

ORIE 630 HOMEWORK 6

SOLUTION SET

I. PROBLEM 1

We will use the revised simplex method to solve the following LP problem.

$$\begin{aligned} \min_x \quad & -6x_1 - 7x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 8, \\ & 3x_1 + x_2 + x_4 = 9, \\ & x \geq 0. \end{aligned}$$

Choose $\beta = \{3, 4\}$ and $\nu = \{1, 2\}$, so that starting with the following known matrices/vectors:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then the followings can be obtained:

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{y} = B^{-T}c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{b} = B^{-1}b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \quad \text{and } \bar{\zeta} = c_B^T \bar{b} = 0.$$

Iteration 1: Start with basic feasible solution $(0, 0, 8, 9)$ and basic variable x_3, x_4 , nonbasic variable x_1, x_2 .

Step 1: Obtain

$$\bar{c}_N = c_N - N^T \bar{y} = \begin{pmatrix} -6 \\ -7 \end{pmatrix}, \quad \text{so } \bar{c} = \begin{pmatrix} -6 \\ -7 \\ 0 \\ 0 \end{pmatrix}$$

Then, choose x_2 to enter the basis, and let $q = 2$.

Step 2: None

Step 3: $q = 2$ such that $\bar{c}_q < 0$. Then

$$\bar{a}_2 = B^{-1}a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} > 0.$$

So we have not met unboundness criteria.

Step 4:

$$\frac{\bar{b}_1}{\bar{a}_{1,2}} = 4, \quad \frac{\bar{b}_2}{\bar{a}_{2,2}} = 9.$$

then use $\frac{\bar{b}_1}{\bar{a}_{1,2}}$ which makes x_3 a leaving variable and $p=1$.

Step 5: Update

$$\begin{aligned} B_+^{-1} &= \left(I - \frac{(\bar{a}_q - e_p)e_p^T}{\bar{a}_{pq}} \right) B^{-1} \\ &= \left(I - \frac{(\bar{a}_2 - e_1)e_1^T}{\bar{a}_{1,2}} \right) B^{-1} \\ &= \left(I - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \right) I = \begin{pmatrix} 0.5 & 0 \\ -0.5 & 1 \end{pmatrix} \end{aligned}$$

$$\bar{b}_+ = B_+^{-1}b = \begin{pmatrix} 0.5 & 0 \\ -0.5 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

$$\zeta_+ = c_{B+}^T \bar{b}_+ = \begin{pmatrix} -7 \\ 0 \end{pmatrix}^T \begin{pmatrix} 4 \\ 5 \end{pmatrix} = -28, \bar{y}_+ = \bar{y} + \frac{\bar{c}_2}{\bar{a}_{1,2}} B^{-T} e_1 = -\frac{7}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e_1 = \begin{pmatrix} -3.5 \\ 0 \end{pmatrix}$$

Iteration 2: With basic variable x_2, x_4 and nonbasic variable x_1, x_3 .

Step 1: Obtain

$$\bar{c}_N = c_N - N^T \bar{y} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}^T \begin{pmatrix} -3.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix}$$

$$\bar{c} = \begin{pmatrix} -2.5 \\ 0 \\ 3.5 \\ 0 \end{pmatrix}$$

Then choose x_1 to enter, and yet $q = 1$.

Step 2: None

Step 3: We have $q = 1$ such that $\bar{c}_1 < 0$, then

$$\bar{a}_1 = B^{-1}a_1 = \begin{pmatrix} 0.5 & 0 \\ -0.5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2.5 \end{pmatrix}$$

Step 4:

$$\frac{\bar{b}_1}{\bar{a}_{1,1}} = 8, \frac{\bar{b}_2}{\bar{a}_{2,1}} = 2.$$

Therefore, $p=2$ and which makes x_4 a leaving variable.

Step 5:

$$B_+^{-1} = (I - \frac{(\bar{a}_1 - e_2)e_2^T}{\bar{a}_{2,1}})B^{-1} = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2.5} \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} (\begin{pmatrix} 0 & 1 \end{pmatrix})) \begin{pmatrix} 0.5 & 0 \\ -0.5 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{pmatrix}$$

$$\bar{b}_+ = B_+^{-1}b = \begin{pmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{pmatrix} \begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\zeta_+ = \bar{c}_{B+}^T \bar{b}_+ = \begin{pmatrix} -7 \\ -6 \end{pmatrix}^T \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -33, \bar{y}_+ = \bar{y} + \frac{\bar{c}_1}{\bar{a}_{2,1}} B^{-T} e_2 = \begin{pmatrix} -3.5 \\ 0 \end{pmatrix} + \frac{-2.5}{2.5} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

Iteration 3: With basic variable x_2, x_1 and nonbasic variable x_4, x_3 .

Step 1: Obtain

$$\bar{c}_N = c_N - N^T \bar{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

which shows the optimality of LP problem. The result should be

$$x_1 = 2, x_2 = 3, x_3 = 0, x_4 = 0$$

with objective function value $\zeta = -33$.

Question 2:

$$\begin{array}{l} \text{min } 4x_1 + 7x_2 + 5x_3 \\ \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 3 & 1 & 5 \\ -1 & 1 & 4 & 2 \end{array} \right) \xrightarrow{-I} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 3 & 1 & 5 \\ 0 & 2 & 3 & 3 \end{array} \right) \xrightarrow{\text{Row } 3 \rightarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 3 & 1 & 5 \\ 0 & 2 & 3 & 3 \end{array} \right) \end{array}$$

$$\xrightarrow{\text{Row } 1 \rightarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & -1 & -1 & 5 \\ 1 & 3 & 1 & 5 \\ 0 & 2 & 3 & 3 \end{array} \right) \xrightarrow{\text{Row } 2 \rightarrow R_2 + 3R_1} \left(\begin{array}{ccc|c} 0 & -1 & -1 & 5 \\ 0 & 2 & 0 & 16 \\ 0 & 2 & 3 & 3 \end{array} \right) \xrightarrow{\text{Row } 3 \rightarrow R_3 - R_2} \left(\begin{array}{ccc|c} 0 & -1 & -1 & 5 \\ 0 & 2 & 0 & 16 \\ 0 & 0 & 3 & -13 \end{array} \right)$$

$$c_B = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{3}{5} & -\frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -1 \end{pmatrix}$$

$$B^{-1} c_B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \bar{y}$$

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\sqrt{B} u = -\frac{2}{5}$$

$$\tilde{B} = B + \lambda \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{B}^{-1} = B^{-1} - \frac{\lambda}{1 - \frac{2}{5}\lambda} \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 0 \end{pmatrix}^T$$

\tilde{B}^{-1} is a matrix if $1 - \frac{2}{5}\lambda \neq 0, \lambda \neq \frac{5}{2}$. (3)

$$\tilde{B}^{-1} b = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} - \mu \cdot 2 \cdot \begin{pmatrix} 4/5 \\ -2/5 \\ 1/5 \end{pmatrix}$$

≥ 0 as long as $\mu \leq \frac{25}{8}$ and
 $\mu \geq -\frac{25}{4}$

$$\mu = \frac{\lambda}{1 - \gamma_5 \lambda}$$

$$\text{if } \lambda < \frac{5}{2}, \quad \lambda \leq \frac{25}{8} - \frac{5}{4}\lambda \quad \lambda \leq \frac{25}{18}$$

$$\lambda \geq -\frac{25}{4} + \frac{5}{2}\lambda \quad \lambda \leq \frac{25}{6}$$

$$\text{if } \lambda > \frac{5}{2} \quad \lambda \geq \frac{25}{8} - \frac{5}{4}\lambda, \quad \lambda \geq \frac{25}{18}$$

$$\lambda \leq -\frac{25}{4} + \frac{5}{2}\lambda, \quad \lambda \geq \frac{25}{6}$$

so primal feasible if $\lambda \leq \frac{25}{18}$
or if $\lambda \geq \frac{25}{6}$.

Also, $\tilde{B}^{-1} c_B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \mu \cdot 4 \cdot \begin{pmatrix} 3/5 \\ -1/5 \\ 0 \end{pmatrix}$

\tilde{y}^{II}

$$\tilde{c}_1 = 0 - 4\lambda + 4\mu \cdot (6_5 - 1_5 - 2_5\lambda) \\ = -4\lambda + \frac{4\lambda}{(-2_5\lambda)} ((-2_5\lambda)) = 0 \quad \checkmark$$

$$\tilde{c}_2 = \dots 0$$

$$\tilde{c}_3 = 2 - 2\lambda + 4\mu \left(\frac{3}{5} - 1_5 - 2_5\lambda \right) \\ = 2 - 2\lambda + \frac{4\lambda}{1 - 2_5\lambda} (2_5 - 2_5\lambda) \\ = (2 - 2\lambda) \left(1 + \frac{4\lambda}{5 - 2\lambda} \right) = \frac{(2 - 2\lambda)(5 + 2\lambda)}{5 - 2\lambda}$$

$$So \quad \tilde{c}_3 \geq 0 \quad for \quad -\frac{5}{2} \leq \lambda \leq 1$$

and for $\lambda > \frac{5}{2}$

$$\tilde{c}_4 = 0 - (-1 + 4\mu \cdot \frac{2}{5}) = 1 - \frac{12\lambda}{5 - 2\lambda}$$

$$\tilde{c}_4 \geq 0 \quad for \quad \lambda < \frac{5}{2} \quad and \quad \lambda \leq \frac{5}{14}$$

or for $\lambda > \frac{5}{2}$.

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$$\tilde{c}_5 = 0 - \left(-2 + 4\mu \left(-\frac{\lambda}{5} \right) \right)$$
$$= 2 + \frac{4\lambda}{5-2\lambda}$$

$$\tilde{c}_5 \geq 0 \quad \text{for} \quad \lambda < \frac{5}{2} \quad \text{and} \quad 0\lambda \geq -5$$
$$\quad \quad \quad \text{or} \quad \lambda > \frac{5}{2} \quad \text{and} \quad 0\lambda \leq -5 \quad \times$$

$$\tilde{c}_6 = 0.$$

So still primal and dual feasible as long as:

$$\lambda < \frac{5}{2} \quad (\tilde{c}_5 \geq 0) \quad \text{and}$$

$$\lambda \leq \frac{25}{14} \quad (\tilde{B}^{-1}b \geq 0)$$

$$\text{and} \quad -\frac{5}{2} \leq \lambda \leq 1 \quad (\tilde{c}_3 \geq 0)$$

$$\text{and} \quad \lambda \leq \frac{5}{14} \quad (\tilde{c}_4 \geq 0).$$

i.e. if $-\frac{5}{2} \leq \lambda \leq \frac{5}{14}$

$$\left(\text{so } \lambda = \frac{1}{3} \text{ ok} \right).$$