Mathematical Programming	**Lecture 11**
OR 630 Fall 2005	**September 29, 2005**
Notes by Serkan Kirac	

Suppose that x_q comes into the basis at the level

$\bar{\alpha} = \frac{\bar{b}_p}{\bar{a}_{pq}} = \min\left[\frac{\bar{b}_i}{\bar{a}_{iq}} : i = 1,..,m \text{ and } \bar{a}_{iq} > 0\right]$ replacing the pth variable; then the new basis matrix is

$$B_+ = \left[\begin{array}{c|c|c} \dots & . & \dots \\ \text{old columns} & a_{pq} & \text{old columns} \\ \dots & . & \dots \end{array}\right] = B \begin{bmatrix} 1 & 0 & \dots & \bar{a}_{1q} & \dots & 0 \\ 0 & 1 & \dots & \bar{a}_{2q} & \dots & 0 \\ . & . & \dots & . & \dots & 0 \\ . & . & \dots & \bar{a}_{pq} & \dots & . \\ . & . & \dots & . & \dots & 0 \\ 0 & . & \dots & \bar{a}_{mq} & \dots & 1 \end{bmatrix}.$$

Also the new objective function value comes from $\zeta = \bar{\zeta} + \bar{e}_q\alpha = \bar{\zeta} + \frac{\bar{b}_p\bar{c}_q}{\bar{a}_{pq}}$.

Theorem 3 ***(Improvement of basic feasible solutions)*** *Suppose $\bar{x}$ is the basic feasible solution corresponding to the basis matrix B, basic indices β, and nonbasic indices ν. Suppose $\bar{y} = B^{-T}c_B$ and $\bar{c}_q = c_q - \bar{a}_q^T\bar{y} < 0$ for some $q \in \nu$. Then, with $\bar{a}_q = B^{-1}a_q$ and $\bar{b} = B^{-1}b$, suppose $\bar{a}_{pq} > 0$ and $p = argmin\{\frac{\bar{b}_i}{\bar{a}_{iq}} : i = 1,..,m \text{ s.t. } \bar{a}_{iq} > 0\}$. Then*

$$\bar{x}_+ = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix} + \frac{\bar{b}_p}{\bar{a}_{pq}}\begin{pmatrix} -\bar{a}_q \\ e_p \end{pmatrix}$$

is a basic feasible solution corresponding to the basic indices where q replaces the pth entry of β and the basis matrix where a_q replaces the pth column of B. Moreover, if $\bar{\zeta} = c^T\bar{x}$, $\bar{\zeta}_+ = c^T\bar{x}_+$ then $\bar{\zeta}_+ = \bar{\zeta} + \frac{\bar{b}_q\bar{c}_q}{\bar{a}_{pq}} \leq \bar{\zeta}$ with strict inequality if $\bar{b}_q > 0$.

These three theorems indicate a schema for a class of algorithms, known collectively as the (primal) simplex algorithm.

Simplex Algorithm

Step 0 Find an initial basic feasible solution $\bar{x}$ to (P) corresponding to basic indices β, nonbasic indices ν and basis matrix B.

Step 1 (Check for optimality) Compute $\bar{y} = B^{-T}c_B$. If $\bar{c}_j = c_j - a_j^T\bar{y} \geq 0$ for all $j \in \nu$, then $\bar{x}$ is an optimal solution. **STOP.**

Step 2 (Choosing entering variable) Otherwise, choose $q \in \nu$ with $\bar{c}_q < 0$.

Step 3 (Checking unboundedness) Compute $\bar{a}_q = B^{-1}a_q$. If $\bar{a}_q \leq 0$ then the objective function is unbounded below on the feasible ray

$$x(\alpha) = \begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -\bar{a}_q \\ e_q \end{pmatrix}, \ \alpha \geq 0;$$

then **STOP.**

Step 4 (Choosing leaving variable) Otherwise choose p with $\bar{a}_{pq} > 0$ and $\frac{\bar{b}_p}{\bar{a}_{pq}} = \min \left[\frac{\bar{b}_i}{\bar{a}_{iq}} : i = 1, .., m \ \text{ and } \ \bar{a}_{iq} > 0 \right]$ where $\bar{b} = B^{-1}b$.

Step 5 (Update) Move to the new basic feasible solution

$$\bar{x}_+ = \begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} + \frac{\bar{b}_p}{\bar{a}_{pq}} \begin{pmatrix} -\bar{a}_q \\ e_q \end{pmatrix}.$$

Update and go back to **Step1**.

Steps 0 and 5 will be covered in the next lecture.
In the algorithm, there are a lot of questions to deal with, but first of all, does it *"converge finitely"*? The other questions are: how to find an initial basic feasible solution; in Step 2, how to choose $q \in \nu$; and in Step 3, how to choose p such that $\bar{a}_{pq} > 0$ and the minimum ratio condition holds, if there is a choice. Also, we need to avoid cycles in the simplex method.

Definition 1 *A basic solution $\bar{x}$ to (P) is a degenerate solution if one of the basic variables takes the value* 0. *Otherwise it is nondegenerate.*

Remark 1 *If there is a basic degenerate solution to (P), b must be in one of finitely many hyperplanes in $\mathbf{R}^m$. So "general" $b \rightarrow$ no degenerate basic solution.*

Theorem 4 ***(Finite convergence under nondegeneracy)*** *Suppose (P) has no degenerate basic feasible solution and we can find an initial basic feasible solution. Then any simplex algorithm starting here will terminate in a finite number of iterations, either with an optimal solution or with an indication of unboundedness.*

Proof: Any such algorithm generates a sequence of basic feasible solutions until it terminates, so we are done if we show that no basic feasible solution can be repeated, i.e., no set of basic indices can repeat, because there are only at most $\begin{pmatrix} m \\ n \end{pmatrix}$ choices for β. But each time we move from one basic feasible solution to the next, the objective function changes by $\frac{\bar{b}_p \bar{c}_q}{\bar{a}_{pq}} < 0$ ($\bar{b}_p > 0$, $\bar{c}_q < 0$, $\bar{a}_{pq} > 0$) by nondegeneracy, so the objective function is strictly decreasing. $\square$

Geometry of the Simplex Method
Each basic feasible solution to (P) satisfies n constraints at equality. When we choose x_q to

Thanks to Jay Schomer
www.halycon.com/
nemain/gallery/gallery.html

www.ehow.com/eHow/
eHow/1,1053,4570,00.html

Figure 1: Geometry of Simplex

increase, we relax the equality of $x_q \geq 0$ giving us $n-1$ equality constraints. This defines an edge of the feasible region. If a ray, we get unboundedness. Otherwise, the edge has another endpoint and this is the new basic feasible solution. So in the space of the feasible polyhedron, we have an edge-following algorithm. Our intuition says this might work very well, if the polyhedron is "like a quartz crystal," with long edges from one side to the other, or very badly, if the polyhedron is "like a disco ball," needing lots of small steps. See the figure.

Dantzig's column geometry. Imagine the problem:

$$\begin{array}{lrcl} \min_x & c^T x & & \\ & \sum \tilde{a}_j x_j & = & \tilde{b}, \\ & \sum x_j & = & 1, \\ & x & \geq & 0. \end{array} \quad A = \begin{bmatrix} \tilde{a}_1 & \tilde{a}_2 & ... & \tilde{a}_n \\ 1 & 1 & ... & 1 \end{bmatrix}$$

Here the additional constraint $\sum x_j = 1$ makes the feasible region bounded.
A simplex with vertices at the $\tilde{a}_j$'s containing $\tilde{b}$ corresponds to a basic feasible solution.
Remember here the $\tilde{a}_j$'s and $\tilde{b}$ lie in $\mathbf{R}^{m-1}$. In our example $m = 3$ and $\tilde{a}_1, \tilde{a}_2, ...$ lie in $\mathbf{R}^2$.
Now consider $\hat{a}_j = \begin{pmatrix} \tilde{a}_j \\ c_j \end{pmatrix} \in \mathbf{R}^m$ and the line $\{ \begin{pmatrix} \tilde{b}_j \\ \zeta \end{pmatrix} : \zeta \in \mathbf{R} \}$.
A simplex defined by m $\hat{a}_j$'s that intersects the line corresponds to a basic feasible solution and the ζ component of the intersection is the current objective function value.

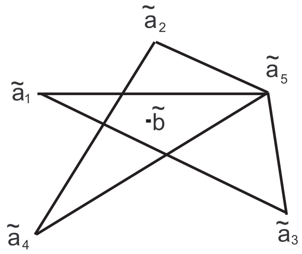

Figure 2: Simplices containing $\tilde{b}$ corresponding to basic feasible solutions

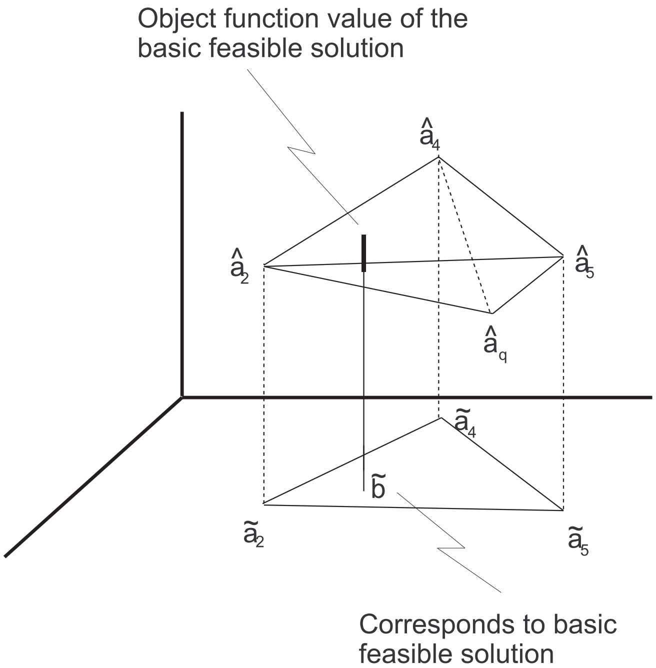

Figure 3: Column geometry

Note that:

$$\begin{array}{rcll} \bar{y} & = & B^{-T}c_B & \\ a_j^T\bar{y} & = & c_j & j \in \beta \\ \tilde{a}_j^T\tilde{y} + \eta & = & c_j & where\ \bar{y} = \begin{pmatrix} \tilde{y} \\ \eta \end{pmatrix} \\ -\tilde{a}_j^T\tilde{y} + c_j & = & \eta. & \end{array}$$

So all the points $\hat{a}_j$ for $j \in \beta$ lie on this hyperplane. Choosing the entering variable x_q corresponds to choosing a new vector $\hat{a}_q$ that lies as far below this hyperplane as possible (if we choose the smallest $\bar{c}_q$). This makes an $(m+1)$-dimensional simplex. We move in this $(m+1)$-simplex as far down the line $\begin{pmatrix} \hat{b} \\ \zeta \end{pmatrix}$ as possible, until we emerge in the face opposite $\hat{a}_j$ for some j. That j (corresponding to the pth basic variable) is dropped. In this geometry the simplex method seems more promising: it appears that we can quickly find m $\hat{a}_j$'s on the "bottom" of the convex hull of all the $\hat{a}_j$'s defining a simplex which meets the line at the point where it leaves the convex hull.