Mathematical Programming	Homework 10
OR 630 Fall 2005	**Due November 22, 2005**

1. a) Let us consider the following modified cutting-stock problem. Instead of a demand for exactly b_i rolls of width w_i, the customer will accept any amount between $.9b_i$ and $1.1b_i$. Also, a total of M large rolls of width W are available. Suppose the profit for each roll of width w_i sold is p_i. If a_j, $j = 1, \ldots, N$, denote the possible cutting patterns, formulate an integer linear programming problem to maximize profit while staying within the sales limits and respecting the limit on large rolls.

b) Explain how you could use column generation to solve the linear programming relaxation of the problem in (a).

2. Suppose you wish to solve the (large-scale) linear programming problem

$$\begin{array}{llcccccl} \min & c_1^T x_1 & + & c_2^T x_2 & + & d^T w & & \\ & A_1 x_1 & & & + & F_1 w & \geq & b_1, \\ (P) & & & A_2 x_2 & + & F_2 w & \geq & b_2, \\ & x_1, & & x_2 & & & \geq & 0. \end{array}$$

Think of this as the problem of a corporation with two divisions, where the corporate variables w impact the constraints of each division.

a) Take the dual of (P), and discuss how you would apply Dantzig-Wolfe decomposition to it. In particular, derive the subproblems that would be solved at each iteration and interpret their duals.

b) Now consider (P) itself, and write the constraints as two separate systems, one involving x_1 and a new vector w_1 and the other x_2 and a new vector w_2, with a constraint linking w_1 and w_2. The result should be a problem in block-angular form, with a "small" number of linking constraints. Discuss how you would apply Dantzig-Wolfe decomposition to this. In particular, derive the subpreoblems that would be solved at each iteration.

c) Compare the "resource-directed" decomposition of (a) (why is it called this?) and the "price-directed" decomposition of (b) (why is it called that?).

3. The *standard simplex* in $\mathbf{R}^m$ is the set $S^m := \{x \in \mathbf{R}^m : e^T x \leq 1, \, x \geq 0\}$, where $e \in \mathbf{R}^m$ is the vector of ones.

a) Consider the simplex $\{x \in \mathbf{R}^m : a^T x \leq 1, \, x \geq 0\}$, where $a \in \mathbf{R}^m$ is a positive vector. Show that there is a nonsingular linear transformation taking this into S^m. (In fact, there is a nonsingular affine transformation taking *any* m-dimensional simplex in $\mathbf{R}^m$ into S^m.)

b) Consider the hyperplane $\{x \in \mathbf{R}^m : e^T x = m/(m+1)\}$ through the centroid (center of gravity) $\bar{x} := e/(m+1)$ of S^m. Show that this hyperplane cuts the simplex into two pieces, with the piece containing the origin having volume $(m/(m+1))^m$ times that of S^m. Show that this is at least $\exp(-1)$.

In fact, *any* hyperplane through the center of gravity of *any* convex body (compact subset with nonempty interior) of $\mathbf{R}^m$ cuts it into two pieces, and each has volume at least $(1-\exp(-1))$ times that of the body itself. (You don't have to prove that!) This constant volume reduction compares to the much slower reduction in the ellipsoid method.