Mathematical Programming	**Homework 1**
OR 630 Fall 2005	**Due September 8, 2005**

1. a) Consider the LP problem

$$\begin{array}{rcrcl} \min & c_w^T w & + & c_x^T x & \\ & A_w w & + & A_x x & = & b, \\ & & & x & \geq & 0. \end{array}$$

Suppose we want to convert this to a problem in standard form, i.e., with all nonnegative variables. The usual way to do this is to replace each unrestricted variable by the difference between two nonnegative variables: this doubles the number of such variables. Devise another technique to obtain an equivalent standard form problem where the number of variables is only increased by one. (Hint: $(-5;-7) = (2;0) - 7(1;1)$.)

b) Consider the LP problem

$$\begin{array}{rrcl} \max & b^T y & & \\ & A_w^T y & = & c_w, \\ & A_x^T y & \leq & c_x. \end{array}$$

We want to convert this into a problem in the form of the dual to a standard form problem, i.e., with all less-than-or-equal-to constraints. The usual way to do this is to replace each equality constraint by two inequality constraints, but this doubles the number of such constraints. Devise another technique that only increases the number of constraints by one. What is the relationship between this technique and that in (a)?

c) Extend the relationship between a primal minimization problem and its dual to allow less-than-or-equal-to constraints and nonpositive variables in the primal problem. Explain your correspondences by examining the chain of inequalities demonstrating weak duality.

2. Formulate $\min \|Aw - b\|_\infty$ as an LP problem. Write its dual and simplify it, to obtain dual constraints involving the 1-norm.

3. Consider the LP problem $\min\{c^T x : Ax = b\}$, i.e., with equality constraints and unrestricted variables. State the dual of this problem. Show that strong duality holds: if either problem has an optimal solution, so does the other and their optimal values are equal. What can you say about optimal solutions of these problems? [This should convince you that *linear* programming problems with no inequality constraints are uninteresting.]

4. Let $\mathcal{S}^n$ denote the space of real symmetric $n \times n$ matrices. This is a finite-dimensional vector space: indeed, by considering just the upper-triangular entries or by taking an appropriate basis, it can be viewed as isomorphic to $\mathbf{R}^{n(n+1)/2}$. A matrix $A \in \mathcal{S}^n$ is called *positive semidefinite* if $x^T A x \geq 0$ for all $x \in \mathbf{R}^n$.

Show that the set of positive semidefinite real symmetric $n \times n$ matrices is a convex cone containing the origin. You can do this directly, or by showing that it is an (infinite) intersection of half-spaces corresponding to hyperplanes through the origin.