SOLUTION SET

Thanks to

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Homework #7
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10p.

1. Solve the following LP problem by the dual simplex method, starting with the all-surplus basis:

$$\begin{array}{lrcrcrcrcrcrcl} \min & 10x_1 & + & 5x_2 & + & 5x_3 & & & & & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & - & x_4 & & & & & = & 5, \\ & x_1 & + & 3x_2 & + & x_3 & & & - & x_5 & & & = & 5, \\ & x_1 & + & x_2 & + & 4x_3 & & & & & - & x_6 & = & 2, \\ & & & & & & & & & & & x & \geq & 0. \end{array} \qquad (1)$$

This is the example of the lecture notes of September 22nd., with the objective function coefficients changed to make it interesting!
Use Bland's least-index rule. Generate the information you need as required using the basis inverse, rather than using the updated equations or tableaus.

We begin with the all-surplus basis, $\beta = \{4,5,6\}$, which gives

$$\begin{aligned} B &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \\ B^{-1} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ x_B &= \bar{b} = B^{-1}b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}, \end{aligned}$$

so we could choose any of our basis elements to leave the basis. Using least-index rule, we will choose x_4 to leave the basis. Since x_4 corresponds to the first element in β, to figure out which variable will enter the basis we calculate

$$\begin{aligned} \bar{y} &= B^{-T}c_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \bar{c}_N &= c_N - N^T\bar{y} = \begin{pmatrix} 10 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 5 \end{pmatrix}, \end{aligned}$$

and so

$$N^T B^{-T} e_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix},$$

$$\frac{\bar{c}_1}{-\bar{a}_{11}} = \frac{10}{2} = 5, \quad \frac{\bar{c}_2}{-\bar{a}_{12}} = \frac{5}{1} = 5, \quad \frac{\bar{c}_3}{-\bar{a}_{13}} = \frac{5}{1} = 5,$$

so we could also choose any of our non-basic variables to enter the basis. Using the least-index rule, we will choose x_1 to enter, so $\beta = \{1, 5, 6\}$, and updating B^{-1} similarly to how we did when using the revised simplex method, we obtain

$$\begin{aligned} B &= \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \\ B^{-1} &= \begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & -1 & 0 \\ 0.5 & 0 & -1 \end{pmatrix}, \\ \bar{x}_{\mathcal{B}}^{+} &= \bar{b} = B^{-1} b = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & -1 & 0 \\ 0.5 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2.5 \\ 0.5 \end{pmatrix}, \end{aligned}$$

so we must choose x_5 to leave the basis. Since x_5 corresponds to the second element in β, to figure out which variable will enter the basis we calculate

$$\begin{aligned} \bar{y} &= B^{-T} c_B = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \\ \bar{c}_N &= c_N - N^T \bar{y} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

and so

$$N^T B^{-T} e_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0.5 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -2.5 \\ -0.5 \end{pmatrix},$$

$$\frac{\bar{c}_4}{-\bar{a}_{24}} = \frac{5}{0.5} = 10, \quad \frac{\bar{c}_2}{-\bar{a}_{22}} = \frac{0}{2.5} = 0, \quad \frac{\bar{c}_3}{-\bar{a}_{23}} = \frac{0}{0.5} = 0,$$

so we could also choose either x_2 or x_3 to enter the basis. Using the least-index rule, we will choose x_2 to enter, so $\beta = \{1,2,6\}$, and updating B^{-1} once again, we obtain

$$
\begin{aligned}
B &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \\
B^{-1} &= \begin{pmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ 0.4 & 0.2 & -1 \end{pmatrix}, \\
x_{B}^{+} &= \bar{b} = B^{-1}b = \begin{pmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ 0.4 & 0.2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},
\end{aligned}
$$

so since $\bar{b} > 0$ we have an optimal solution, with objective function value 25. ■

10 pt.

2. Consider the upper-bounded problem

$$
\begin{aligned}
\min\ & c^T x \\
Ax &= b, \\
0 \le x &\le u,
\end{aligned}
\tag{2}
$$

where A is $m \times n$ and has rank m. Suppose we write this as a standard form problem

$$
\begin{aligned}
&\min\ c^T x \\
&\quad Ax \qquad\quad = b, \\
(P) &\quad x \ + \ w = u, \\
&\quad x, \quad\ \ w \ge 0,
\end{aligned}
\tag{3}
$$

with $m+n$ equality constraints. Then a basis matrix $\bar{B}$ will correspond to the choice of $m+n$ basic indices and variables. Let β consist of those j's in $\{1,2,\ldots,n\}$ with both x_j and w_j basic; λ those j's with w_j but not x_j basic; and ν those j's with x_j but not w_j basic.

a) Show that $|\beta| = m$, and let B be the corresponding submatrix of A. Also, let N_U denote the submatrix of A corresponding to ν.

b) Write $\bar{B}$ in terms of B and N_U. For simplicity, assume that $\nu = \{1,2,\ldots,k\}, \lambda = \{k+1,k+2,\ldots,n-m\}$, and $\beta = \{n-m+1,n-m+2,\ldots,n\}$, and that the basic variables are listed in the order $x_j, j \in \beta, x_j, j \in \nu, w_j, j \in \lambda$, and $w_j, j \in \beta$.

c) Obtain $\bar{B}^{-1}$ in terms of B^{-1} and N_U. (This shows how the simplex method for (P) could be efficiently implemented using only B^{-1}. But of course we can also just use the bounded-variable simplex method.)

a) If there are exactly $|\beta|$ indices, j, where x_j and w_j are both basic, then this accounts for $2|\beta|$ of the basic variables. Since there must be a total of $n+m$ basic variables, then there are $n+m-2|\beta|$ basic variables with index not in β. Note that for each index j of the remaining $n - |\beta|$ indices not in β, we cannot have both x_j and w_j basic. Thus there are at most $n - |\beta|$ remaining basic variables, so

$$n + m - 2|\beta| \leq n - |\beta| \quad \Rightarrow \quad |\beta| \geq m.$$

Also due to the $x_j + w_j = u_j$ constraints, we need at least one of x_j or w_j to be nonbasic for each index j. Having accounted for $|\beta|$ of these constraints, we require at least $n - |\beta|$ more basic variables, so

$$n + m - 2|\beta| \geq n - |\beta| \quad \Rightarrow \quad |\beta| \leq m,$$

therefore $|\beta| = m$.

b) The coefficient matrix $\bar{A}$ of this LP is of the form

$$\begin{bmatrix} A & \mathbf{0}_{m\times n} \\ I_{n\times n} & I_{n\times n} \end{bmatrix}$$

where $\mathbf{0}_{m\times n}$ is the $m \times n$ matrix of all zeroes. We can express $\bar{B}$ using the convention defined in the problem statement as

$$\bar{B} = \left[\begin{pmatrix} B \\ \mathbf{0}_{(n-m)\times m} \\ I_{m\times m} \end{pmatrix} \begin{pmatrix} N_U \\ I_{k\times k} \\ \mathbf{0}_{(n-k)\times k} \end{pmatrix} \begin{pmatrix} \mathbf{0}_{(m+k)\times(n-k)} \\ I_{(n-k)\times(n-k)} \end{pmatrix} \right]$$

c) By inspection we find that $\bar{B}^{-1}$ is given by

$$\bar{B}^{-1} = \left[\begin{pmatrix} B^{-1} \\ \mathbf{0}_{n+m\times m} \\ -B^{-1} \end{pmatrix} \begin{pmatrix} -B^{-1}N_U \\ I_{k\times k} \\ B^{-1}N_U \end{pmatrix} \begin{pmatrix} \mathbf{0}_{(m+k)\times(n-k)} \\ I_{(n-k)\times(n-k)} \end{pmatrix} \right]$$

■

10 pt.

3. Consider a 3×3 transportation problem, with all the supplies and demands equal to 1 (also known as an assignment problem), with costs given by the matrix

$$C = (c_{ij}) := \begin{bmatrix} 4 & 6 & 3 \\ 5 & 4 & 2 \\ 4 & 3 & 2 \end{bmatrix}.$$

a) Suppose the current basic variables are $x_{11}, x_{12}, x_{22}, x_{23}$, and x_{32}. Write down the equations determining the basic solution, omitting the equation for demand 3, with the rows and columns ordered so that the coefficient matrix is triangular. Hence determine the basic solution.

b) Determine the corresponding dual solution. Are the optimality conditions satisfied?

c) Draw the tree corresponding to the basic solution in (a). Suppose you want to increase x_{13}. What is the pattern of changes to the basic variables?

d) Do (a) and (b) again, where x_{13} becomes a basic variable in place of x_{12}. Comment on the differences between your answers regarding the optimality conditions.

a) We have

$$
\begin{array}{rrrrrrcr}
-x_{11} & & & & & & = & -1 \\
x_{11} & +x_{12} & & & & & = & 1 \\
 & & x_{32} & & & & = & 1 \\
 & -x_{12} & -x_{32} & -x_{22} & & & = & -1 \\
 & & & x_{22} & +x_{23} & & = & 1
\end{array}
$$

so the current basic solution is

$$
\begin{bmatrix} x_{11} \\ x_{12} \\ x_{32} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
$$

which has objective function value 9.

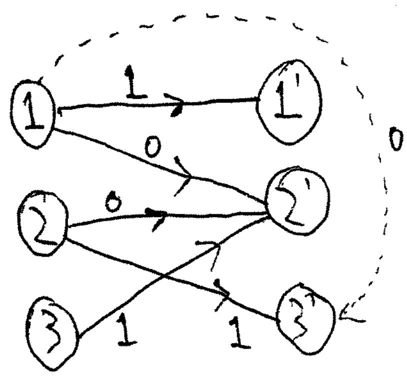
b) From the ordering of our primal variables, we obtain the corresponding dual solution

$$
\begin{bmatrix} w_1 \\ u_1 \\ u_3 \\ w_2 \\ u_2 \end{bmatrix} = B^{-T} c_B = \begin{bmatrix} -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ -2 \\ 2 \end{bmatrix}
$$

which has objective function value 5,

but $\overline{c_{13}} = c_{13} - u_1 + w_3 = 3 - 4 + 0 = -1 < 0$
so not optimal.

c)



Intuitively,
If we try and increase x_{13}, we would have to decrease x_{11} or x_{12}. Decreasing x_{11} is a problem because neither x_{21} nor x_{31} are basic variables, so we could not increase any basic variables to compensate. Decreasing x_{12} is a problem because it is already at 0. This means that if x_{13} is added to the basis, it will continue to have value 0, and x_{12} will be removed.

I.e. to increase X_{13}, we want to change flows on path from 1 to 3′ in tree which means decrease flow on 12′, increase on 22′, decrease on 23′.

d) We have

$$\begin{array}{rrrrrcr} -x_{11} & & & & & = & -1 \\ x_{11} & +x_{13} & & & & = & 1 \\ & & x_{32} & & & = & 1 \\ & & -x_{32} & -x_{22} & & = & -1 \\ & & & x_{22} & +x_{23} & = & 1 \end{array}$$

so this basic solution is

$$\begin{bmatrix} x_{11} \\ x_{13} \\ x_{32} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

which has objective function value 9. The corresponding dual solution is given by

$$\begin{bmatrix} w_1 \\ u_1 \\ u_3 \\ w_2 \\ u_2 \end{bmatrix} = B^{-T} c_B = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \\ 2 \end{bmatrix}$$

which has objective function value 9, Even though this is essentially the same basic solution as in part (a), the change of basis allows for a better dual solution which proves optimality. ■

$$\bar{c}_N = c_N - N^T \bar{y} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \geq 0 \quad \rightarrow \text{optimal.}$$