Mathematical Programming	Homework 3
OR 630 Fall 2005	**Due September 23, 2005**

1. Find the polars of the following sets:

a) $B_\infty := \{x \in \mathbf{R}^n : \|x\|_\infty \leq 1\}$, $B_1 := \{x \in \mathbf{R}^n : \|x\|_1 \leq 1\}$, and $B_2 := \{x \in \mathbf{R}^n : \|x\|_2 \leq 1\}$;

b) the ellipsoid $\{Mx : x \in \mathbf{R}^n, \|x\|_2 \leq 1\}$, where M is a nonsingular matrix in $\mathbf{R}^{n\times n}$;

c) the linear subspace $\mathcal{N}(A) = \{x \in \mathbf{R}^n : Ax = 0\}$;

d) the affine subspace $\{x \in \mathbf{R}^n : Ax = b\}$, given that it is nonempty and contains the point w;

e) the cone $\{y \in \mathbf{R}^m : A^T y \leq 0\}$; and

f) the nonnegative orthant $\{y \in \mathbf{R}^m : y \geq 0\}$.

Also,

g) show that the polar of a convex cone $C \subseteq \mathbf{R}^n$ is $\{z \in \mathbf{R}^n : x^T z \leq 0 \text{ for all } x \in C\}$. (You may want to do this before cases (e) and (f) above.)

2 a) Show that, if there is a solution to $A^T y < 0$, $b^T y > 0$, then $\max\{b^T y : A^T y \leq c\}$ is (feasible and) unbounded.

b) Find an alternative system (as in the Farkas Lemma) to $A^T y < 0$, $b^T y > 0$.

3. Suppose $C \subseteq \mathbf{R}^n$ is closed and convex, and $x \in \partial C$, the *boundary* of C. This means that x lies in C, but there are points arbitrarily close to x that do not. Show that there is a supporting hyperplane to C at x, i.e., a nonzero $a \in \mathbf{R}^n$ with $a^T z \leq a^T x$ for all $z \in C$. (Hint: any sequence of points a_k in $\mathbf{R}^n$ all of Euclidean length 1 has a convergent subsequence.)

4. Give an example of a symmetric dual pair of problems (i.e., both are in inequality form with nonnegative variables) with $m = n = 1$ and with both problems infeasible. Show that an arbitrarily small perturbation of the data (A, b, c, all 1×1) can give problems where the primal is infeasible and the dual unbounded or vice versa.