Mathematical Programming	**Homework 4**
OR 630 Fall 2005	**Due September 30, 2005**

1. Here's another approach to proving strong duality. By the Fundamental Theorem of LP, (D) is either infeasible, or unbounded, or has an optimal solution y_*. The proof technique we used takes care of the first two cases: so let us suppose (D) has an optimal solution, with value ζ_*. This means that there is no feasible solution to $A^T y \leq c$, $b^T y > \zeta_*$. This is like system (II) in the Farkas Lemma, except that the right-hand sides are not all zero. Modify this system to get a new system like (II) that does not have a feasible solution. Now apply the Farkas Lemma to show that (P) has an optimal solution with value ζ_*.

2. (Strict Complementary Slackness) Suppose that (P) and (D) have optimal solutions, with objective values equal to ζ_*.

a) Show that the sets of optimal solutions of both (P) and (D) are convex sets.

b) By considering the LP problem

$$\min\{-e_j^T x : \, Ax = b, \, -c^T x \geq -\zeta_*, \, x \geq 0\},$$

show that either there is an optimal solution to (P) with its jth component positive, or there is an optimal solution to (D) with its jth inequality holding strictly.

c) Show that there are optimal solutions x_* and y_* to (P) and (D) so that, with $s_* := c - A^T y_*$, $s_* + x_* > 0$. (These are so-called strictly complementary solutions.)

3. We used the Farkas Lemma to prove strong duality. Suppose we had derived the Strong Duality Theorem another way. Show how you could prove the Farkas Lemma from it.

4. I claimed in class that strong duality fails for more general conic programming problems, where the nonnegative orthant is replaced by another closed convex cone. Here is an example, showing that optimal solutions may not exist even if the problem is feasible with bounded objective function value.

Recall that $C \bullet X$ denotes trace$(C^T X)$ for any equally-dimensioned matrices C and X. Also note that a 2×2 symmetric matrix is positive semidefinite iff its diagonal entries are nonnegative and its determinant is nonnegative.

Consider the problem

$$\begin{array}{lrcl} \min & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X & & \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X & = & 2, \\ & X & \text{is} & \text{positive semidefinite.} \end{array}$$

a) Show that this problem is feasible.

b) Show that any feasible solution has nonnegative objective function value.

c) Show that there are feasible solutions with arbitrarily small positive objective function value, but none with value 0.