$ satisfying $B^T \bar{y} = c_B$, $N^T \bar{y} \leq c_N$.

To move, we relax some constraint indexed by $j \in \beta$ in (D). The order of operations has changed compared to the original method: in the dual simplex algorithm we first choose a $j \in \beta$ to take out and then figure out which $l \in \nu$ we bring in. In the primal simplex algorithm we first choose $l \in \nu$ to bring in and then choose which $j \in \beta$ leaves.

• Which $j \in \beta$ should we remove? Which constraint (say the i'th) shall we relax?
If we relax the i'th constraint, it is equivalent to moving to a new $\bar{y}$ with $B^T y_{\text{new}} = c_B - \delta e_i$: Let

$$y_{\text{new}} = B^{-T} c_B - \delta B^{-T} e_i = \bar{y} - \delta B^{-T} e_i,$$

and choose i such that $b^T y_{\text{new}} > b^T \bar{y}$ for positive δ, i.e., $b^T B^{-T} e_i < 0$.
This implies that $\left(B^{-1} b\right)_i < 0 \Rightarrow$ we're looking for infeasibility of the *primal* (in the original method we looked for infeasibility of the dual). Hence we get
Dual Simplex Criteria for Leaving Variable: Choose some $k \in \beta$ with x_k the p'th basic variable such that $\left(B^{-1} b\right)_p < 0$ (cf. choose q with $\bar{c}_q < 0$).
Now we move as far as possible so we remain feasible.
$B^T y_{\text{new}} \le c_B \Rightarrow$ all the components indexed by $j \in \beta$ except for the k'th remain satisfied with equality. The k'th component is reducing – so we're o.k. Also, we require $N^T y_{\text{new}} \le c_N$, so we want

$$N^T \left(B^{-T} c_B - \delta B^{-T} e_p\right) \le c_N,$$

for the maximum possible δ, i.e.,

$$-\delta N^T B^{-T} e_p \le c_N - N^T \bar{y} = \bar{c}_N.$$

Consider now $a_j^T \left(B^{-T} e_p\right)$: if this is positive, we do not need to worry: δ can be as large as we wish. Thus we choose $j \in \nu$ to go into β by considering all j's with

$$a_j^T \left(B^{-T} e_p\right) = e_p^T \left(B^{-1} a_j\right) = e_p^T \bar{a}_j = \bar{a}_{pj} < 0,$$

and choosing $q \in \nu$ with $a_q^T B^{-T} e_p < 0$ and

$$\frac{\bar{c}_q}{-a_q^T B^{-T} e_p} = \min \left\{ \frac{\bar{c}_j}{-a_j^T B^{-T} e_p} : j \in \nu, \;\; a_j^T B^{-T} e_p < 0 \right\}.$$

This is the Dual Simplex Criterion for the Entering Variable.
The quantity above is the most we can move $\bar{y}$ in the direction $-B^{-T} e_p$.
Then to perform the iteration we make x_q basic in place of the current pth basic variable x_k.
• What if $a_j^T B^{-T} e_p \ge 0$ for all $j \in \nu$?
Then we can move $\bar{y}$ as far as we want in direction $-B^{-T} e_p$, and $b^T y \to +\infty$, so (D) is unbounded and (P) is infeasible (we can use $\bar{y} = -B^{-T} e_p$ to show that (P) is infeasible by the Farkas lemma).