Let us consider standard form problems. If the feasible region is

$$Q = \{x \in \Re^n : Ax = b, x \geq 0\},$$

then Q is pointed because every line $\{x + \alpha d : x \in \Re\}$, $0 \neq d \in \Re^n$ leaves the nonnegative orthant for some α. What are the basic solutions of this system?

Assume, without loss of generality, that $\text{rank}(A) = m$ where $A \in \Re^{m \times n}$. Let $a_{i.}^T$, $i = 1, \dots, m$ be the ith row of A. The constraints, therefore, are

$$\begin{aligned} a_{i.}^T x &= b_i, \qquad i = 1, \dots, m, \\ -x_j &\leq 0, \qquad j = 1, \dots, n. \end{aligned}$$

A basic solution x satisfies all equality constraints and satisfies n linearly independent constraints at equality. Let us say that the constraints that are satisfied with equality are indexed by $i = 1, \dots, m$ and $j \in \nu$, $\nu \subseteq \{1, 2, \dots, n\}$ with $|\nu| = n - m$. Let $\beta = \{1, \dots, n\} \setminus \nu$. Thus, $a_{i.}$, $i = 1, \dots, m$, and $-e_j$, $j \in \nu$ are linearly independent. Recall that e_j is the vector of length n with all entries zero except for the jth entry being a 1.

Let us consider this another way. Let B be the $m \times m$ submatrix of A consisting of columns indexed by $j \in \beta$, and N be the $m \times (n - m)$ submatrix of A consisting of columns indexed by $j \in \nu$. After re-ordering the columns, the matrix of active constraints is

$$\begin{pmatrix} B & N \\ 0 & -\mathbf{I}_{n-m} \end{pmatrix}.$$

x is a basic solution if and only if this is a $n \times n$ nonsingular matrix, i.e., B is a $m \times m$ nonsingular matrix. If we also reorder the components of x to get x_B and x_N, then $Ax = b$ is equivalent to

$$\begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b.$$

The active equality constraints can thus be written as

$$\begin{pmatrix} B & N \\ 0 & -\mathbf{I}_{n-m} \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

which implies that $x_N = 0$ and $x_B = B^{-1}b$.

Proposition 1 *For a standard form system of equations with $A \in \Re^{m \times n}$ with rank$(A) = m$ and nonnegativity constraints, each basic solution corresponds to a partition of A into $[B \quad N]$ with B being a $m \times m$ matrix and nonsingular. The basic solution is $x_B = B^{-1}b$ and $x_N = 0$. It is a basic feasible solution if $B^{-1}b \geq 0$.* $\square$

Note that the choice of B for a given basic solution x is not unique in general.

Last week, we showed that every bounded polyhedron could be represented as a polytope. As well, every pointed polyhedron can be represented as a polytope plus a recession cone. Now let us consider the converse question: Is every polytope a bounded polyhedron?

Let us motivate this question with some examples from combinatorial optimization. We want to optimize a function defined on a finite, but large, set of objects. Very often, this can be expressed as optimizing a linear function over a finite, but large, set of vectors in $\Re^p$ for some p not too large. This is equivalent to optimizing a linear function over a polytope, the convex hull of these vectors. So if we can represent this polytope as a polyhedron, we get an linear programming problem. We will show that this is possible in theory.

Example 1 (The Assignment Problem) *We have n jobs to be assigned to n machines, one to each. We want to minimize the total processing time where job i takes c_{ij} units of time on machine j. Each such assignment is a one-to-one mapping from $\{1, 2, \ldots, n\}$ to itself, where $\pi(i) = j$ means assigning job i to machine j. There are $n!$ such permutations. Each permutation π corresponds to an $n \times n$ matrix $\mathbf{X} = (x_{ij})$ where*

$$x_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\mathbf{X}$ may be a 3×3 matrix (corresponding to a vector in $\Re^{3^2}$) of the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The cost of any permutation will thus be

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}.$$

The polytope is the convex combination of all permutation matrices. Can we express the convex hull of all the permutation matrices as a polyhedron?

Obvious necessary conditions:

$$\begin{aligned} \sum_{i=1}^{n} x_{ij} &= 1, \qquad j = 1, \ldots, n, \\ \sum_{j=1}^{n} x_{ij} &= 1, \qquad i = 1, \ldots, n, \\ x_{ij} &\geq 0, \qquad \text{all } i, j. \end{aligned}$$

It turns out that these conditions are also sufficient. (We'll show this later in the course.) Therefore, we were able to change a problem with $n!$ discrete feasible solutions into a LP problem with $2n$ equality constraints in nonnegative variables.

[Note that this example is very degenerate: there are $2n$ equality constraints (actually, just $2n-1$ linearly independent equality constraints), so we expect basic solutions to have this many nonzero components: but permutation matrices have just n nonzero components, their "1" entries.]

Example 2 (The Travelling Salesman Problem) *There are n cities, including the salesman's home. He must visit all n cities in some order and return home. We want to minimize the total length of the tour, i.e., the sum of all chosen c_{ij}, where c_{ij} is the cost of going directly from city i to city j. For each such tour, consider an $n \times n$ matrix $\mathbf{X} = (x_{ij})$, where*

$$x_{ij} = \begin{cases} 1 & \textit{if the salesman goes directly from } i \textit{ to } j \\ 0 & \textit{otherwise.} \end{cases}$$

The cost of such an $\mathbf{X}$ is just

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}.$$

Can we represent the convex hull of all such tour vectors as a polyhedron? Yes, but not easily. It is difficult because even though all tours are permutation matrices, not all permutation matrices are tours. For example, in a four city tour, a matrix $\mathbf{X}$ that has the sales person go from $1 \to 2, 2 \to 1, 3 \to 4, 4 \to 3$ is not a valid tour.

If we consider the symmetric case, where $c_{ji} = c_{ij}$ and we do not distinguish between the two directions a tour can be traversed, there are $(n-1)!/2$ tours of n cities. For $n = 7$, the TSP polytope, therefore, has 360 vertices, but it needs $3,437$ linear inequalities to define it; for $n = 8$, these numbers become $2,520$ and $194,187$. So it is not a good idea to solve the problem by obtaining all *necessary inequalities and solving the resulting LP problem. Instead, partial descriptions, where linear inequalities are added "as needed" on the "important" side of the polytope can be very useful.*

For the solution of various TSP problems, see the TSP page at `http://www.tsp.gatech.edu/` *and the milestones in computation at* `http://www.tsp.gatech.edu/history/milestone.html` *The current record is a problem with over 24,000 cities.*

Now let us consider the Separating Hyperplane Theorem. It is important since it will be used in proving the Farkas Lemma as well as in the proof of Strong Duality. We need the following theorem in proving the Separating Hyperplane Theorem:

Theorem 1 (Weierstrass's Theorem) *A real-valued continuous function defined on a non-empty compact set attains its minimum and maximum.*

Now we can state the Separating Hyperplane Theorem:

Theorem 2 (Separating Hyperplane Theorem) *Let $C \subseteq \Re^n$ be closed and convex. Let $x \in \Re^n$ and $x \notin C$. Then, there exists $0 \neq a \in \Re^n$, $\beta \in \Re$, such that $a^T x > \beta$ and $a^T z < \beta$ for all $z \in C$.*