Mathematical Programming	**Homework 8**
OR 630 Fall 2005	**Due November 4, 2005**

1. An undirected graph is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite set of nodes and $\mathcal{E}$ a set of (unordered) pairs $\{i, j\}$ of nodes, called edges. The edge $\{i, j\}$ is said to be incident on i and j. The node-edge incidence matrix A of the graph has rows corresponding to each node and columns corresponding to each edge, with a $+1$ if the edge is incident on the node and a 0 otherwise.

a) Consider the *fractional node-covering problem*: find a set of nonnegative weights for the edges so that the total weight is minimized while the sum of the weights of the edges incident on each node is at least one. Show that this can be formulated as a linear programming problem whose coefficient matrix is the node-edge incidence matrix of the graph.

b) Show an example (it can be very small!) where the optimal solution is not integer-valued.

c) Suppose now that the graph is bipartite: $\mathcal{N}$ can be partitioned into $\mathcal{N}_1$ and $\mathcal{N}_2$ such that each edge is incident on one node in $\mathcal{N}_1$ and one in $\mathcal{N}_2$. Show that the linear programming problem in (a) can be written as a network flow problem and hence that it has an integer-valued optimal solution.

2. Consider the dual simplex algorithm for a network flow problem. So suppose you have the basic solution $\bar{x}$ corresponding to some spanning tree, and all reduced costs $\bar{c}_{jk}$ are nonnegative, but some basic variable, say the pth $\bar{x}_{hi}$, is negative. So we want to remove this variable from the basis, i.e., remove the arc (h, i) from the tree.

a) What happens to the spanning tree when arc (h, i) is removed?

b) In the dual simplex method, we want to choose some x_q to enter the basis where $\bar{a}_{pq}$ is negative. In our case, what arcs (j, k) have $\bar{a}_{p,(j,k)}$ negative, and what is $\bar{a}_{p,(j,k)}$ for such arcs?

c) Which arc is chosen by the minimum ratio test to enter the basis?

3. a) Show that a network flow problem can have a degenerate basic solution only if $\sum_{i \in I} b_i = 0$ for some proper subset I of nodes.

b) Consider a transportation problem with supplies s_i and demands d_j, all integer. Suppose there are m sources and n sinks, and we modify the supplies and demands as follows: $\hat{s}_i := ns_i$ for $i < m$, $\hat{s}_m := ns_m + n$; $\hat{d}_j = nd_j + 1$ for all j. Show that the new transportation problem has all basic solutions nondegenerate.

c) Suppose you have an optimal basic feasible solution for the modified problem, corresponding to a particular spanning tree. Show that the same spanning tree gives an optimal basic feasible solution (possibly degenerate) to the original problem. (Hint: express each basic variable in terms of the supplies and demands in part of the tree.)