Mathematical Programming	**Lecture 5**
OR 630 Fall 2005	**September 8, 2005**
Notes by Gurmeet Singh	

Last time we saw that every bounded polyhedra is a polytope in the set of convex combination of its vertices.
Now we will extend the theory to pointed polyhedra (i.e., those that contain no lines).

Definition 1 *Let C be a nonempty convex set: then the recession cone of $C, rec(C)$, is*

$$\{d \in R^m : \forall x \in C, \forall \alpha \geq 0, x + \alpha d \in C\}.$$

Proposition 1 *If C is a nonempty set then rec(C) is a nonempty convex cone.*

Proof:
Let $d_1, d_2 \in \text{rec}(C), \lambda_1, \lambda_2 \geq 0$. We want to show that $\lambda_1 d_1 + \lambda_2 d_2 \in \text{rec}(C)$. For any $x \in C$ and any $\alpha \geq 0$

$$x + \alpha(\lambda_1 d_1 + \lambda_2 d_2) \;=\; [x + (\alpha\lambda_1)d_1] \;+\; (\alpha\lambda_2)d_2.$$

The quantity in brackets lies in C since $\alpha\lambda_1 \geq 0$ and $d_1 \in \text{rec}(C)$, and then the desired vector lies in C because $\alpha\lambda_1 \geq 0$ and $d_2 \in \text{rec}(C)$. Also, $0 \in \text{rec}(C)$ by definition. $\square$

Proposition 2 *For $Q := \{y \in \mathbf{R}^m : A_x^T y \leq c_x, A_w^T y = c_w\}$ then (if Q is nonempty)*

$$rec(Q) = \{d \in \mathbf{R}^m : A_x^T d \leq 0, A_w^T d = 0\}.$$

Proof:
$\supseteq$:
if $A_x^T d \leq 0, A_w^T d = 0$ then for any $y \in Q, \alpha \geq 0$.

$$\begin{aligned} A_x^T(y + \alpha d) &= A_x^T y + \alpha A_x^T d \\ &\leq c_x + 0 = c_x, \\ \text{and similarly} & \\ A_w^T(y + \alpha d) &= c_w, \end{aligned}$$

hence $(y + \alpha d) \in Q$.

$\subseteq$:
Suppose $d \in \text{rec}(Q)$, and choose any $y \in Q$. Then $\forall \alpha \geq 0$

$$\begin{aligned} A_x^T(y + \alpha d) &= A_x^T y + \alpha A_x^T d \leq c_x; \\ \text{and then} & \\ A_x^T y &\leq c_x \Rightarrow A_x^T d \leq 0 \end{aligned}$$

(otherwise, the inequality would fail for large α); similarly

$$A_w^T d = 0.$$

$\square$

Theorem 1 *(Representation of Pointed Polyhedra). Let Q (defined as in Proposition 2) be a nonempty pointed polyhedron, and let P be the set of all convex combinations of its vertices and K be its recession cone. Then*

$$Q = P + K := \{p + d : p \in P, d \in K\}.$$

Proof:
$\supseteq$:
Every vertex of Q satisfies all linear constraints of Q so p also does for any $p \in P$.
So any $p + d \in P + K$ has

$$\begin{aligned} A_x^T(p+d) &= A_x^T p + A_x^T d \leq c_x + 0 = c_x; \\ A_w^T(p+d) &= A_w^T p + A_w^T d = c_w + 0 = c_w. \end{aligned}$$

$\subseteq$:
The proof is by induction on $\{m - ra(y)\}$.

True for $\{m - ra(y) = 0\} \Leftrightarrow y$ is itself a vertex of Q and $d = 0 \in \text{rec}(C)$.

Suppose true if $\{m - ra(y) < k\}$ for some $k > 0$ and consider $y \in Q$ with $ra(y) = m - k < m$. Choose $0 \neq d \in \mathbf{R}^m$ with $\{a_j^T d = 0, \forall j \in I(y)\}$ and consider $y + \alpha d, \alpha \in \mathbf{R}$. Since Q is pointed there are three cases to consider.

(1) α is bounded above and below, say by $\underline{\alpha} < 0$ & $\overline{\alpha} > 0$.
As in the previous theorem

$$y \quad = \quad \tfrac{\overline{\alpha}}{\overline{\alpha}-\underline{\alpha}}(y + \underline{\alpha}d) \quad + \quad \tfrac{-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}}(y + \overline{\alpha}d),$$

and $(y + \overline{\alpha}d)$ has $m - ra(y + \overline{\alpha}d) < k$, so

$$(y + \overline{\alpha}d) \quad = \quad \overline{p} \quad + \quad \overline{d} \quad , \quad \overline{p} \in P \quad , \quad \overline{d} \in K,$$
and similarly
$$(y + \underline{\alpha}d) \quad = \quad \underline{p} \quad + \quad \underline{d} \quad , \quad \underline{p} \in P \quad , \quad \underline{d} \in K,$$

so

$$\begin{aligned} y \quad &= \quad \tfrac{\overline{\alpha}}{\overline{\alpha}-\underline{\alpha}}(\underline{p} + \underline{d}) \quad + \quad \tfrac{-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}}(\overline{p} + \overline{d}) \\ &= \quad [\tfrac{\overline{\alpha}}{\overline{\alpha}-\underline{\alpha}}\underline{p} + \tfrac{-\underline{\alpha}}{\overline{\alpha}-\underline{\alpha}}\overline{p}] \quad + \quad \{\ldots \underline{d} + \ldots \overline{d}\}. \end{aligned}$$

The vector in brackets is a point of P and that in braces a point in K.

(2) α is bounded below but not above. Then $d \in K$ and $y = [y + \underline{\alpha}d] + (-\underline{\alpha})d$, with $\underline{\alpha}$ defined as before. The vector in brackets lies in $P + K$ as in the first part by the inductive hypothesis. Therefore

$$
\begin{aligned}
y &= (\underline{p} + \underline{d}) + (-\underline{\alpha})d \\
&= \underline{p} + (\underline{d} + (-\underline{\alpha})d)
\end{aligned}
$$

lies in $P + K$.

(3) α is bounded above but not below. Then we can simply switch d to $-d$ and $\overline{\alpha}$ to $-\underline{\alpha}$, and we get back to case(2).
This completes the proof.
□

Theorem 2 *(Fundamental theorem of LP). Consider the LP problem* $\max\{b^T y : y \in Q\}$ *with* Q *being a pointed polyhedron. Then*

1. *if there is a feasible solution, there is a vertex solution (basic feasible solution);*
2. *if there is a feasible solution and* $b^T y$ *is unbounded above on* Q*, then there is a ray or halfline:* $\{y + \alpha d : \alpha \geq 0\} \in Q$ *on which* $b^T y$ *is unbounded above; and*
3. *if* $b^T y$ *is bounded above on* Q*, then the max is attained and attained at a vertex* Q*.*

Proof:
(1): If $Q \neq \emptyset, P \neq \emptyset$, so there exists a vertex.

(2)& (3):
Assume $P \neq \emptyset$ & P is a set of convex combinations of $v_1, v_2, v_3, ..., v_k$.

$$
\begin{aligned}
\sup\{b^T y : y \in Q\} &= \sup\{b^T y : y \in P + K\} \\
&= \sup\{b^T p + b^T d : p \in P, d \in K\} \\
&= \sup\{b^T p : p \in P\} + \sup\{b^T d : d \in K\}.
\end{aligned}
$$

If there is some $\overline{d} \in K$ with $b^T \overline{d} > 0$ then by considering $\alpha \overline{d}$, $\alpha \rightarrow +\infty$, see that $\sup\{b^T d : d \in K\} = +\infty$. Then $b^T y$ is unbounded above on Q and clearly unbounded above on $\{y + \alpha \overline{d}\ ,\ \alpha \geq 0\}$ for any $y \in Q$.
If there is no such $\overline{d} \in K$, then $\sup\{b^T d : d \in K\} = 0$, attained by $d = 0$. Then

$$
\begin{aligned}
\sup\{b^T y : y \in Q\} &= \sup\{b^T p : p \in P\} \\
&= \sup\{\textstyle\sum_{i=1}^k \lambda_i (b^T v_i) : \sum_{i=1}^k \lambda_i = 1,\ \text{all } \lambda_i \geq 0\} \\
&= \max_{1 \leq i \leq k} b^T v_i
\end{aligned}
$$

In this case $\sup\{b^T y : y \in Q\}$ is attained by $y = v_i$ where i attains the maximum.
□