Mathematical Programming	**Lecture 12**
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Today, we will go on discussing the simplex method. We still have some unfinished business to take care of as follows:

- Getting an initial basic feasible solution;
- Implementation;
- Finite termination (related to pivot rules).

We will analyze the first problem and a bit about the second one.

The LP problem in standard form is:

$$(P) \qquad \begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b, \\ & x & \geq & 0. \end{array}$$

First problem: Obtaining an initial basic feasible solution.

1. Suppose the initial problem had inequality constraints: $Ax \leq b, x \geq 0$, so we add slack variables to get
$$\begin{array}{rcl} Ax + Is & = & b, \\ x & \geq & 0, \\ s & \geq & 0. \end{array}$$
If $b \geq 0$, we can choose $B = I$, so $B^{-1}b = b \geq 0$. This is the *all-slack basis*. Similarly if the initial formulation is $Ax \geq b$, $x \geq 0$ and $b \leq 0$, then we can do the same with surplus variables.

2. Suppose the initial formulation had constraints $Ax \geq b$, $x \geq 0$. Suppose some component of b is positive, but A has a column that is positive. Subtract surplus variables to get
$$\begin{array}{rcl} Ax - It & = & b, \\ x & \geq & 0, \\ t & \geq & 0. \end{array}$$
Make all surplus variables basic, so the basis is $-I$ and we get $-Ax + It = -b$. This is a basic solution but not feasible. Choose x_q to become basic, where $a_q > 0$. Then $\bar{a}_q < 0$. Now let
$$x_q = \frac{b_p}{a_{pq}} = \max_i \{ \frac{b_i}{a_{iq}} : i = 1, 2, \ldots, m \},$$
i.e., increase x_q until the last t_i hits zero, say t_p. Then make x_q basic and t_p nonbasic, and you have a basic feasible solution.

3. General case: attack the feasibility problem as an optimization problem by constructing an artificial problem.

   (a) Add slack/surplus variables to get the problem into the standard form (P).

   (b) Multiply rows with negative b_is by -1, so the RHS (right hand side) becomes nonnegative. Choose
   $$S = \mathrm{Diag}(\pm 1, \pm 1, \ldots, \pm 1) \in \Re^{m \times m}$$
   so that $Sb \geq 0$ and work with $SAx = Sb$.

   (c) Add an artificial variable z_i to each row to get $SAx + Iz = Sb$ where $x \geq 0, z \geq 0$. *Note: If the ith row has a suitable slack/surplus variables, we do not need z_i.*

   (d) Solve the optimization problem (with $e = (1, 1, \ldots, 1)^T$):
   $$\begin{array}{lrcrcl} \min_{x,z} \quad \omega = & & & e^T z & & \\ & SAx & + & Iz & = & Sb, \\ & x & & & \geq & 0, \\ & & & z & \geq & 0, \end{array}$$
   called the **Phase I problem**, by the simplex method starting with the initial basis I corresponding to all the artificial variables. The **Phase I problem** is feasible and bounded, so it has an optimal solution. Reach an optimal basic feasible solution to this problem.

      i. The optimal value is positive. Then the original problem (P) is infeasible. Find a certificate of this infeasibility as follows: We have an optimal dual solution, say $\tilde{y}$, to the **Phase I problem**, so
      $$\begin{array}{rcl} (SA)^T \tilde{y} & \leq & 0, \\ I\tilde{y} & \leq & e, \\ (Sb)^T \tilde{y} & > & 0. \end{array}$$
      Then $y = S^T \tilde{y}$ satisfies
      $$\begin{array}{rcl} A^T y & \leq & 0, \\ b^T y & > & 0, \end{array}$$
      which shows by the Farkas Lemma that (P) is infeasible.

      ii. The optimal value is zero, and all the artificial variables are nonbasic. Then, the current basic feasible solution (after eliminating the artificial variables) is also a basic feasible solution to the original problem (P) (and has basis B instead of SB, so it's easy to get B^{-1}). Then solve the original problem (now called the **Phase II problem**) from this initial basic feasible solution.

      iii. The optimal value is zero, but one or more artificial variables are still basic (at degenerate level zero). Consider each basic z_i in turn. If z_i is the pth basic variable, the current equation p expresses z_i in terms of the nonbasic variables. If any $(e_p^T B^{-1}) a_j = (B^{-T} e_p)^T a_j$ is nonzero, we can make x_j basic instead of

z_i (at level 0), and can proceed. If all these coefficients are zero, then the pth updated equation is something like $z_i - \sum_{h=i} \lambda_h z_h = 0$. So

$$(SAx - Sb)_i - \Sigma_{i \neq h} \lambda_h (SAx - Sb)_h \equiv 0.$$

This means that the original equation i is a linear combination of the remaining equations in $Ax = b$, so we can eliminate it. (This is how we discover and deal with the case that rank$(A) < m$.) Then eliminate row p and variable z_i, and the original equation i. So eliminate the pth row and ith column of B^{-1} to get the new basis inverse. For example, if $p = i = m$, we get

$$B = \begin{bmatrix} Q & 0 \\ \cdots & 1 \end{bmatrix} \Longrightarrow B^{-1} = \begin{bmatrix} Q^{-1} & 0 \\ \cdots & 1 \end{bmatrix}.$$

Second problem: Implementation.

We need an efficient way to update quantities when we move from one basic feasible solution to the next. The key is the basis inverse: how can we update B^{-1}?

Recall,

$$B_+ = \begin{array}{c} \begin{array}{ccc} & p\text{th} & \end{array} \\ \begin{bmatrix} \vdots & a_q & \vdots \end{bmatrix} \end{array} = BE_p = B \begin{array}{c} \begin{array}{ccccccc} &&& p\text{th} &&& \end{array} \\ \left[\begin{array}{ccccccc} 1 & & & \bar{a}_{1q} & & & \\ & \ddots & & \bar{a}_{2q} & & 0 & \\ & & 1 & \vdots & & & \\ & & & \bar{a}_{pq} & & & \\ & & & \bar{a}_{p+1,q} & 1 & & \\ & 0 & & \vdots & & \ddots & \\ & & & \bar{a}_{mq} & & & 1 \end{array} \right] \end{array}.$$

So,

$$B_+^{-1} = E_p^{-1} B^{-1} = \begin{array}{c} \begin{array}{ccccccc} &&& p\text{th} &&& \end{array} \\ \left[\begin{array}{ccccccc} 1 & & & -\frac{\bar{a}_{1q}}{\bar{a}_{pq}} & & & \\ & \ddots & & -\frac{\bar{a}_{2q}}{\bar{a}_{pq}} & & 0 & \\ & & 1 & \vdots & & & \\ & & & \frac{1}{\bar{a}_{pq}} & & & \\ & & & -\frac{\bar{a}_{p+1,q}}{\bar{a}_{pq}} & 1 & & \\ & 0 & & \vdots & & \ddots & \\ & & & -\frac{\bar{a}_{mq}}{\bar{a}_{pq}} & & & 1 \end{array} \right] \end{array} B^{-1}.$$

This takes $O(m^2)$ time to update. Now, we want a clean simple way to deal with elementary (E_p) matrices and corresponding basis changes.

Key: rank-one update

$$E_p = I + \underbrace{(\bar{a}_q - e_p) e_p^T}_{\text{rank}-\text{one matrix}}.$$

Here, $e_p = \begin{array}{c} \\ \left[\begin{array}{ccccccc} 0 & \cdots & 0 & \overset{p\text{th}}{1} & 0 & \cdots & 0 \end{array}\right]^T \end{array}$. And we should notice that $u^T v$ is an inner product and hence a number, while uv^T is an outer product and hence an $m \times m$ matrix. Similarly, $B_+ = B + \underbrace{(\bar{a}_q - Be_p)e_p^T}_{\text{rank-one modification}}$.

Next time: If M is nonsingular, u,v are vectors, is $M + uv^T$ nonsingular? and if so, what is its inverse?

Here are some references for the material we have been discussing (and will soon discuss):

For Dantzig's column geometry: Section 3.6 of Bertsimas-Tsitsiklis.
For a discussion of the edge-following interpretation of the simplex method and its discussion by Fourier in the 1820s (with a translation of Fourier's description): Geometric Interpretation of the Simplex Method in Chapter 17 of Chvátal.
For initialization: Section 3.5 of Bertsimas-Tsitsiklis and the sections Initialization in Chapter 3 and The Two-Phase Simplex Method in Chapter 8 in Chvátal.
For implementation: Section 3.3 of Bertsimas-Tsitsiklis and the section Eta Factorization of the Basis and Chapter 24 (for triangular factorizations) in Chvátal.
For finite termination: Section 3.4 of Bertsimas-Tsitsiklis and the section Termination: Cycling in Chapter 3 of Chvátal.