Mathematical Programming	Lecture 9
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Today we will start talking about how to actually solve linear programming problems. The first technique of solving linear programs was the *Simplex Method* developed in 1947 by George Dantzig (1914-2005). It is based on the idea of "travelling" between the vertices of a feasible region.

For simplicity we will work with the problem in the standard form

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b, \\ & x & \geq & 0, \end{array} \tag{P}$$

and we will also assume that matrix A has a full rank, and that we have a basic feasible solution available. Later we will talk about how to deal with problems when this is not the case.

We will discuss the method in general, but we will also demonstrate the method on a specific example:

$$\begin{array}{lrcrcrrrrcl} \min & c_1x_1 & + & 7x_2 & + & 5x_3 & & & & & \\ & & & & & & & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & -x_4 & & & = & 5, \\ & x_1 & + & 3x_2 & + & x_3 & & -x_5 & & = & 5, \\ & x_1 & + & x_2 & + & 4x_3 & & & -x_6 & = & 2, \\ & & & & & & & & x & \geq & 0. \end{array} \tag{E}$$

Let's choose basic indices $\beta = \{4, 1, 6\}$. Note that the ordering is important here and that it transforms into the ordering of vectors in matrix B and into all later calculations. The non-basic indices are then $\nu = \{2, 3, 5\}$. With this base our matrices B and N are

$$B = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & 4 & 0 \end{pmatrix}.$$

Now $B^{-1}b$ is our basic feasible solution, so let's calculate it:

$$B^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{and} \qquad B^{-1}b = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} \geq 0.$$

Our next goal is to express everything in terms of the non-basic variables. In general, for all feasible solutions we have the relationship

$$Ax = b \qquad \Longleftrightarrow \qquad Bx_B + Nx_N = b. \tag{1}$$

As B is a basis, it is regular and hence we can multiply (1) by B^{-1} to obtain

$$
Ix_B + B^{-1}Nx_N = B^{-1}b = \overline{x}_B, \tag{2}
$$

where

$$
\overline{x} = \begin{pmatrix} \overline{x}_B \\ \overline{x}_N \end{pmatrix} = \begin{pmatrix} \overline{x}_B \\ 0 \end{pmatrix}
$$

is our current basic feasible solution. This equation tells us how the basic variables respond to changes in the non-basic variables. Now, let's return back to our example and compute

$$
B^{-1}N = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 1 & -1 \\ 2 & -3 & -1 \end{pmatrix}.
$$

Hence we know that

$$
\begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} + \begin{pmatrix} 5 & 1 & 2 \\ 3 & 1 & -1 \\ 2 & -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}.
$$

Now it's time to take a look at the objective function. Let's denote it $\zeta = c^T x$ and write the equality as $\zeta - c^T x = 0$. Similarly as we partition A and x, we can partition the vector c into its basic and non-basic parts. Then we can write

$$
0 = \zeta - c_B^T x_B - c_N^T x_N = \zeta - c_B^T(B^{-1}b - B^{-1}Nx_N) - c_N^T x_N,
$$

or

$$
\zeta - (c_N - N^T B^{-T} c_B)^T x_N = c_B^T B^{-1} b. \tag{3}
$$

Back to our example. Let's try to write equation (3) for our example, and let's start with $c_1 = 1$.

$$
B^{-T}c_B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
$$

$$
c_N - N^T(B^{-T}c_B) = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}.
$$

So the equation (3) looks like

$$
\zeta - \begin{pmatrix} 4 & 4 & 1 \end{pmatrix} . \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} . \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} = 5.
$$

This equation holds for every feasible solution, so we can see that $\zeta \geq 5$ for any feasible $(x_2; x_3; x_5)$. So, we have found the optimum.

Note that equations (2) and (3) exhibit the current BFS and its cost, and setting $(x_2; x_3; x_5) = (0; 0; 0)$ gives us $\overline{x} = \binom{B^{-1}b}{0}$ and $\overline{\zeta} = c_B^T B^{-1} b = c^T \overline{x}$. However, these equations also show how x_B and ζ depend on x_N for any feasible solution.

Specifically in our case $\zeta = 5 + 4x_2 + 4x_3 + x_5$ and $x \geq 0$ so current solution is optimal. Moreover, to achieve optimality we need $x_2 = x_3 = x_5 = 0$ so we see our optimal solution $\binom{\overline{x}_B}{0}$ is unique.

Optimality Criterion

In general, as we can deduce from our example, whether the solution is optimal or not depends on the vector $c_N - N^T B^{-T} c_B$. If all of its elements are nonnegative, the current BFS is optimal solution. If all elements are positive, the current BFS is the unique optimal solution. Let's denote $\overline{y} = (B^{-T} c_B)$. Then the optimality condition is equivalent to

$$c_j - a_j^T \overline{y} \geq 0 \qquad \text{for all} \quad j \in \nu.$$

Note that it is the same condition as dual constraints. So what about $j \in \beta$? Let's write $c_j - a_j^T \overline{y}$ for all $j \in \beta$ – together we obtain

$$c_B - B^T \overline{y} = c_B - B^T (B^{-T} c_B) = c_B - c_B = 0.$$

So $\overline{y}$ satisfies dual constraints. Moreover, $\overline{\zeta} = x_B^T c_B = b^T B^{-T} c_B = b^T \overline{y}$, and this is the "dual value" of $\overline{y}$. So, we can understand current value of $\overline{y}$ to be a "trial" solution to the dual problem, which may or may not be feasible. If it happens to be feasible, we have (recall weak duality) an optimal primal solution $\overline{x}$ and also an optimal dual solution $\overline{y}$. Let's summarize what we have already shown into a theorem.

Theorem 1 Optimality Criterion *Suppose $\overline{x} = \binom{\overline{x}_B}{\overline{x}_N}$ is a basic feasible solution corresponding to the basis matrix B for* (P). *Let $\overline{y} = B^{-T} c_B$. Then if $c_N - N^T \overline{y} \geq 0$ then $\overline{x}$ is an optimal solution for (P) and $\overline{y}$ is an optimal solution to (D). Further if $c_N - N^T \overline{y} > 0$ then $\overline{x}$ is the unique optimal solution.*

Remark 1 *We always have such $\overline{x}$ and $\overline{y}$ that satisfy the complementary slackness condition, so that anytime $\overline{x}_j > 0$ we have $a_j^T \overline{y} = c_j$. But $\overline{y}$ may be infeasible!!!*

Example continues... Let's now try to work on our example with a different parameter $c_1 = -1$. Luckily most of our calculations are usable, so that we can express

$$\overline{y} = B^{-T} c_B = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

$$c_N - N^T B^{-T} c_B = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ -1 \end{pmatrix}.$$

Our equation (3) looks like

$$\zeta - \begin{pmatrix} 10 & 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} = -5 \qquad \text{or} \qquad \zeta = -5 + 10x_2 + 6x_3 - x_5.$$

So, ζ decreases as x_5 increases. How much could we increase x_5 to remain feasible? Remark the formula (2) that states $x_B = B^{-1}b - B^{-1}Nx_N$, or, in our example,

$$\begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 & 1 & -2 \\ 3 & 1 & -1 \\ 2 & -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix}.$$

Let's denote $x_\alpha = (5; 0; 0; 5; 0; 3) + \alpha(1; 0; 0; 2; 1; 1) = (5 + \alpha; 0; 0; 5 + 2\alpha; \alpha; 3 + \alpha)$. As we can see, x_α is feasible for every $\alpha \geq 0$ and value $c^T x_\alpha = -1 \cdot (5+\alpha) = -5 - \alpha$ goes to minus infinity as α increases. We have just found a feasible ray on which the objective function is unbounded below, and hence our problem is unbounded below too.

Note that the key to unboundedness is the negative signs in the 3$^{\text{rd}}$ column of the matrix $B^{-1}N$, because otherwise increasing x_5 too much would necessarily result in infeasibility.

Next time we will discuss the most interesting situation, when there is neither optimal solution nor unboundedness.