Mathematical Programming	**Homework 2**
OR 630 Fall 2005	**Due September 16, 2005**

1. Suppose S is an arbitrary subset of $\mathbf{R}^n$. There are two plausible ways to define the *convex hull* $\mathrm{conv}(S)$ of S, by looking at convex combinations of elements of S or by looking at convex sets containing S. Prove that the set of all convex combinations of elements of S is equal to the intersection of all convex sets $C \subseteq \mathbf{R}^n$ with $S \subseteq C$ (and this is defined to be the convex hull of S).

2 (Carathéodory's theorem). Show that, if $x \in \mathbf{R}^n$ is a convex combination of $v_1, v_2, \ldots, v_k$, then it is also a convex combination of at most $n+1$ of these points.

3. Let Q be a nonempty pointed polyhedron in $\mathbf{R}^n$, and let $f : \mathbf{R}^n \to \mathbf{R}$ be *concave*: for all $x, y \in \mathbf{R}^n$ and all $0 \leq \lambda \leq 1$,

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y).$$

Suppose $\inf\{f(x) : x \in Q\}$ is attained; show that it is attained by a vertex of Q.

4. Consider the LP problem

$$\max_y \; b^T y, \quad a^T y \leq \gamma, \quad y \geq 0,$$

where $a, b, y \in \mathbf{R}^m$ and $\gamma > 0$. Suppose first that all components of the vectors a and b are positive.

a) Show that the feasible region is bounded. Find all its basic feasible solutions.

b) Using part (a), find an optimal solution of the problem. Give an economic interpretation of the solution in terms of "the biggest bang for the buck."

c) Write the dual LP problem, and solve it by inspection. Hence confirm that your solution in part (b) is indeed optimal.

d) Now consider the general case where the components of a and b can take any sign. Show how you can determine whether the LP problem above is bounded or not, and if it is, an optimal solution.