OR 630: Mathematical Programming I. Fall 2005.

Comments on the final exam.

1. a) Let $\hat{x}$ be the solution to (1) with all components positive, and let B be any basis matrix for A, which is given to have rank m. Then $\hat{x}_B = B^{-1}b - B^{-1}N\hat{x}_N > 0$. Define $\hat{x}(\epsilon)$ by $\hat{x}_N(\epsilon) := \hat{x}_N$ and $\hat{x}_B(\epsilon) := B^{-1}b(\epsilon) - B^{-1}N\hat{x}_N$. Then $\hat{x}_N(\epsilon) > 0$, $A\hat{x}(\epsilon) = b(\epsilon)$, and $\hat{x}_B(\epsilon) \rightarrow \hat{x}_B$ as $\epsilon \rightarrow 0$. So for all sufficiently small positive ϵ, say for $0 < \epsilon \leq \bar{\epsilon}$, $\hat{x}(\epsilon)$ is positive and hence feasible for (2). (Note: it is possible that all basic feasible solutions to (1) are degenerate, but that such a $\hat{x}$ exists: e.g., consider the assignment problem.)

b) Consider every basis matrix B such that the corresponding basic solution to (1) is *not* feasible. For any such B, there is some index i such that $(B^{-1}b)_i$ is negative, so there is some positive $\hat{\epsilon}_B$ such that $(B^{-1}b(\epsilon))_i$ is negative for $0 < \epsilon \leq \hat{\epsilon}_B$. There are only a finite number of basis matrices, so only a finite number of such B's: let $\hat{\epsilon} > 0$ be the minimum of these $\hat{\epsilon}_B$'s. Then for $0 < \epsilon \leq \hat{\epsilon}$, if $B^{-1}b(\epsilon)$ is nonnegative, then B cannot be any of these infeasible bases, so $B^{-1}b$ is nonnegative also.

c) $\bar{x}$ is an extreme point of the feasible region of (1), so it is a vertex of this region, and there is some c so that $\bar{x}$ is the unique solution of the optimization problem $\min\{c^T x : Ax = b, x \geq 0\}$. Then there is an optimal dual solution, say $\hat{y}$.

Now choose any $0 < \epsilon \leq \tilde{\epsilon}$, where $\tilde{\epsilon} := \min\{\bar{\epsilon}, \hat{\epsilon}\}$, and consider the optimization problem $\min\{c^T x : Ax = b(\epsilon), x \geq 0\}$. By (a) this has a feasible solution, and $\hat{y}$ is a feasible solution to its dual, so it has an optimal solution, and an optimal basic feasible solution, say corresponding to the optimal basis matrix B. Then the corresponding dual solution $\bar{y}$ has $c_B - B^T\bar{y} = 0$, $c_N - N^T\bar{y} \geq 0$. By (b), the basis matrix B corresponds to a basic feasible solution, say $\hat{x}$, to (1). Then $\hat{x}$ and $\bar{y}$ are feasible for the original optimization problem and its dual, and satisfy complementary slackness, so $\hat{x}$ is optimal and hence equal to $\bar{x}$. Hence B is the desired basis matrix.

(Note: (a) to (c) show the relationship between basic feasible solutions to (1) and to (2), and this can be used to show that the maximum diameter of a polyhedron with certain m and n can be realized by one with no degenerate basic feasible solutions.)

2. We want feasible x and (y,s) to the perturbed (P) and (D) with the duality gap $x^T s$ "small." Note that the solutions given could have been obtained by the simplex method (before completion, maybe), the ellipsoid method, or an interior-point method.

a) If c_j becomes $c_j + \Delta$, we can correct feasibility in the equations by changing s_j to $\bar{s}_j := s_j + \Delta$, keeping x and y unchanged. Then we are still feasible as long as $\bar{s}$ is nonnegative, i.e., as long as $\Delta \geq -s_j$, and the new duality gap is $x^T \bar{s} = x^T s + x_j \Delta \leq \epsilon + x_j \Delta$.

So we have ϵ-optimal solutions for $-s_j \leq \Delta \leq 0$ ($-s_j \leq \Delta$ if $x_j = 0$), and $(\epsilon + \eta)$-optimal solutions for $0 \leq \Delta \leq \eta / x_j$.

Note that $x_j s_j \leq \epsilon$, so if $x_j > 0$, $\epsilon / x_j \geq s_j$. So we get (2ϵ)-optimal solutions for $-s_j \leq \Delta \leq s_j$, a reasonable range if s_j is large.

b) Now $\hat{A}\hat{x} \geq b, \hat{x} \geq 0$ is converted to $Ax := [\hat{A}, -I](\hat{x}; t) = b, x \geq 0$. If b_i becomes $b_i + \Delta$, we can correct feasibility in the equations by changing t_i to $\bar{t}_i := t_i - \Delta$, keeping $\hat{x}$, y, and s unchanged. Then we are still feasible as long as $\Delta \leq t_i$, and the new duality gap is $\bar{x}^T s = x^T s - \Delta s_k$, where $k := n - m + i$.

So we have ϵ-optimal solutions for $0 \leq \Delta \leq t_i$ ($\Delta \leq t_i$ if $s_k = 0$), and $(\epsilon + \eta)$-optimal solutions for $-\eta / s_k \leq \Delta \leq 0$.

Again, $t_i s_k \leq \epsilon$, so if $s_k > 0$, $\epsilon / s_k \geq t_i$. So we get (2ϵ)-optimal solutions for $-t_i \leq \Delta \leq t_i$, a reasonable range if t_i is large.

(So we can do some sensitivity analysis without basis matrices if we have both primal and dual near-optimal solutions. We can't do much if we only have a primal near-optimal solution or a dual one, so primal-dual methods are more useful. Also, while it seems that a basis matrix can give much more information, this can be unreliable if the basis is primal or dual degenerate.)

3. a) We need primal and dual feasibility and complementary slackness:

$$Ax = b, x \geq 0;$$

$$A^T y + s = c, \quad s \geq 0;$$

$$XSe = 0, \text{ or } x_j s_j = 0, \text{ all } j \text{ , or } c^T x = b^T y, \text{ or } c^T x \leq b^T y.$$

b) The primal simplex method maintains primal feasibility and complementary slackness, but relaxes dual feasibility (or just $s \geq 0$) until optimality is reached. It strives for dual feasibility.

The dual simplex method maintains dual feasibility and complementary slackness, but relaxes primal feasibility (or just $x \geq 0$) until optimality is reached. It strives for primal feasibility.

Primal-dual path-following methods maintain primal and dual feasibility (strictly), but relax the complementary slackness condition $XSe = 0$ to $XSe \approx \mu e$ for positive μ and strive for $\mu = 0$.

4. a) Set $x = e$, $y = 0$, $\tau = 1$, and $\theta = 1$ and check all the equations and (strict) inequalities.

b) Let the dual variables be w, z, λ, and μ corresponding to the first, second, third, and fourth (sets of) constraints. Then writing down all the appropriate constraints (with $\leq$ and equality constraints) and nonnegativities, we find that the feasible region of the dual of (H) is *exactly the same* as that of (H) if we identify w with x, z with y, λ with τ, and μ with θ. Then the dual objective to maximize $-(n+1)\mu$ is equivalent to minimizing $(n+1)\mu$, so that the dual of (H) is *completely equivalent* to (H) itself (we say (H) is "self-dual").

c) Since (H) and its dual have feasible solutions by (a) and (b), they both have optimal solutions by strong duality. If (x, y, τ, θ) is optimal for (H), it is also optimal for its dual by (b), and since the objective values are equal, $(n+1)\theta = -(n+1)\theta$ so $\theta = 0$. So the optimal value is 0.

d) Let (x, y, τ, θ) be optimal with $\tau > 0$ and $\theta = 0$ by (c). Let $\bar{x} := x/\tau$ and $\bar{y} = y/\tau$. Then $-A^T y + c\tau + 0 \geq 0$ implies $A^T y \leq c\tau$ and so $A^T \bar{y} \leq c$.

Next, $Ax - b\tau + 0 = 0$ implies $Ax = b\tau$ so $A\bar{x} = b$. Also, $x \geq 0$ and $\tau > 0$ imply $\bar{x} \geq 0$.

Finally, $-c^T x + b^T y + 0 \geq 0$ implies $c^T x \leq b^T y$ and so $c^T \bar{x} \leq b^T \bar{y}$.

So $\bar{x}$ is feasible in (P), $\bar{y}$ is feasible in (D), and $c^T \bar{x} \leq b^T \bar{y}$ shows with weak duality that $\bar{x}$ and $\bar{y}$ are optimal in (P) and (D) respectively.

e) Now let (x, y, τ, θ) be optimal with $\tau = 0$, $-c^T x + b^T y + \bar{\zeta}\theta > 0$ (and $\theta = 0$ by (c)). Then $c^T x < b^T y$, and so either $c^T x < 0$ or $c^T x \geq 0$ and then $b^T y > 0$.

In the second case we have $-A^T y + 0 + 0 \geq 0$ and $b^T y > 0$, so $A^T y \leq 0$ and $b^T y > 0$, which shows that (P) is infeasible by the Farkas Lemma.

In the first case we have $Ax + 0 + 0 = 0$, $x \geq 0$, and $c^T x < 0$, so $Ax = 0$, which shows that (D) is infeasible by a corollary to the Farkas Lemma.

(Note that variations of primal-dual interior-point methods can find (approximations to) optimal solutions to (H) that either have $\tau > 0$ or $\tau = 0$ but $-c^T x + b^T y > 0$.)