Complexity of Simplex

Klee and Minty (1975) proved that the Simplex algorithm may use exponential time

$$
\begin{array}{l}
\text{maximize} \\
2^{n-1}x_1 \;+\; 2^{n-2}x_2 \;+\; \dots \;+\; 2x_{n-1} \;+\; 1x_n \\
\text{subject to} \\
\begin{array}{rcrcccccccccl}
1x_1 & + & & + & & + & & + & & \leq & 5 \\
4x_1 & + & 1x_2 & + & & + & & + & & \leq & 5^2 \\
8x_1 & + & 4x_2 & + & 1x_3 & + & & + & & \leq & 5^3 \\
\vdots & + & & + & & + & & + & & \leq & \vdots \\
2^n x_1 & + & 2^{n-1}x_2 & + & \dots & + & 4x_{n-1} & + & 1x_n & \leq & 5^n
\end{array} \\
x_i \geq 0, i = 1, \dots, n
\end{array}
$$

The problem has

- n variables
- n constraints
- 2^n extreme points
- Simplex, starting at $x = (0, \dots, 0)$, visits all extreme points
- optimal solution $(0, 0, \dots, 0, 5^n)$

Complexity of Simplex

For $n = 3$ simplex visits $2^3 = 8$ extreme points
Assume (s_1, s_2, s_3) slack variables:

	nonbasis			
basis	x_1	x_2	x_3	RHS
s_1	1^*			5
s_2	4	1		25
s_3	8	4	1	125
$-z$	4	2	1	0

	nonbasis			
basis	s_1	x_2	x_3	RHS
x_1	1			5
s_2	-4	1^*		5
s_3	-8	4	1	85
$-z$	-4	2	1	-20

	nonbasis			
basis	s_1	s_2	x_3	RHS
x_1	1^*			5
x_2	-4	1		5
s_3	8	-4	1	65
$-z$	4	-2	1	-30

	nonbasis			
basis	x_1	s_2	x_3	RHS
s_1	1			5
x_2	4	1		25
s_3	-8	-4	1^*	25
$-z$	-4	-2	1	-50

	nonbasis			
basis	x_1	s_2	s_3	RHS
s_1	1^*			5
x_2	4	1		25
x_3	-8	-4	1	25
$-z$	4	2	-1	-75

	nonbasis			
basis	s_1	s_2	s_3	RHS
x_1	1			5
x_2	-4	1^*		5
x_3	8	-4	1	65
$-z$	-4	2	-1	-95

	nonbasis			
basis	s_1	x_2	s_3	RHS
x_1	1^*			5
s_2	-4	1		5
x_3	-8	4	1	85
$-z$	4	-2	-1	-105

	nonbasis			
basis	x_1	x_2	s_3	RHS
s_1	1^*			5
s_2	4	1		25
x_3	8	4	1	125
$-z$	-4	-2	-1	-125