Mathematical Programming	Lecture 1
OR 630 Fall 2005	August 25, 2005

In memory of George B. Dantzig (1914–2005) and Leonid G. Khachiyan (1952–2005): two giants in the development of linear programming.

Much of the course will be devoted to linear programming (LP), the study of the optimization of a linear function of several variables subject to linear equality and inequality constraints. Here "programming" should be understood in the sense of planning — more like TV programming than computer programming — and linear refers to the types of functions involved. Nevertheless, because of the inequalities, these problems are much more subtle and harder to solve than the related problems of linear equations and linear least squares: perhaps they should be called piecewise-linear.

There are many forms such a problem can take. We first consider an inequality-constrained problem:

$$\begin{array}{rrcl} \max\limits_{y} & b^T y & & \\ & A^T y & \leq & c, \end{array}$$

where $y \in \mathbf{R}^m$ is a vector of decision variables (also indicated by the subscript to "max"), and $A \in \mathbf{R}^{m \times n}$ (the space of real $m \times n$ matrices), $b \in \mathbf{R}^m$, and $c \in \mathbf{R}^n$ constitute the data. There are n inequality constraints, each of the form $a_j^T y \leq c_j$, where a_j is the jth column of A, and an *objective function* $b^T y$, here to be maximized. We call the set

$$Q := \{y \in \mathbf{R}^m : A^T y \leq c\}$$

of points satisfying all the constraints the *feasible region* or *feasible set*. Note that all our vectors are columns, and that a subscripted letter could be a component of a vector (like c_j) or itself a vector (like a_j). Also, vector inequalities are interpreted componentwise.

Example 1 *(Product Mix): The Marie-Antoinette bakery makes high-end bread and cakes. Each loaf requires 3 pounds of flour and 2 hours of oven time, while each cake requires just 1 pound of flour but 4 hours of oven time. There are 7 pounds of flour and 8 hours of oven time available, and all other ingredients are in ample supply. (Note that this is a very small operation, and the oven can handle only one bakery product at a time!) Each loaf and each cake can be sold for \$5. How many loaves and how many cakes should be made to maximize revenue?*

If we let y_1 and y_2 denote the numbers of loaves and cakes made (our decision variables), then the objective function is $5y_1 + 5y_2$, to be maximized. The flour constraint is $3y_1 + 1y_2 \leq 7$, while the oven constraint is $2y_1 + 4y_2 \leq 8$. Are these all the constraints? No: the numbers of loaves and cakes cannot be negative, so we get

$$\begin{array}{lllllll} \max_y & 5y_1 & + & 5y_2 & & \\ & 3y_1 & + & 1y_2 & \leq & 7, \\ & 2y_1 & + & 4y_2 & \leq & 8, \\ & y_1 \geq 0, & & y_2 \geq 0. & & \end{array}$$

We might argue that y_1 and y_2 should be integers, but this makes our problem an integer linear programming problem, which is potentially much harder to solve. So for now we allow y_1 and y_2 to take on any real values. This might be a reasonable approximation for a problem instance of more realistic size: perhaps y_j is the number of batches (say of 100 loaves or 100 cakes) made, so fractions are possible.

Our problem above is of the form $\max\{b^T y : A^T y \leq c\}$, where $A = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 1 & 4 & 0 & -1 \end{bmatrix}$ (note that the columns of A give the coefficients of the constraints), $b = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = (5,5)^T = (5;5)$ and $c = (7,8,0,0)^T = (7;8;0;0)$.

We can solve such a small problem graphically, by drawing the feasible region in $\mathbf{R}^2$:

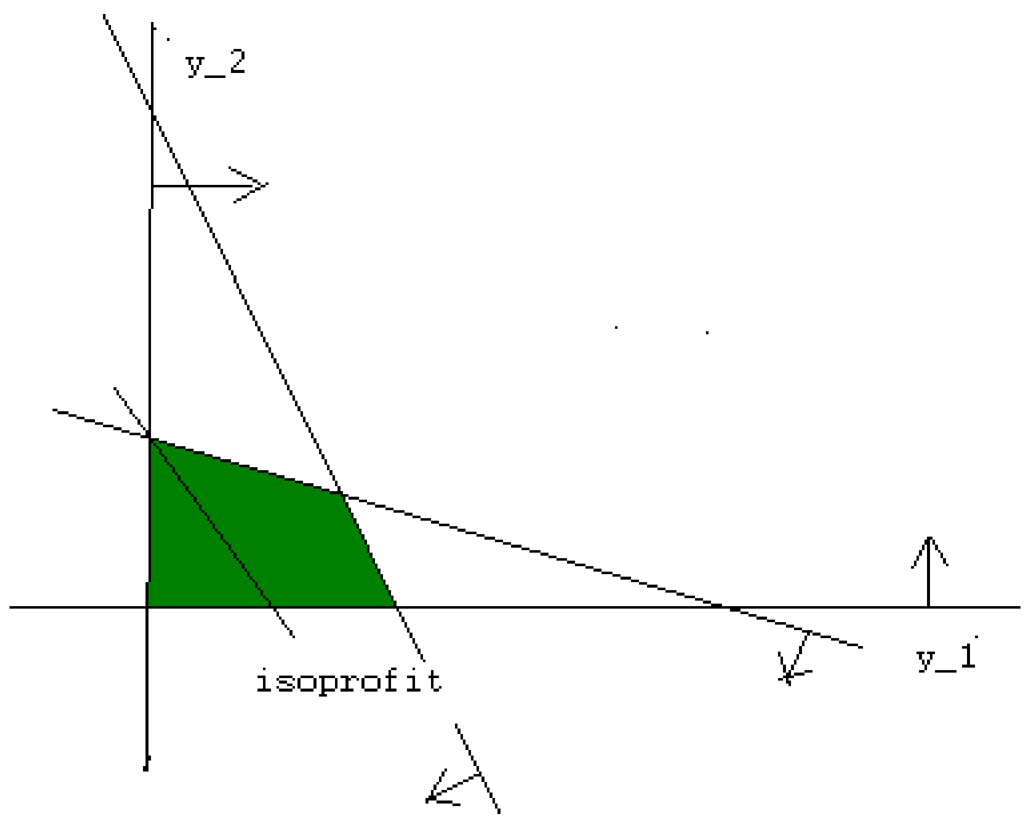

Figure 1: *The feasible region and an isoprofit line for Example 1.*

We can also draw the line $5y_1+5y_2 = 10$; parallel lines show points of equal profit. There are clearly an infinite number of feasible points (by contrast, there are only finitely many *integer* feasible points: so why is integer linear programming harder than LP??). But moving the "isoprofit" line up as much as possible, we see that $(2; 1)$ looks like a good point; it gives a profit of \$15.

This is pretty convincing: but can we get an *algebraic* proof that this is optimal, which might work even if we can't draw a picture? Yes! *Any* feasible point must satisfy the two constraints

$$\begin{array}{rcrcl} 3y_1 & + & 1y_2 & \leq & 7, \\ 2y_1 & + & 4y_2 & \leq & 8, \end{array}$$

so satisfies their sum: $5y_1 + 5y_2 \leq 15$. But we also have a feasible point, $y = (2; 1)$, which gives objective function value 15: so it must be optimal!

Let us modify our example a bit. Suppose a competitor, Josephine, offers comparable cakes, so Marie and Antoinette have to decrease the price of cakes to \$4. They compensate by increasing the price of each loaf to \$7 (if you think this is outrageous, tough: as Marie and Antoinette said, "Let them eat cake!" (see `http://www.straightdope.com/classics/a2_334.html`)). The objective function is now $7y_1 + 4y_2$. Adding the constraints no longer works, but we could take positive multiples of them first:

$$\begin{array}{rrcrcrcrcr} & 2 & \times & 3y_1 & + & 1y_2 & \leq & 7, \\ + & 1/2 & \times & 2y_1 & + & 4y_2 & \leq & 8, \\ & & & -- & - & -- & -- & -- \\ & & & 7y_1 & + & 4y_2 & \leq & 18, \end{array}$$

and the feasible point $(2; 1)$ gives exactly \$18 revenue, so is still optimal.

But what if Josephine also sells bread, so b_1 decreases to \$1 per loaf, and the objective function becomes $1y_1 + 4y_2$? Simple algebra suggests

$$\begin{array}{rrcrcrcrcr} & -2/5 & \times & 3y_1 & + & 1y_2 & \leq & 7, \\ + & 11/10 & \times & 2y_1 & + & 4y_2 & \leq & 8, \\ & & & -- & - & -- & -- & -- \\ & & & 1y_1 & + & 4y_2 & \leq & 6?? \end{array}$$

Is this valid? No!! Multiplying an inequality by a negative number changes its sense, and we can't then add the resulting inequalities.

Instead, we can proceed as follows:

$$\begin{array}{rrcrcrcrcr} & 1 & \times & 2y_1 & + & 4y_2 & \leq & 8, \\ + & 1 & \times & -1y_1 & & & \leq & 0, \\ & & & -- & - & -- & -- & -- \\ & & & 1y_1 & + & 4y_2 & \leq & 8, \end{array}$$

and $y = (0; 2)$ is feasible and gives objective function value \$8. (With such a low revenue, Marie-Antoinette may go out of business, or even worse: `http://www2.lucidcafe.com/lucidcafe/library/95nov/antoinette.html`.)

This little example illustrates quite a bit about our study of LP. First, it may be helpful to view it three ways:

- economics/modelling: what is the value of another pound of flour? What happens if a coefficient changes?
- theory/structure: how can we characterize an optimal solution? What can we say about the structure of the feasible region, or of an optimal solution. (The picture above suggests several possibilities, but beware: "Our intuition in higher dimensions may not be worth a damn," according to Dantzig.)
- computation: how can we efficiently compute an optimal solution when one exists?

In particular, the method we used to derive upper bounds on feasible objective function values leads to the very important topic of *duality*. In more generality, if y satisfies

$$a_1^T y \le c_1, \quad \ldots \quad , a_n^T y \le c_n,$$

it also satisfies $(\sum_j a_j x_j)^T y \le \sum_j c_j x_j = c^T x$ for any nonnegative $x \in \mathbf{R}^n$. So if we choose x so that $\sum_j a_j x_j = b$, we can conclude that $b^T y \le c^T x$. To get the best possible bound, we choose x to make $c^T x$ as small as possible. So together with

$$\begin{array}{lrcl} \max_y & b^T y & & \\ & A^T y & \le & c, \end{array} \tag{1}$$

we consider

$$\begin{array}{lrcl} \min_x & c^T x & & \\ & Ax & = & b, \\ & x & \ge & 0, \end{array} \tag{2}$$

and we have

Theorem 1 *(Weak Duality) If y is feasible in (1) and x in (2), then*

$$b^T y \le c^T x.$$

Proof: We have

$$b^T y = y^T b = y^T (Ax) = (y^T A)x = (A^T y)^T x \le c^T x.$$

□

Corollary 1 *(Sufficient Conditions for Optimality) If y_* and x_* are feasible in (1) and (2), and $b^T y_* = c^T x_*$, then each is optimal in its respective problem.*

□

We shall see later that the converse is true: optimal solutions to dual problems have equal objective function values (strong duality). Problems (1) and (2) are our first examples of dual problems.