APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 2, April 1992, Pages 315-316

A QUASI-POLYNOMIAL BOUND FOR THE DIAMETER OF GRAPHS OF POLYHEDRA

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Abstract. The diameter of the graph of a d-dimensional polyhedron with n facets is at most $n^{\log d+2}$

Let P be a convex polyhedron. The *graph* of P denoted by $\mathrm{G}(P)$ is an abstract graph whose vertices are the extreme points of P and two vertices u and v are adjacent if the interval $[v, u]$ is an extreme edge (= 1-dimensional face) of P. The diameter of the graph of P is denoted by $\delta(P)$.

Let $\Delta(d, n)$ be the maximal diameter of the graphs of d-dimensional polyhedra P with n facets. (A facet is a $(d-1)$-dimensional face.) Thus, P is the set of solutions of n linear inequalities in d variables. It is an old standing problem to determine the behavior of the function $\Delta(d, n)$. The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for Dantzig's simplex algorithm for linear programming with any pivot rule.

In 1957 Hirsch conjectured [2] that $\Delta(d, n) \leq n-d$. Klee and Walkup [6] showed that the Hirsch conjecture is false for unbounded polyhedra. They proved that for $n \geq 2d, \Delta(d, n) \geq n-d+[d/5]$. This is the best known lower bound for $\Delta(d, n)$. The statement of the Hirsch conjecture for bounded polyhedra is still open. For a recent survey on the Hirsch conjecture and its relatives, see [5].

In 1967 Barnette proved [1, 3] that $\Delta(d, n) \leq n3^{d-3}$. An improved upper bound, $\Delta(d, n) \leq n2^{d-3}$, was proved in 1970 by Larman [7]. Barnette's and Larman's bounds are linear in n but exponential in the dimension d. In 1990 the first author [4] proved a subexponential bound $\Delta(d, n) \leq 2^{\sqrt{(n-d)\log(n-d)}}$.

The purpose of this paper is to announce and to give a complete proof of a quasi-polynomial upper bound for $\Delta(d, n)$. Such a bound was proved by the first author in March 1991. The proof presented here is a substantial simplification that was subsequently found by the second author. See [4] for the original proof and related results. The existence of a polynomial (or even linear) upper bound for $\Delta(d, n)$ is still open. Recently, the first author found a randomized pivot rule for linear programming which requires an expected $n^{4\sqrt{d}}$ (or less) arithmetic operations for every linear programming problem with d variables and n constraints.

1991 *Mathematics Subject Classification*. Primary 52A25, 90C05.
Received by the editors July 1, 1991
The first author was supported in part by a BSF grant by a GIF grant. The second author was supported by an AFOSR grant

Theorem 1.

$$\Delta(d,n) \le n^{\log d+2}. \tag{1}$$

Proof. Let P be a d-dimensional polyhedron with n facets, and let v and u be two vertices of P. Let k_v $[k_u]$ be the maximal positive number such that the union of all vertices in all paths in $\mathrm{G}(P)$ starting from v $[u]$ of length at most k_v $[k_u]$ are incident to at most $n/2$ facets. Clearly, there is a facet F of P so that we can reach F by a path of length k_v+1 from v and a path of length k_u+1 from u. We claim now that $k_v \le \Delta(d,[n/2])$. Indeed, let Q be the polyhedron obtained from P by ignoring all the inequalities that correspond to facets that cannot be reached from v by a path of length at most k_v. Let ω be a vertex in $\mathrm{G}(P)$ whose distance from v is k_v. We claim that the distance of ω from v in $G(Q)$ is also k_v. To see this consider the shortest path between v and ω in $G(Q)$. If the length of this path is smaller than k_v there must be an edge in the path that is not an edge of P. Consider the first such edge E. Since E is not an edge of P, it intersects a hyperplane H that corresponds to one of the inequalities that was ignored. This gives a path in P of length smaller than k_v from v to the facet of P determined by H, which is a contradiction.

We obtained that the distance from v to u is at most $\Delta(d-1,n-1)+2\Delta(d,[n/2])+2$. This gives the inequality $\Delta(d,n) \le \Delta(d-1,n-1)$ $+2\Delta(d,[n/2])+2$, which implies the statement of the theorem.

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