Mathematical Programming	**Lecture 14**
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1 Least-index Rule

Least-index Rule: At each iteration, choose as entering variable x_j where x_j is eligible to enter the basis and j is minimal; choose to leave the basis the ith basic variable x_j which is eligible to leave the basis ($\bar{a}_{iq} > 0$ and minimum ratio test candidate) and has j minimal.

Theorem 1 *(R.G. Bland) The simplex method with the* **Least-index Rule** *terminates in a finite number of iterations.*

Proof: By contradiction. If the Least-index Rule fails to terminate, it leads to a cycle of degenerate pivots. Say the sequence of sets of basic indices is

$$\beta_0, \beta_1, ..., \beta_h, ..., \beta_l, \beta_h, ..., \beta_l, ...$$

where $\beta_h, ..., \beta_l$ is a cycle of degenerate pivots. In this cycle of pivots, the objective function is staying constant and all variables keep the same value.

Consider the set of all indices j such that j is in some β_k, $h \leq k \leq l$, and j is <u>not</u> in some β_i, $h \leq i \leq l$. Call these <u>fickle</u>. These variables are always zero in the cycle. Let t be the largest fickle index. It leaves the basis at some iteration, say the <u>ith</u>, and re-enters at some other iteration, say the <u>kth</u>.

Consider first the ith iteration, when x_t leaves the basis. Say x_s enters the basis (s is fickle, so $s < t$). Consider $\bar{c}$, the reduced costs at iteration i and d, the direction of "movement." It follows that

$$\begin{array}{ll} \bar{c}_s < 0 & (x_s \text{ enters}), \\ d = \begin{pmatrix} -\bar{a}_s \\ e_s \end{pmatrix} & \begin{array}{l} \leftarrow \text{basic variables} \\ \leftarrow \text{nonbasic variables} \end{array} \end{array}.$$

Then,

$d_t < 0$ ($\bar{a}_{ps} > 0$, if x_t is the pth basic variable, because it is eligible to leave the basis),
$d_s = 1 > 0$,
$d_j \geq 0$ if $j \in \beta_i$, and j is fickle (would have chosen j if $d_j < 0$; j not chosen but $j < t$),
$d_j =??$ if $j \in \beta_i$ and j not fickle,
$d_j = 0$ if $j \notin \beta_i$ and $j \neq s$.

Finally, $Ad = 0$, thus, $d \in \mathcal{N}(A)$.

Next consider β_k, where x_t comes back into the basis. Let $\hat{c}$ be the vector of reduced costs of this iteration. Then

$$\hat{c} = c - A^T\hat{y}, \text{ some } \hat{y}$$

and

$$\bar{c} = c - A^T\bar{y}, \text{ some } \bar{y}.$$

Define,

$$\tilde{c} = \hat{c} - \bar{c} = A^T(\bar{y} - \hat{y}),$$

so

$$\tilde{c} \in \mathcal{R}(A^T) = (\mathcal{N}(A))^{\perp}.$$

Then,

$$\begin{array}{ll}
\tilde{c}_t = \hat{c}_t - \bar{c}_t < 0 & (\hat{c}_t \geq 0 \text{ because } t \text{ chosen and } \bar{c}_t = 0), \\
\tilde{c}_s = \hat{c}_s - \bar{c}_s > 0 & (\hat{c}_s \geq 0 \text{ because } x_s \text{ either basic or not chosen over } x_t), \\
\tilde{c}_j = \hat{c}_j - \bar{c}_j \geq 0 & \text{if } j \in \beta_i \text{ and fickle} \quad (\hat{c}_j \geq 0 \text{ for same reason as } \hat{c}_s;\ \bar{c}_j = 0), \\
\tilde{c}_j = \hat{c}_j - \bar{c}_j = 0 & \text{if } j \in \beta_i \text{ and not fickle}, \\
\tilde{c}_j = ?? & \text{if } j \notin \beta_i \text{ and } j \neq s.
\end{array}$$

But then

$$0 = \tilde{c}^T d = \tilde{c}_t d_t + \tilde{c}_s d_s + \sum_{j \in \beta_i, j \text{ fickle}} \tilde{c}_j d_j + \sum_{j \in \beta_i, j \text{ not fickle}} \tilde{c}_j d_j + \sum_{j \notin \beta_i, j \neq s} \tilde{c}_j d_j > 0$$

which is a contradiction. $\square$

Since the simplex method (with appropriate pivot rules) terminates finitely, we can use it as a constructive proof technique to show

- If A has rank m, then if (P) has a feasible solution, it has a basic feasible solution;
- Strong duality; and
- the Farkas Lemma.

2 Sensitivity Analysis

The simplex method generate an optimal solution to (P)

$$\begin{array}{ll}
\min & c^T x \\
 & Ax = b, \\
 & x \geq 0,
\end{array}$$

and its dual. Then $\bar{x}, \bar{y}$ and $\bar{\zeta}$ are optimal for the problem given by data (A, b, c), but clearly depend on the data. **Sensitivity** or **post-optimality** analysis asks how (or if) $\bar{x}, \bar{y}$, and $\bar{\zeta}$ change if some elements of the data change.

Key Idea: If we can construct feasible primal and dual solutions satisfying complementary slackness conditions, then they must be optimal!

Case:

(a) Change one component of c; then $\bar{x}$ is still a feasible primal solution.

(i) Change c_j, j nonbasic.

Then c_B unchanged so $\bar{y} = B^{-T} c_B$ is unchanged. Our only concern is the feasibility of $\bar{y}$, i.e., the feasibility of $\bar{y}$ in the jth constraint. So the old $\bar{x}$ and $\bar{y}$ remain optimal as long as new $c_j \geq a_j^T \bar{y}$ (i.e. new $\bar{c}_j \geq 0$). Note: even if a_j also changes, still optimal as long as (new c_j)$\geq$(new a_j)$^T \bar{y}$.

(ii) Change c_j, $j \in \beta$, e.g., $c_j \leftarrow c_j + \delta$. Say x_j is the ith basic variable.

Then $c_B \leftarrow c_B + \delta e_i$, and $\bar{y} \leftarrow B^{-T}(c_B + \delta e_i) = \bar{y} + \delta B^{-T} e_i$. This $\bar{y}$ still satisfies complementary slackness, but have to check feasibility, i.e., check $a_k^T \bar{y} + \delta(a_k^T B^{-T} e_i) \leq c_k$ for $k \in \nu$. For each k with $a_k^T(B^{-T} e_i) > 0$, get an upper bound on δ, and for each k with $a_k^T(B^{-T} e_i) < 0$, get a lower bound on δ. Thus, get the range for δ (including 0). Thus, get range for c_j (including its nominal value). Hence $\bar{x}$ and the adjusted $\bar{y}$ remain optimal as long as c_j varies within this range.

Remark 1 *Consider more general changes, e.g., $c \leftarrow c + \delta \hat{c}$ (parametric analysis). The analysis is the same as in case (a)(ii).*

Remark 2 *If in any subcase, we make a larger than permitted change, then some reduced cost becomes negative, but $\bar{x}$ is still a primal basic feasible solution, so can continue applying the simplex method from here (reoptimization).*

Remark 3 *In case (i), $\bar{\zeta}$ doesn't change, while in case (ii), it increases by $\delta \bar{x}_j$ as expected, with both cases subject to $\bar{x}$ remaining optimal.*