Mathematical Programming	**Lecture 8**
OR 630 Fall 2005	**September 20, 2005**
Notes by Yinan Huang	

Today we will discuss one of the central theorems in Linear Programming, the Strong Duality Theorem. We will concentrate on the standard form primal LP

$$(P)\qquad \begin{array}{rrcl} \min_x & c^Tx & & \\ & Ax & = & b, \\ & x & \geq & 0, \end{array}$$

and its dual

$$(D)\qquad \begin{array}{rrcl} \max_y & b^Ty & & \\ & A^Ty & \leq & c. \end{array}$$

First, recall that last time we had the Farkas Lemma, which states that exactly one of (I) and (II) below has a feasible solution

$$\begin{array}{rrclrcl} (I) & Ax & = & b, & x & \geq & 0, \\ (II) & A^Ty & \leq & 0, & b^Ty & > & 0. \end{array}$$

Since (I) defines the feasible region in our primal LP (P), we see that Farkas Lemma provides a short proof for the infeasibility of (P). All we need is a certificate y that satisfies (II). Similarly, we would naturally like to have a certificate x that provides a short proof for the infeasibility of the dual (D). We have the following Corollary of the Farkas Lemma:

Corollary 1 *(Corollary to Farkas) Exactly one of (III) and (IV) below has a feasible solution.*

$$\begin{array}{rrclrclrcl} (III) & A^Ty & \leq & c, & & & & & & \\ (IV) & Ax & = & 0, & x & \geq & 0, & c^Tx & < & 0. \end{array}$$

The proof for this corollary is straightforward. We can rewrite (III) in its equivalent standard form as (I), and apply the Farkas Lemma. The detailed proof will be given in the recitation.

Now we are ready to present the Strong Duality Theorem.

Theorem 1 *(Strong Duality) For the primal (P) and dual (D) as given above, one and only one of the following four possibilities can occur:*
(a): (P) and (D) are both infeasible;
(b): (P) is infeasible, (D) is unbounded;
(c): (P) is unbounded, (D) is infeasible; or
(d): (P) and (D) are both feasible, they have optimal solutions x_* and y_*, respectively, and there is no duality gap at optimality, i.e., $c^Tx_* = b^Ty_*$.

We have a few comments here

Remark

- A direct result from Strong Duality is that, if either (P) or (D) has an optimal solution, so does the other, and only case (d) applies.

- Strong Duality holds for any specific form of Linear Programming, not just the standard form. Since any LP has its equivalent standard form, it suffices to prove the theorem in our (P)-(D) context.

- In fact, all of the four possible cases can occur. In a sense, case (a) is the only weird thing in Linear Programming, where it is very unstable to have linear systems that are infeasible for both (P) and (D). A small pertubation in the data will change the system to case (b) or (c). More discussions can be found in the recitation and homework problems for this week.

- We say (P) is infeasible if it has no feasible solution, and (P) is unbounded if it has some sequence of feasible solutions that make the objective value $c^T x$ unbounded below. Similarly, (D) is unbounded if it has some sequence of feasible solutions that make the objective value $b^T y$ unbounded above. An unbounded problem is definitely different from an unbounded feasible region. (In fact, Clark's Theorem already tells us that the feasible regions for (P) and (D) if nonempty cannot be both bounded!) But Strong Duality just means that it does not matter to have an unbounded region; we will still have a bounded objective value, guaranteed by case (d).

Proof: We will enumerate all possible cases.

First, consider the case that (P) is infeasible. The feasible region of (P) is (I), so we directly apply the Farkas Lemma. (I) is infeasible means that there exists a solution $\bar{y}$ to (II), i.e., $A^T\bar{y} \leq 0$ and $b^T\bar{y} > 0$. Here we still need to discuss the feasibility of (D):

1. If (D) is infeasible, then we are in case (a). We are done.

2. If (D) is feasible, say (D) has a feasible solution $\hat{y}$, then $\hat{y} + \alpha\bar{y}$ is feasible for (D) for all $\alpha \geq 0$, but $b^T(\hat{y} + \alpha\bar{y}) = b^T\hat{y} + \alpha b^T\bar{y} \to +\infty$ as $\alpha \to +\infty$. So, (D) is unbounded from above. Case (b) holds.

Second, consider the case that (D) is infeasible. The feasible region of (D) is (III), so we directly apply the Corollary to Farkas. (III) is infeasible means that there exists a solution $\bar{x}$ to (IV), i.e: $A\bar{x} = 0$, $c^T\bar{x} < 0$ and $\bar{x} \geq 0$. Here we discuss the feasibility of (P), as we did at first:

1. If (P) is infeasible, then we are again in case (a). We are done.

2. If (P) is feasible, say (P) has a feasible solution $\hat{x}$, then $\hat{x} + \alpha\bar{x}$ is feasible for (P) for all $\alpha \geq 0$, but $c^T(\hat{x} + \alpha\bar{x}) = c^T\hat{x} + \alpha c^T\bar{x} \to -\infty$ as $\alpha \to +\infty$ So, (P) is unbounded from below. Case (c) holds.

Next, we consider the remaining case, where both (P) and (D) are feasible. We need to show that only case (d) holds under this assumption. Our idea is: we construct a "larger" linear system whose feasibility implies case (d), and then we apply the Corollary to Farkas to establish its feasibility. We combine all constraints in (P) and (D), together with an additional constraint from Weak Duality, to get our auxiliary linear system as:

$$\begin{array}{rcrcl} Ax & & & \leq & b, \\ -Ax & & & \leq & -b, \\ -x & & & \leq & 0, \\ & & A^T y & \leq & c, \\ c^T x & - & b^T y & \leq & 0. \end{array} \tag{1}$$

We recognize that if the system (1) has a feasible solution (x_*, y_*), then x_* is feasible for (P) and y_* is feasible for (D). Besides, $c^T x_* \leq b^T y_*$. From Weak Duality, this means that x_* is optimal for (P) and y_* is optimal for (D) and their optimal values are the same. Case (d) holds. That is our goal.

We treat our new system (1) as a huge instance of $A^T y \leq c$ as in the Corollary to Farkas. (1) is of the form $\bar{A}^T \bar{y} \leq \bar{c}$, where

$$\bar{A} = \begin{bmatrix} A^T & -A^T & -I & 0 & c \\ 0 & 0 & 0 & A & -b \end{bmatrix}$$

and $\bar{y} = (x; y)$, $\bar{c} = (b; -b; 0; c; 0)$.

We prove by contradiction that (1) indeed has a feasible solution. If otherwise (1) is infeasible, then according to the Corollary to Farkas, there exists a feasible solution to its alternative system:

$$\begin{array}{rcrcrcrcrcl} A^T s & - & A^T t & - & u & & & + & c\omega & = & 0, \\ & & & & & & Av & - & b\omega & = & 0, \\ b^T s & - & b^T t & & & + & c^T v & & & < & 0, \\ s, & & t, & & u, & & v, & & \omega & \geq & 0. \end{array} \tag{2}$$

The scalar $\omega \geq 0$ looks like a slack variable. Since this system (1) has a feasible solution, we have either $\omega > 0$ or $\omega = 0$.

1. If $\omega > 0$, then we can scale s, t, u, v, ω to make $\omega = 1$. Then we have
   (i) $A^T(t - s) + u = c$ and $u \geq 0$, so $t - s$ is feasible for (D).
   (ii) $Av = b$ and $v \geq 0$, so v is feasible for (P).
   However, $c^T v < b^T(t - s)$, this contradicts Weak Duality!

2. If $\omega = 0$, then the system (2) simplifies to
   (i) $A^T(t - s) + u = 0$ and $u \geq 0$,
   (ii) $Av = 0$ and $v \geq 0$,
   (iii) $c^T v < b^T(t - s)$.

If $c^T v < 0$, then $Av = 0$ and $v \geq 0$, vector v shows that (D) is infeasible by the Corollary to Farkas. This contradicts our assumption that (D) is feasible.
If $c^T v \geq 0$, then $A^T(t-s) + u = 0$, $b^T(t-s) > 0$, and $u \geq 0$, and vector $t - s$ shows that (P) is infeasible by the Farkas Lemma. This contradicts our assumption that (P) is feasible.

All possible choices of a feasible solution to the alternative system (2) lead to a contradiction. Therefore we conclude that our constructed system (1) indeed has a feasible solution, which finishes our proof that case (d) holds when (P) and (D) are both feasible.

This completes the proof of the theorem. $\square$
Now we look at some corollaries and examples.

Corollary 2 *(Complementary Slackness) If either (P) or (D) has an optimal solution, then so does the other, and their optimal solutions x_* and y_* satisfy complementary slackness, i.e.: if we define $s_* = c - A^T y_*$, then $s_*^T x_* = 0$. Equivalently, for each j, either the jth component of s_* or the jth component of x_* must be zero.*

This corollary means that slackness in a nonnegativity of (P) implies tightness in the corresponding constraint of (D), and vice versa.
Proof:
If (P) or (D) has an optimal solution, then according to Strong Duality, this must correspond to case (d). Therefore, both (P) and (D) have optimal solutions, and there is no duality gap. (I.e., their optimal objective values are the same.)
Let's look at the chain of inequalities for x_* and y_*:
$c^T x_* = (A^T y_* + s_*)^T x_* = y_*^T(Ax_*) + s_*^T x_* = b^T y_* + s_*^T x_* \geq b^T y_*$.
Since x_* and y_* are optimal, there is no duality gap. So $c^T x_* = b^T y_*$, and we must have equality throughout the chain. Therefore $s_*^T x_* = 0$.
Remark

- Complementary Slackness provides a way to check optimality. If we have a feasible x for (P), a feasible y for (D), and they satisfy Complementary Slackness, then we can conclude that they are both optimal.
- For any feasible x for (P) and feasible y for (D), we can only say that they have a nonnegative duality gap $s^T x \geq 0$, where $s = c - A^T y$. The value $s^T x$ can be regarded as a bound or criterion for the distance to optimality.

We can generalize our discussion to the symmetric primal-dual pair system:

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & \geq & b, \\ & x & \geq & 0, \end{array} \tag{3}$$

and its dual

$$
\begin{array}{rrcl}
\max_y & b^T y & & \\
& A^T y & \leq & c, \\
& y & \geq & 0.
\end{array} \tag{4}
$$

Similarly, define
$s = c - A^T y$, the slack variable in the dual (4), and
$t = b - Ax$, the surplus variable in the primal (3), then
our chain of inequalities becomes
$c^T x = (A^T y + s)^T x = y^T Ax + s^T x = y^T(b + t) + s^T x = b^T y + s^T x + t^T y \geq b^T y$.
At optimality, we must have $s^T x = 0$ and $t^T y = 0$. This is Complementary Slackness!

Economic Interpretation

We look at two examples.

Example 1 *(Product Mix)* We give the following interpretation for (3) and (4):
Suppose now there is a producer who wishes to produce m products. He has n resources by hand, each with a limited amount.
x_i is the price for the ith scarce resource (the shadow price), $i \in \{1, 2, ..., n\}$;
y_j is the amount of the jth product he will produce,$j \in \{1, 2, ..., m\}$;
b_j is the price for the jth product in the market;
c_i is the amount of the ith resource available;
Then, the dual (4) is explained as: The producer wishes to use his limited resources to produce goods and sell them for maximum profit.
Correspondingly, the primal (3) becomes: An entrepreneur wants to buy the resources from the producer. The entrepreneur will try to convince the producer by offering a higher price for the resources that are used by each good than the amount the producer can sell the good for, and at the same time minimizes her total cost of the purchase.

Complementary Slackness tells us:
At optimality, if $s_i > 0$, then $x_i = 0$ i.e: if the ith resource is in excess supply at maximal profit (more than the producer needs), then the excess supply has no additional value at all, and the entrepreneur as a rational buyer is willing to pay 0 dollars!
Also, at optimality, if $t_j > 0$, then $y_j = 0$ i.e: if the entrepreneur offers a strictly higher purchasing price for the resources that are used by product j than the current market price for the jth good, then the producer as a rational seller will not use any of his resources to produce the jth good!

Example 2 *(Diet Mix)* We give the following interpretation for (3) and (4):
x gives the amount of each different food;
y gives the price of each nutrient pill;
b gives the amount of each nutrient required;
c gives the price of each foods;
Then, the primal (3) is explained as: A person wants to eat different foods to satisfy nutrient requirements, and minimize the total amount of money to buy those foods.
Correspondingly, the dual (4) is explained as: An entrepreneur will instead try to sell nutrient pills. In order to be competitive in the market, the prices for the combination of pills that provide the same nutrients as any food should not exceed the current prices of that food. At the same time, he wants to maximize his profit.
Complementary Slackness tells us:
At optimality, if $s_j > 0$, then $x_j = 0$, i.e.: if the person can buy the nutrient pills that provide the same nutrients as food j at a strictly lower price than the market price for that food, then he will not buy any amount of that food!
Also, at optimality, if $t_i > 0$, then $y_i = 0$, i.e.: if the person has excessive amount of the ith nutrient, then the excess supply will not be needed in the market, so the entrepreneur will set his price for the ith nutrient pill at 0!