Mathematical Programming	**Homework 11**
OR 630 Fall 2005	**Due December 2, 2005**

1. a) Suppose that $E(z_+, B_+)$ is the minimum volume ellipsoid containing

$$\{x \in E(z, B) : a^T x \leq a^T z - \alpha(a^T Ba)^{\frac{1}{2}}\},$$

where $\alpha > -1/m$ and $0 \neq a \in \mathbf{R}^m$. Show that

$$a^T z - \alpha(a^T Ba)^{\frac{1}{2}} = a^T z_+ + \frac{1}{m}(a^T B_+ a)^{\frac{1}{2}},$$

i.e., the "depth" of the constraint that was used to make the cut is exactly $-1/m$ in the new ellipsoid.

b) Suppose we apply the ellipsoid method to try to find a point in

$$\{x \in \mathbf{R}^2 : x_1 \leq \frac{1}{2}, -x_1 \leq -\frac{1}{2}, -x_2 \leq -\frac{1}{4}, x_2 \leq \frac{1}{2}\},$$

starting with $E_0 := \{x \in \mathbf{R}^2 : \|x\| \leq 1\}$. At each iteration, we choose as the cut to define the new ellipsoid the constraint $a_i^T x \leq b_i$ with maximum depth

$$\alpha_i := \frac{a_i^T z - b_i}{(a_i^T B a_i)^{\frac{1}{2}}},$$

stopping if all α_i's are nonpositive, and using the deep cut method (i.e., the ellipsoid is updated as in (a)).

(i) What are the depths of all the constraints, and what cut is chosen, at the first iteration?

(ii) What are the depths of all the constraints, and what cut is chosen, at the second iteration?

2. Let $A \in \mathbf{R}^{m \times n}$ have rank m, and let $P_A := I - A^T(AA^T)^{-1}A$.

a) Show that $P_A = P_A^T = P_A^2$ and hence that $u^T P_A u = \|P_A u\|^2$ for every $u \in \mathbf{R}^n$. (So P_A is positive semidefinite: $u^T P_A u \geq 0$ for all u.)

b) Show that $P_A v = 0$ for every v in the range space of A^T, and $P_A v = v$ for every v in the null space of A.

3. Consider the standard-form LP problem and its dual, where $A \in \mathbf{R}^{m \times n}$ has rank m, and suppose $x \in \mathcal{F}^0(P)$ and $(y, s) \in \mathcal{F}^0(D)$. Let $\mu = x^T s/n$, and suppose that $x_j s_j \geq \gamma\mu$ for all j, for some positive γ. Suppose $(\Delta x, \Delta y, \Delta s)$ is the solution to

$$\begin{array}{rcrcl} & & A^T\Delta y \;+ & \Delta s & = 0, \\ A\Delta x & & & & = 0, \\ S\Delta x & & + & X\Delta s & = \sigma\mu e - XSe, \end{array}$$

for some $0 \leq \sigma \leq 1$. Let $(x(\alpha), y(\alpha), s(\alpha)) := (x, y, s) + \alpha(\Delta x, \Delta y, \Delta s)$ for $0 \leq \alpha \leq 1$.

a) Show that $\Delta x^T \Delta s = 0$ and that $\mu(\alpha) := x(\alpha)^T s(\alpha)/n = (1 - \alpha + \alpha\sigma)\mu$.

b) Let $\bar{\alpha} := \max\{\hat{\alpha} \in [0, 1] : X(\alpha)S(\alpha)e \geq \gamma\mu(\alpha)e \text{ for all } \alpha \in [0, \hat{\alpha}]\}$, and let $(x_+, y_+, s_+) := (x(\bar{\alpha}), y(\bar{\alpha}), s(\bar{\alpha}))$. Show that either x_+ is optimal in (P) and (y_+, s_+) in (D), or $x_+ \in \mathcal{F}^0(P)$ and $(y_+, s_+) \in \mathcal{F}^0(D)$, with only the second possibility if $\sigma > 0$.